On the Function which denotes the difference between the number of $(4m+1)$-divisors and the number of $(4m+3)$-divisors of a Number. By J. W. L. GLAISHER, M.A., F.R.S.

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1. The excess of the number of the divisors of a number n which have the form $4m+1$ over the number of divisors which have the form $4m+3$, is a quantity which occurs in researches connected with the Theory of Numbers, and also as coefficient in certain systems of q-series in Elliptic Functions.

If we denote this quantity by $E(n)$, so that

$$E(n) = \left.\begin{array}{l}\text{number of divisors of } n \text{ of the form } 4m+1 \\ - \quad\quad ,, \quad\quad ,, \quad\quad ,, \quad\quad 4m+3\end{array}\right\},$$

then it is obvious that, if $n = 2^p r$, where r is uneven (so that r is the greatest uneven divisor of n), we have $E(n) = E(r)$.

It is also easy to see that, if n be a number of the form $4m+3$, then $E(n) = 0$, for to every divisor of the form $4m+1$ there must correspond a conjugate divisor of the form $4m+3$. It can be shown also that $E(n)$ cannot be negative (see § 2).

2. In § 40 of the *Fundamenta Nova*, Jacobi states that, if $n = 2^p uv$, where u is an uneven number having all its prime factors of the form $4m+1$, and v an uneven number having all its prime factors of the form $4m+3$, then $E(n) = 0$ unless v is a square number, in which case

$$E(n) = \phi(u),$$

where $\phi(u)$ denotes the number of the divisors of u.

This important theorem may be proved in the following manner. The case of n uneven need alone be considered, as the uneven divisors of $2^p uv$ are evidently identical with those of uv.

As already remarked, the theorem is obviously true if n is of the form $4m+3$, for the product of two factors, both of which are of the form $4m+1$ or $4m+3$, is of the form $4m+1$; so that, in the case of a number of the form $4m+3$, there corresponds to every divisor of the form $4m+1$ a conjugate divisor of the form $4m+3$; and the number of divisors of the one form is therefore equal to the number of divisors of the other.

If n is of the form $4m+1$, suppose, first, that it $= a^\alpha b^\beta c^\gamma \ldots$, where $a, b, c, \ldots$ are all prime factors of the form $4m+3$. Then the divisors of n are the terms in the developed expression obtained by multiplying out the factors in the product

$$(1-a+a^2-\ldots \pm a^\alpha)(1-b+b^2\ldots \pm b^\beta)(1-c+c^2\ldots \pm c^\gamma)\ldots$$

The sign of each term in the developed expression is positive in the case of a divisor of the form $4m+1$, and negative in the case of a divisor of the form $4m+3$. Thus $E(n)$ is equal to the value of this product when $a, b, c, \ldots$ are all replaced by unity; whence it follows that $E(n) = 0$, unless $\alpha, \beta, \gamma, \ldots$ are all even, in which case $E(n)=1$. Next, suppose that $n = a^{\alpha}b^{\beta}c^{\gamma} \ldots r^{\rho}s^{\sigma}t^{\tau} \ldots$ where $a, b, c, \ldots$ are, as before, prime factors of the form $4m+3$, and where $r, s, t, \ldots$ are prime factors of the form $4m+1$. Then, reasoning as above, we see that $E(n)$ is equal to the value of

$$(1-a+a^2 \ldots \pm a^{\alpha})(1-b+b^2 \ldots \pm b^{\beta}) \ldots$$
$$\times (1+r+r^2 \ldots + r^{\rho})(1+s+s^2 \ldots + s^{\sigma}) \ldots,$$

when $a, b, \ldots r, s, \ldots$ are all replaced by unity.

Denoting, as above, by $\phi(p)$ the number of divisors of p, we have therefore $$E(n) = E(a^{\alpha}b^{\beta}c^{\gamma} \ldots) \times \phi(r^{\rho}s^{\sigma}t^{\tau} \ldots),$$

which, by means of the result found in the first case,

$$= \phi(r^{\rho}s^{\sigma}t^{\tau} \ldots) \quad \text{or} \quad 0,$$

according as $\alpha, \beta, \gamma, \ldots$ are all even, or are not all even.

It has thus been shown that $E(n) = 0$ unless all the prime factors of n, which are of the form $4m+3$, occur with even exponents; in which case, if $n = 2^p uv^2$, all the prime factors of u being of the form $4m+1$, and all the prime factors of v of the form $4m+3$, then $E(n) = \phi(u)$. We see also that $E(n)$ cannot ever be negative.

3. It follows from the preceding investigation that, if $n = n_1 n_2 n_3 \ldots$, where $n_1, n_2, n_3, \ldots$ are any relatively prime numbers, then

$$E(n) = E(n_1)\, E(n_2)\, E(n_3) \ldots$$

It is evident that, if p be a prime of the form $4m+1$, then

$$E(p^r) = r+1,$$

and that, if p be a prime of the form $4m+3$, then

$$E(p^r) = 1 \quad \text{or} \quad 0,$$

according as r is even or uneven.

Also $$E(2^r) = 1.$$

By means of these formulæ we may write down at once the value of $E(n)$, when n has been resolved into its prime factors. For example, since $$495000 = 2^3 \times 3^2 \times 5^4 \times 11,$$
we have $$E(495000) = 1 \times 1 \times 5 \times 2 = 10.$$

4. The following table, which was calculated in the manner just explained, contains the values of $E(n)$ for all the values of n, up to $n = 1000$, for which $E(n)$ is not equal to zero.

TABLE OF THE VALUES OF $E(n)$ FROM $n=1$ TO $n=1000$.

n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$
1	1	136	2	293	2	464	2	640	2	820	4
2	1	137	2	296	2	466	2	641	2	821	2
4	1	144	1	298	2	468	2	648	1	829	2
5	2	145	4	305	4	477	2	650	6	832	2
8	1	146	2	306	2	481	4	653	2	833	2
9	1	148	2	313	2	482	2	656	2	841	3
10	2	149	2	314	2	484	1	657	2	842	2
13	2	153	2	317	2	485	4	661	2	845	6
16	1	157	2	320	2	488	2	666	2	848	2
17	2	160	2	324	1	490	2	673	2	850	6
18	1	162	1	325	6	493	4	674	2	853	2
20	2	164	2	328	2	500	4	676	3	857	2
25	3	169	3	333	2	505	4	677	2	865	4
26	2	170	4	337	2	509	2	680	4	866	2
29	2	173	2	338	3	512	1	685	4	872	2
32	1	178	2	340	4	514	2	689	4	873	2
34	2	180	2	346	2	520	4	692	2	877	2
36	1	181	2	349	2	521	2	697	4	881	2
37	2	185	4	353	2	522	2	698	2	882	1
40	2	193	2	356	2	529	1	701	2	884	4
41	2	194	2	360	2	530	4	706	2	890	4
45	2	196	1	361	1	533	4	709	2	898	2
49	1	197	2	362	2	538	2	712	2	900	3
50	3	200	3	365	4	541	2	720	2	901	4
52	2	202	2	369	2	544	2	722	1	904	2
53	2	205	4	370	4	545	4	724	2	905	4
58	2	208	2	373	2	548	2	725	6	909	2
61	2	212	2	377	4	549	2	729	1	914	2
64	1	218	2	386	2	554	2	730	4	916	2
65	4	221	4	388	2	557	2	733	2	925	6
68	2	225	3	389	2	562	2	738	2	928	2
72	1	226	2	392	1	565	4	740	4	929	2
73	2	229	2	394	2	569	2	745	4	932	2
74	2	232	2	397	2	576	1	746	2	936	2
80	2	233	2	400	3	577	2	754	4	937	2
81	1	234	2	401	2	578	3	757	2	941	2
82	2	241	2	404	2	580	4	761	2	949	4
85	4	242	1	405	2	584	2	765	4	953	2
89	2	244	2	409	2	585	4	769	2	954	2
90	2	245	2	410	4	586	2	772	2	961	1
97	2	250	4	416	2	592	2	773	2	962	4
98	1	256	1	421	2	593	2	776	2	964	2
100	3	257	2	424	2	596	2	778	2	965	4
101	2	260	4	425	6	601	2	784	1	968	1
104	2	261	2	433	2	605	2	785	4	970	4
106	2	265	4	436	2	610	4	788	2	976	2
109	2	269	2	441	1	612	2	793	4	977	2
113	2	272	2	442	4	613	2	794	2	980	2
116	2	274	2	445	4	617	2	797	2	981	2
117	2	277	2	449	2	625	5	800	3	985	4
121	1	281	2	450	3	626	2	801	2	986	4
122	2	288	1	452	2	628	2	808	2	997	2
125	4	289	3	457	2	629	4	809	2		
128	1	290	4	458	2	634	2	810	2		
130	4	292	2	461	2	637	2	818	2		

The number of arguments for which $E(n)$ is not zero, and the sum of the values of $E(n)$ for each hundred numbers, are as follows :—

	Number of arguments.	Sum of values.
0—99	42	76
100—199	36	79
200—299	35	82
300—399	31	74
400—499	32	80
500—599	32	80
600—699	31	81
700—799	30	75
800—899	28	73
900—999	30	79
Total 0—999	327	779

5. The following two formulæ serve to express $E(n)$ linearly in terms of the E's of numbers less than n.

I.

If n be any uneven number, then

$$E(n)-2E(n-4)+2E(n-16)-2E(n-36)+\&c.$$
$$=0 \text{ or } (-1)^{\frac{1}{2}(\sqrt{n}-1)}\times\sqrt{n},$$

according as n is not, or is, a square number.

Every term in this formula is zero if n is of the form $4m+3$, so that no generality is lost by restricting n to the form $4m+1$.

II.

If n be any number,

$$E(n)-E(n-1)-E(n-3)+E(n-6)+E(n-10)-\&c.$$
$$=0 \text{ or } (-1)^n\times\tfrac{1}{4}\{(-1)^{\frac{1}{2}[\sqrt{(8n+1)}-1]}\times\sqrt{(8n+1)}-1\},$$

according as n is not, or is, a triangular number.

The numbers 1, 3, 6, 10, ..., which occur in the second formula, are the triangular numbers, given by the formula $\frac{1}{2}r(r+1)$, and, if n be itself a triangular number, the last term is $E(n-n)=0$. The signs of the terms after the first are negative and positive in pairs, the terms involving even triangular numbers in the argument having the positive sign, and those involving uneven triangular numbers having the negative sign.

Both formulæ are to be continued up to the term preceding the first term in which the argument becomes negative, *i.e.*, a term with negative argument is to be treated as zero. It may be noticed that, if we *define* $E(n-n)$ to denote

$$(-1)^{n-1} \times \tfrac{1}{4} \left\{(-1)^{\frac{1}{2}[\sqrt{(8n+1)}-1]} \times \sqrt{(8n+1)} - 1\right\},$$

then it follows, from II., that the expression

$$E(n) - E(n-1) - E(n-3) + E(n-6) + E(n-10) - \&c.$$

is equal to zero for *all* values of n; and this is perhaps the most convenient form in which to enunciate the theorem.

The following are examples of the formulæ. Putting $n = 77$ and 81, the formula I. gives

$$E(77) - 2E(73) + 2E(61) - 2E(41) + 2E(13) = 0,$$

and $E(81) - 2E(77) + 2E(65) - 2E(45) + 2E(17) = (-1)^{\frac{1}{2}(9-1)} \times 9$;

that is, $0 - 2\times 2 + 2\times 2 - 2\times 2 + 2\times 2 = 0,$

$$1 - \ \ 0 \ \ + 2\times 4 - 2\times 2 + 2\times 2 = 9.$$

Putting $n = 20$ and 21, the formula II. gives

$$E(20) - E(19) - E(17) + E(14) + E(10) - E(5) = 0,$$

and $E(21) - E(20) - E(18) + E(15) + E(11) - E(6)$

$$= (-1)^{21} \times \tfrac{1}{4}\{(-1)^{6} \times 13 - 1\},$$

that is, $2-0-2+0+2-2 = 0,$

and $0-2-1+0+0-0 = -3.$

6. The formulæ in the last section are of the same kind as Euler's celebrated formula

$$\psi(n) - \psi(n-1) - \psi(n-2) + \psi(n-5) + \psi(n-7) - \ldots = 0,$$

where $\psi(n)$ denotes the sum of the divisors of n; the numbers 1, 2, 5, 7, ..., which occur in the arguments, are the pentagonal numbers given by the formula $\frac{1}{2}r(3r \pm 1)$; and $\psi(n-n)$ is defined to denote n. Euler regarded this formula as very remarkable, since it affords a means of calculating the sum of the divisors of n by a process in which none of the operations have any reference to the divisors, which remain unknown.*

7. The equations I. and II. were obtained by Elliptic Functions as follows:—

Denoting $\dfrac{2K}{\pi}$ by ρ, we have the following formulæ:

$$k^{\frac{1}{2}}\rho^{\frac{1}{2}} = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + 2q^{\frac{49}{4}} + \&c.,$$

$$k'^{\frac{1}{2}}\rho^{\frac{1}{2}} = 1 - 2q + 2q^{4} - 2q^{9} + 2q^{16} - \&c.,$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2q^{\frac{1}{4}} - 6q^{\frac{9}{4}} + 10q^{\frac{25}{4}} - 14q^{\frac{49}{4}} + \&c.,$$

* "Hoc autem merito eo mirabilius videtur, cum nulla operatio sit instituta, quæ ad rationem divisorum ullo modo referri queat; quin etiam divisores, quorum summa per hanc regulam reperitur, ipsi manent incogniti, etiamsi sæpe ex consideratione ipsius summæ concludi possint." — *Opera Arithmetica Collecta*, Vol. i., p. 150.

and also $$k^{\frac{1}{2}}\rho = \frac{2q^{\frac{1}{4}}}{1+q^{\frac{1}{2}}} + \frac{2q^{\frac{3}{4}}}{1+q^{\frac{3}{2}}} + \frac{2q^{\frac{5}{4}}}{1+q^{\frac{5}{2}}} + \frac{2q^{\frac{7}{4}}}{1+q^{\frac{7}{2}}} + \&c.$$
$$= 2\{E(1)\,q^{\frac{1}{4}} + E(5)\,q^{\frac{5}{4}} + E(9)\,q^{\frac{9}{4}} + E(13)\,q^{\frac{13}{4}} + \&c.\},$$

$$k'^{\frac{1}{2}}\rho = 1 - \frac{4q^2}{1+q^2} + \frac{4q^6}{1+q^6} - \frac{4q^{10}}{1+q^{10}} + \frac{4q^{14}}{1+q^{14}} - \&c.$$
$$= 1 - 4E(1)\,q^2 + 4E(2)\,q^4 - 4E(3)\,q^6 + 4E(4)\,q^8 - \&c.;$$

whence we deduce the identical equations

$$2\{E(1)\,q^{\frac{1}{4}} + E(5)\,q^{\frac{5}{4}} + E(9)\,q^{\frac{9}{4}} + \&c.\} = \frac{2q^{\frac{1}{4}} - 6q^{\frac{9}{4}} + 10q^{\frac{25}{4}} - 14q^{\frac{49}{4}} + \&c.}{1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \&c.},$$

$$1 - 4E(1)\,q^2 + 4E(2)\,q^4 - 4E(3)\,q^6 + \&c. = \frac{2q^{\frac{1}{4}} - 6q^{\frac{9}{4}} + 10q^{\frac{25}{4}} - 14q^{\frac{49}{4}} + \&c.}{2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + 2q^{\frac{49}{4}} + \&c.}.$$

Dividing by $q^{\frac{1}{4}}$ in the first equation, and writing $q^{\frac{1}{2}}$ for q in the second, we have

$$E(1) + E(5)\,q + E(9)\,q^2 + \&c. = \frac{1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} - \&c.}{1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \&c.} \quad \ldots \text{(i.)},$$

$$1 - 4E(1)\,q + 4E(2)\,q^2 - 4E(3)\,q^3 + \&c. = \frac{1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \&c.}{1 + q + q^3 + q^6 + q^{10} + \&c.} \quad \ldots\ldots\ldots\ldots \text{(ii.)}$$

Thus $$\Sigma_0^\infty E(4n+1)\,q^n \times \Sigma_{-\infty}^{\infty}(-1)^n q^{n^2} = \Sigma_0^\infty (-1)^n (2n+1)\,q^{n(n+1)},$$
$$\{1 + 4\Sigma_1^\infty (-1)^n E(n)\,q^n\} \times \Sigma_0^\infty q^{\frac{1}{2}n(n+1)} = \Sigma_0^\infty (-1)^n (2n+1)\,q^{\frac{1}{2}n(n+1)}.$$

The formulæ I. and II. may be deduced from these results by equating the coefficients of q^n.

8. The following investigation is given here on account of the similarity of the resulting formula (which involves the difference between the sums of the even and uneven divisors of a number) to formula II. of § 5. We have

$$\frac{(1-q^2)(1-q^4)(1-q^6)\ldots}{(1-q)(1-q^3)(1-q^5)\ldots} = 1 + q + q^3 + q^6 + q^{10} + \&c.;$$

whence, taking the logarithm and differentiating,

$$\frac{q}{1-q} + \frac{3q^3}{1-q^3} + \frac{5q^5}{1-q^5} + \&c. - \frac{2q^2}{1-q^2} - \frac{4q^4}{1-q^4} - \frac{6q^6}{1-q^6} - \&c.$$
$$= \frac{q + 3q^3 + 6q^6 + 10q^{10} + 15q^{15} + \&c.}{1 + q + q^3 + q^6 + q^{10} + q^{15} + \&c.}.$$

Now, if we denote by $\zeta(n)$ the excess of the sum of the uneven divisors of n over the sum of the even divisors [so that $\zeta(n) = \psi(n)$

when n is uneven, and $\zeta(n)$ is negative when n is even], then it is evident that the left-hand member of this equation

$$= \zeta(1)\,q+\zeta(2)\,q^2+\zeta(3)\,q^3+\zeta(4)\,q^4+\&c.,$$

and we therefore find

$$\zeta(1)\,q+\zeta(2)\,q^2+\zeta(3)\,q^3+\zeta(4)\,q^4+\&c. = \frac{q+3q^3+6q^6+10q^{10}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.} \qquad \text{.........(iii.)}.$$

Writing this equation in the form

$$\Sigma_1^\infty\,\zeta(n)\,q^n \times \Sigma_0^\infty\,q^{\frac{1}{2}n(n+1)} = \Sigma_1^\infty\,\tfrac{1}{2}n\,(n+1)\,q^{\frac{1}{2}n(n+1)},$$

and equating the coefficients of q^n, we find that

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\zeta(n-10)+\&c.$$
$$= 0 \text{ or } n,$$

according as n is not, or is, a triangular number.

If, therefore, we define $\zeta(n-n)$ to denote $-n$, we have

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\zeta(n-10)+\&c. = 0$$

for all values of n.

As examples of the formula, putting $n = 20$ and 21, as in Formula II., § 5, we find

$$\zeta(20)+\zeta(19)+\zeta(17)+\zeta(14)+\zeta(10)+\zeta(5) = 0,$$
$$\zeta(21)+\zeta(20)+\zeta(18)+\zeta(15)+\zeta(11)+\zeta(6) = 21;$$

that is

$$-30+20+18-8-6+6 = 0,$$
$$32-30-13+24+12-4 = 21.$$

9. The formula (ii.) may be expressed in a form corresponding to (iii.); viz., we have

$$E(1)\,q-E(2)\,q^2+E(3)\,q^3-E(4)\,q^4+\&c. = \frac{q-q^3+2q^6-2q^{10}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.} \qquad \text{............(iv.)},$$

$$\zeta(1)\,q+\zeta(2)\,q^2+\zeta(3)\,q^3+\zeta(4)\,q^4+\&c. = \frac{q+3q^3+6q^6+10q^{10}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.} \qquad \text{............(iii.)},$$

the numerator in the first equation being

$$q-q^3+2q^6-2q^{10}+3q^{15}-3q^{21}+4q^{28}-4q^{36}+\&c.$$

This expression is evidently divisible by $1-q^2$, and, when divided by this factor, the quotient is

$$q+2q^6+2q^8+3q^{15}+3q^{17}+3q^{19}+4q^{28}+4q^{30}+4q^{32}+4q^{34}+5q^{45}+\&c.,$$

where the law of the terms is that, if $t_1, t_2, t_3, t_4, \ldots$ denote the triangular numbers 1, 3, 6, 10, ..., then the only exponents that occur are the numbers

$$t_{2r-1},\quad t_{2r-1}+2,\quad t_{2r-1}+4,\ \ldots\ t_{2r-1}+2r-2$$

(*i.e.*, the r even numbers intermediate to $t_{2r-1}-2$ and t_{2r} if t_{2r-1} and t_{2r} are even, and the r uneven numbers intermediate to these limits when t_{2r-1} and t_{2r} are uneven), and the coefficient of each of these terms is r.

10. Replacing $\frac{1}{1-q^2}$ by $1+q^2+q^4+q^6+\&c.$, we thus find

$$\{E(1)q-E(2)q^2+E(3)q^3-E(4)q^4+\&c.\}\times\{1+q^2+q^4+q^6+q^8+\&c.\}$$
$$=\frac{q+2q^6+2q^8+3q^{15}+3q^{17}+3q^{19}+\&c.}{1+q+q^3+q^6+q^{10}+q^{15}+\&c.}.$$

Now, let $S(2n-1)=E(1)+E(3)+E(5)\ldots+E(2n-1)$,

$$S(2n)=E(2)+E(4)+E(6)\ldots+E(2n);$$

then the left-hand member of this equation is

$$S(1)q-S(2)q^2+S(3)q^3-S(4)q^4+\&c.,$$

and therefore

$$S(1)q-S(2)q^2+S(3)q^3-S(4)q^4+\&c.=\frac{q+2q^6+2q^8+3q^{15}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.}$$
............(v.)

Equating the coefficients of q^n in the equation

$$\Sigma_1^\infty(-1)^{n-1}S(n)q^n\times\Sigma_0^\infty q^{\frac{1}{2}n(n+1)}=q+2q^6+2q^8+3q^{15}+\&c.,$$

we find that

$$S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\&c.$$
$$=0 \text{ or } (-1)^{n-1}r,$$

according as n is not, or is, one of the numbers

$$t_{2r-1},\quad t_{2r-1}+2,\quad \ldots,\quad t_{2r-1}+2r-2.$$

This theorem may also be enunciated in the following singular form:

Counting $S(0)$, when it occurs, as a term, though assigning to it the value zero, then

$$S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\&c.$$

is zero, (i.) if the number of terms is uneven, or (ii.) if the argument of the last term is uneven; but (iii.), if the number of terms is even and the argument of the last term is also even (0 being regarded as an even argument), then it is equal to

$$(-1)^{n-1}\times\tfrac{1}{2}\text{ number of terms.}$$

As examples of the three cases, let $n=90$, 98, and 99. The theorem gives

(i.)

$$S(90)-S(89)-S(87)+S(84)+S(80)-S(75)-S(69)$$
$$+S(62)+S(54)-S(45)-S(35)+S(24)+S(12)\qquad=0,$$

(ii.)

$$S(98)-S(97)-S(95)+S(92)+S(88)-S(83)-S(77)$$
$$+S(70)+S(62)-S(53)-S(43)+S(32)+S(20)-S(7) = 0,$$

(iii.)

$$S(99)-S(98)-S(96)+S(93)+S(89)-S(84)-S(78)$$
$$+S(71)+S(63)-S(54)-S(44)+S(33)+S(21)-S(8) = 7.$$

In (i.) the positive terms are $36+34+32+24+22+9+5 = 162$,
and the negative terms are $37+35+30+28+19+13 = 162$.

In (ii.) the positive terms are $37+36+34+27+24+12+9 = 179$,
and the negative terms are $39+37+31+30+22+17+3 = 179$.

In (iii.) the positive terms are $39+37+37+28+24+13+8 = 186$,
and the negative terms are $37+36+34+30+22+17+3 = 179$;

the difference being 7, as it should be.

The formula affords a complete verification of the accuracy of the values of a table of $E(n)$, for it involves all the arguments less than any given number n, and in such a manner that all the even-argument terms have one and the same sign, and all the uneven-argument terms have one and the same sign. Whenever, therefore, a term $E(r)$ enters, it occurs with the same sign, and an error in it would produce an increased effect (and could not be neutralised) by its repeated occurrence in the S-terms.

This will appear also from the developed form of the S-expression given in the next section.

11. The expression

$$(-1)^{n-1}\{S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\dots\}$$

is identically equal to the coefficient of q^n in the product

$$\{E(1)\,q-E(2)\,q^2+E(3)\,q^3-\&c.\}$$
$$\times\{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}+\&c.\}$$
$$\times\{1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\&c.\}.$$

Now we find by multiplication

$$(1+q^2+q^4+q^6+q^8+\&c.)\times(1+q+q^3+q^6+q^{10}+\&c.)$$
$$=$$
$$1$$
$$+q+q^2$$
$$+2q^3+q^4+2q^5$$
$$+2q^6+2q^7+2q^8+2q^9$$
$$+3q^{10}+2q^{11}+3q^{12}+2q^{13}+3q^{14}$$
$$+3q^{15}+\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$

where the coefficients are

$$\begin{array}{c} 1 \\ 1,\ 1 \\ 2,\ 1,\ 2 \\ 2,\ 2,\ 2,\ 2 \\ 3,\ 2,\ 3,\ 2,\ 3 \\ 3,\ 3,\ 3,\ 3,\ 3,\ 3 \\ 4,\ 3,\ 4,\ 3,\ 4,\ 3,\ 4 \\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4 \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \end{array}$$

The successive groups of terms contain one, two, three, four, &c. members. The groups containing an even number of members consist of *repetitions; e.g.*, the group of four consists of 4 twos, the group of six of 6 threes, &c.; the groups containing an uneven number of members consist of *alternations, e.g.*, the group of three is 2, 1, 2; the group of five is 3, 2, 3, 2, 3; and, in general, in a group containing $2r$ members, each coefficient is equal to r, and in a group containing $2r+1$ members, the coefficients are

$$r+1,\ r;\ r+1,\ \dots\ r,\ r+1.;$$

i.e., they are $r+1$ and r alternately, the first and last being $r+1$.

It follows, therefore, that

$$S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\dots$$

is identically equal to

$$\begin{array}{c} E(n) \\ -E(n-1)+E(n-2) \\ -2E(n-3)+E(n-4)-2E(n-5) \\ +2E(n-6)-2E(n-7)+2E(n-8)-2E(n-9) \\ +3E(n-10)-\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ \dots\ \dots\ \dots\ \dots\ \dots\ \pm rE(1). \end{array}$$

We notice that no term in this expression has a zero coefficient, which is in accordance with the remark made at the end of the last section.

12. Since the triple product considered in the preceding section is equal to $q+2q^6+2q^8+3q^{15}+3q^{17}+3q^{19}+\&c.$,

we find, by equating coefficients, the value of the E-expression for any given value of n, and we thus obtain a theorem which may be enunciated in the following curious form:

If n be any number, then

$$
\begin{array}{c}
E(n) \\
-E(n-1)+E(n-2) \\
-2E(n-3)+E(n-4)-2E(n-5) \\
+2E(n-6)-2E(n-7)+2E(n-8)-2E(n-9) \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \pm sE(0)
\end{array}
$$

is equal to zero, if we assign to $E(0)$ the value 0 or 1 in accordance with the following rule: (i.) if $sE(0)$ is a *term of an alternation*, or (ii.) if $sE(0)$ is a term of a sequence and n and s are one even and the other uneven, then $E(0) = 0$; (iii.) if $sE(0)$ is a term of a sequence and n and s are both even or both uneven, then $E(0) = 1$.

For example, putting $n = 5, 6, 7$, we have

$$
(\alpha) \qquad \begin{array}{c}
E(5) \\
-E(4)+E(3) \\
-2E(2)+E(1)-2E(0) = 0,
\end{array}
$$

$$
(\beta) \qquad \begin{array}{c}
E(6) \\
-E(5)+E(4) \\
-2E(3)+E(2)-2E(1) \\
+2E(0) \qquad\qquad = 0,
\end{array}
$$

$$
(\gamma) \qquad \begin{array}{c}
E(7) \\
-E(6)+E(5) \\
-2E(4)+E(3)-2E(2) \\
+2E(1)+2E(0) \qquad = 0.
\end{array}
$$

In (α), $E(0)$ falls in an alternation, so that, by (i.), we put $E(0) = 0$; in (β), $E(0)$ falls in a sequence and 6 and 2 are both even, therefore, by (iii.), we put $E(0) = 1$; in (γ), $E(0)$ falls in a sequence, but 7 and 2 are one even and the other uneven, whence, by (ii.), we put $E(0) = 0$.

Substituting for the E's their values, the equations (α), (β), (γ) become

$$
\begin{array}{ll}
(\alpha) \; \dots\dots & 2-1+0-2+1-0 = 0, \\
(\beta) \; \dots\dots & 0-2+1-0+1-2+2 = 0, \\
(\gamma) \; \dots\dots & 0-0+2-2+0-2+2+0 = 0.
\end{array}
$$

The theorem contained in this section affords a complete verification of a table of $E(n)$, and it also serves to express $E(n)$ in terms of the E's of *all* the numbers inferior to n.

Other formulæ of the same class, but in which the coefficients are regulated by a more simple law, will be investigated in the next two sections.

13. It was shown in § 7 that

$$E(1)+E(5)q+E(9)q^2+E(13)q^3+\&c.$$
$$=\frac{1-3q^2+5q^6-7q^{12}+9q^{20}-11q^{30}+13q^{42}-\&c.}{1-2q+2q^4-2q^9+2q^{16}-2q^{25}+2q^{36}-\&c.}.$$

If we divide the numerator of this expression by $1-q$, we obtain as quotient the expression

$$\begin{array}{c} 1+q \\ -2q^2-2q^3-2q^4-2q^5 \\ +3q^6+3q^7+3q^8+3q^9+3q^{10}+3q^{11} \\ -4q^{12}-4q^{13}-4q^{14}-4q^{15}-4q^{16}-4q^{17}-4q^{18}-4q^{19} \\ +5q^{20}+\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots \\ \dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots \end{array}$$

where the successive groups of two, four, six, ... terms have opposite signs, and each term in a group of $2r$ terms has the coefficient r.

If we divide the denominator by $1-q$, we obtain as quotient the expression

$$\begin{array}{c} 1 \\ -q-q^2-q^3 \\ +q^4+q^5+q^6+q^7+q^8 \\ -q^9-q^{10}-q^{11}-q^{12}-q^{13}-q^{14}-q^{15} \\ +q^{16}+\;\dots\;\dots\;\dots\;\dots\;\dots \\ \dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots \end{array}$$

where the successive groups of one, three, five, ... terms have opposite signs, and all the terms have the coefficient unity.

We thus find

$$E(1)+E(5)q+E(9)q^2+E(13)q^3+\&c.$$
$$=\frac{1+q-2(q^2+q^3+q^4+q^5)+3(q^6+q^7+q^8+q^9+q^{10}+q^{11})-\&c.}{1-q-q^2-q^3+q^4+q^5+q^6+q^7+q^8-q^9-q^{10}-q^{11}-q^{12}-q^{13}-\&c.},$$

and, by equating coefficients, we may deduce the following theorem:

If p denote any number of the form $4m+1$, then

$$\begin{array}{c} E(p) \\ -E(p-4)-E(p-8)-E(p-12) \\ +E(p-16)+E(p-20)+E(p-24)+E(p-28)+E(p-32) \\ -E(p-36)-\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots \\ \dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots\;\dots \\ \dots\;\dots\;\dots\;\dots\;\dots\pm E(1) \\ =(-1)^{\frac{1}{2}(r-1)}\times\tfrac{1}{2}(r+1), \end{array}$$

where r^2 is the greatest uneven square which does not exceed p. Thus, if p is itself a square, $r=\sqrt{p}$.

For example, let $p = 29$; the theorem gives

$$\begin{aligned} &E(29) \\ &-E(25)-E(21)-E(17) \\ &+E(13)+E(9)+E(5)+E(1) \\ &= (-1)^{\frac{1}{2}(5-1)} \times \tfrac{1}{2}(5+1), \end{aligned}$$

that is, $$2-3-0-2+2+1+2+1 = 3.$$

The table in § 4 was verified by means of this formula, the values of the positive and negative terms in

$$\begin{matrix} & & & E(997) & & & \\ & & -E(993) & -E(989) & -E(985) & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & -E(5)-E(1) & & \end{matrix}$$

being found to be 188 and 204 respectively, so that the value of this expression is -16. The greatest uneven square not exceeding 997 is 31^2 and $(-1)^{\frac{1}{2}(31-1)} \times \frac{1}{2}(31+1) = -16$.

The preceding formula affords a simple and complete verification of a table of $E(n)$; for the terms are combined by mere addition and subtraction, and all the arguments are of the form $4m+1$. As $E(n)$ vanishes when n is of the form $4m+3$, the formula is thus free from the presence of a number of terms which must necessarily be zero. Also, the absence of the even-argument terms is not a disadvantage, for we may regard the definition of $E(n)$ as applying primarily to the case of n uneven, the extension to n even being made by means of the formula $E(2^m p) = E(p)$. In actually verifying a table of $E(n)$, the accuracy of the even-argument terms is completely proved by merely verifying that, for every even argument $2m$, $E(2m) = E(m)$.

14. From (iv.) of § 9 we have

$$E(1)q-E(2)q^2+E(3)q^3-\&c. = \frac{q(1-q^2)+2q^6(1-q^4)+3q^{15}(1-q^6)+\&c.}{1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+q^{28}+\&c.}$$

and, dividing both numerator and denominator by $1-q$, we find

$$E(1)q-E(2)q^2+E(3)q^3-\&c. = \frac{q+q^2+2q^6+2q^7+2q^8+2q^9+3q^{15}+\&c.}{1+2q+2q^2+3q^3+3q^4+3q^5+4q^6+\&c.},$$

the numerator being

$$\begin{aligned} &q+q^2 \\ &+2q^6+2q^7+2q^8+2q^9 \\ &+3q^{15}+3q^{16}+3q^{17}+3q^{18}+3q^{19}+3q^{20} \\ &+4q^{28}+ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

and the denominator being

$$\begin{array}{l} 1 \\ +2q+2q^2 \\ +3q^3+3q^4+3q^5 \\ +4q^6+4q^7+4q^8+4q^9 \\ +5q^{10}+ \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}$$

We may thus, by equating coefficients, obtain the theorem:

If n be any number, then

$$\begin{array}{l} E(n) \\ -2E(n-1)+2E(n-2) \\ -3E(n-3)+3E(n-4)-3E(n-5) \\ +4E(n-6)-4E(n-7)+4E(n-8)-4E(n-9) \\ +5E(n-10)- \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad +(-1)^{n-1}rE(1) \\ = (-1)^{n-1}\times\tfrac{1}{2}s \text{ or } 0, \end{array}$$

according as s is even or uneven, where s is what the coefficient of $E(0)$ would be, if the formula were continued one term further. Thus $s=r$, unless $rE(1)$ is the last term in a group, in which case $s=r+1$.

Taking as examples $n=5$ and 6, the formula gives

$$\begin{array}{l} E(5) \\ -2E(4)+2E(3) \\ -3E(2)+3E(1) \qquad = 0, \end{array}$$

and

$$\begin{array}{l} E(6) \\ -2E(5)+2E(4) \\ -3E(3)+3E(2)-3E(1) = (-1)^5\times 2, \end{array}$$

since the coefficient of $E(0)$, if the formula were continued one term further, would be 4.

Substituting for the E's their values, these equations become

$$2-2+0-3+3 \quad = 0,$$
$$0-4+2-0+3-3 = -2.$$

15. If we divide by $1-q$ (*i.e.*, multiply by $1+q+q^2+q^3+\&c.$) the numerator and denominator of the fraction in equation (iii.) of § 9,

viz.,
$$\frac{q+3q^3+6q^6+10q^{10}+15q^{15}+\&c.}{1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\&c.},$$

we find

$$\zeta(1)q+\zeta(2)q^2+\zeta(3)q^3+\&c. = \frac{q+q^2+4q^3+4q^4+4q^5+10q^6+10q^7+\&c.}{1+2q+2q^2+3q^3+3q^4+3q^5+4q^6+\&c.},$$

and, by equating coefficients as before, we obtain the theorem:

If n be any number, then

$$\begin{array}{c} \zeta(n) \\ +2\zeta(n-1)+2\zeta(n-2) \\ +3\zeta(n-3)+3\zeta(n-4)+3\zeta(n-5) \\ +4\zeta(n-6)+\dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad +r\zeta(1) \\ = \tfrac{1}{6}(s^3-s), \end{array}$$

where s is the coefficient of $\zeta(0)$ if the series be continued one term further. As before, $s=r$ unless $r\zeta(1)$ is the last term of a group, in which case $s=r+1$. The right-hand member of the equation

$$= \tfrac{1}{6}(s+1)\,s\,(s-1),$$

and is therefore clearly integral, since $s-1$, s, and $s+1$ are consecutive numbers.

Putting, as before, $n=5$ and 6, the formula gives

$$\begin{array}{ll} \zeta(5) & \\ +2\zeta(4)+2\zeta(3) & \\ +3\zeta(2)+3\zeta(1) & = \tfrac{1}{6}(3^3-3), \end{array}$$

and

$$\begin{array}{l} \zeta(6) \\ +2\zeta(5)+2\zeta(4) \\ +3\zeta(3)+3\zeta(2)+3\zeta(1) = \tfrac{1}{6}(4^3-4); \end{array}$$

that is, $6+2\times-5+2\times4+3\times-1+3=4,$

$$-4+2\times6+2\times-5+3\times4+3\times-1+3\times1=10.$$

16. It may be mentioned that a formula,* similar to the E-formula and the ζ-formula which have just been given, exists also in the case of the function ψ, where $\psi(n)$ denotes the sum of the divisors of n; viz., we have, if n be any number,

$$\begin{array}{c} \psi(n) \\ -2\psi(n-1)-2\psi(n-2) \\ +3\psi(n-3)+3\psi(n-4)+3\psi(n-5) \\ -4\psi(n-6)- \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad +(-1)^{r-1}r\psi(1) \\ = (-1)^s\times\tfrac{1}{6}(s^3-s), \end{array}$$

where s has the same meaning as in the two preceding sections.

* This formula was communicated to the Cambridge Philosophical Society in February, 1884. The paper in which it occurs is in course of publication in the Society's *Transactions*.

Putting $n = 5$ and 6, the formula gives

$$\begin{aligned}&\psi(5)\\ &-2\psi(4)-2\psi(3)\\ &+3\psi(2)+3\psi(1) = -\tfrac{1}{6}(3^3-3),\end{aligned}$$

and

$$\begin{aligned}&\psi(6)\\ &-2\psi(5)-2\psi(4)\\ &+3\psi(3)+3\psi(2)+3\psi(1) = \tfrac{1}{6}(4^3-4),\end{aligned}$$

that is,

$$6 - 2\times 7 - 2\times 4 + 3\times 3 + 3 = -4,$$

$$12 - 2\times 6 - 2\times 7 + 3\times 4 + 3\times 3 + 3 = 10.$$

17. The formulæ contained in the three preceding sections may be enunciated in the following convenient form by including the term involving $E(0)$.

If n be any number, then

I.

$$\begin{aligned}&E(n)\\ &-2E(n-1)+2E(n-2)\\ &-3E(n-3)+3E(n-4)-3E(n-5)\\ &+4E(n-6)- \dots\\ &\dots\\ &\dots + (-1)^n sE(0) = 0,\end{aligned}$$

where $E(0)$ denotes $\frac{1}{2}$ or 0 according as s is even or uneven.

II.

$$\begin{aligned}&\zeta(n)\\ &+2\zeta(n-1)+2\zeta(n-2)\\ &+3\zeta(n-3)+3\zeta(n-4)+3\zeta(n-5)\\ &+4\zeta(n-6)+ \dots\\ &\dots\\ &\dots + s\zeta(0) = 0,\end{aligned}$$

where $\zeta(0)$ denotes $-\frac{1}{6}(s^2-1)$.

III.

$$\begin{aligned}&\psi(n)\\ &-2\psi(n-1)-2\psi(n-2)\\ &+3\psi(n-3)+3\psi(n-4)+3\psi(n-5)\\ &-4\psi(n-6)- \dots\\ &\dots\\ &\dots + (-1)^{s-1} s\psi(0) = 0,\end{aligned}$$

where $\psi(0)$ denotes $\frac{1}{6}(s^2-1)$.

Expressions for $E(n)$, &c., as determinants.

18. If $P_1x - P_2x^2 + P_3x^3 - \&c. = \dfrac{b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \&c.}{1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \&c.}$,

then it can be shown that

$$P_n = \begin{vmatrix} b_1, & b_2, & b_3, & b_4, & b_5, & \dots \\ 1, & a_1, & a_2, & a_3, & a_4, & \dots \\ 0, & 1, & a_1, & a_2, & a_3, & \dots \\ 0, & 0, & 1, & a_1, & a_2, & \dots \\ 0, & 0, & 0, & 1, & a_1, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \; (n \text{ rows}),$$

where the elements of the third row are the same as those of the second, but shifted one place to the right, the first element being a cipher; the elements of the fourth row are the same as those of the third, but shifted one place to the right, with two ciphers prefixed; and so on. Excepting only the first row, all the elements in any diagonal parallel to the principal diagonal are the same. In writing determinants of this form, it is sufficient therefore to give the first two rows.

By means of this theorem we may deduce at once from the formulæ given in this paper the following expressions for $E(n)$, &c., in which, for brevity, only the first two rows of the determinants are written:

(i.)

$$E(n)$$

$$=$$

$$\begin{vmatrix} 1, 0, -1, 0, 0, 2, 0, 0, 0, -2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, -3, \dots \\ 1, 1, \quad 0, 1, 0, 0, 1, 0, 0, \quad 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, \quad 0, \dots \end{vmatrix}$$

$$=$$

$$\begin{vmatrix} 1, 1, 0, 0, 0, 2, 2, 2, 2, 0, 0, 0, 0, 0, 3, 3, 3, 3, 3, 3, 0, 0, 0, 0, 0, \dots \\ 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, \dots \end{vmatrix}$$

$$=$$

$$\begin{vmatrix} 1, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 3, 0, 3, 0, 3, 0, 0, 0, 0, 0, 0, \dots \\ 1, 1, 1, 2, 1, 2, 2, 2, 2, 2, 3, 2, 3, 2, 3, 3, 3, 3, 3, 3, 3, 4, 3, 4, 3, \dots \end{vmatrix},$$

each determinant containing n rows.

(ii.)

$$E(4n+1)$$

$$=$$

$$\begin{vmatrix} 1, 0, -3, 0, 0, 0, 5, 0, 0, 0, 0, 0, -7, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, \dots \\ 1, 2, \quad 0, 0, 2, 0, 0, 0, 0, 2, 0, 0, \quad 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, \dots \end{vmatrix}$$

$$=$$

$$(-)^n \begin{vmatrix} 1, \quad 1, -2, -2, -2, -2, 3, 3, 3, \quad 3, \quad 3, \quad 3, -4, -4, \dots \\ 1, -1, -1, -1, \quad 1, \quad 1, 1, 1, 1, -1, -1, -1, -1, -1, \dots \end{vmatrix},$$

each determinant containing $n+1$ rows.

(iii.)

$$S(n) = \begin{vmatrix} 1, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 3, 0, 3, 0, 3, 0, 0, 0, 0, 0, 0, \dots \\ 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, \dots \end{vmatrix},$$

the determinant containing n rows.

(iv.)

$$(-)^{n-1}\zeta(n) = \begin{vmatrix} 1, 0, 3, 0, 0, 6, 0, 0, 0, 10, 0, 0, 0, 0, 15, 0, 0, 0, 0, 0, 21, 0, 0, 0, \dots \\ 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, \dots \end{vmatrix}$$

$$= \begin{vmatrix} 1, 1, 4, 4, 4, 10, 10, 10, 10, 20, 20, 20, 20, 20, 35, 35, 35, 35, 35, \dots \\ 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, \dots \end{vmatrix},$$

each determinant containing n rows.

It may be added that

$$(-)^{n-1}\psi(n) = \begin{vmatrix} 1, 1, -4, -4, -4, 10, 10, 10, 10, -20, -20, -20, -20, \dots \\ 1, -2, -2, 3, 3, 3, -4, -4, -4, -4, 5, 5, 5, \dots \end{vmatrix},$$

the determinant containing n rows.

These determinant-values are, of course, quite inappropriate for purposes of calculation, being at best but inconvenient forms of expressing the results given by the formulæ I. and II. of § 5, and the similar formulæ in §§ 8—16. They seem, however, worth notice, as affording, though in an impracticable form, definite numerical expressions for $E(n)$, &c.

The functions $E(n)$, $\chi(n)$, $\lambda(n)$.

19.* If n be uneven, the number of primary complex numbers having n as their norm is equal to $E(n)$. In the *Quarterly Journal of Mathematics* for June, 1884, Vol. xx., pp. 97—167, I have considered the function $\chi(n)$, which denotes the sum of the primary complex numbers having n as norm, and also $\lambda(n)$ the sum of their squares. These functions are connected with one another and with $E(n)$ and $\psi(n)$ by a number of relations in which the terms follow laws similar to those which occur in this paper. The following two formulæ are perhaps the most curious of these relations; they would afford complete verifications of a table giving the values of $E(n)$ and $\chi(n)$.

* This section has been added since the paper was read.

If p be any number of the form $4m+1$, then

(i.)

$$\begin{aligned}&E(p)\\&-2E(p-4)-2E(p-8)\\&+3E(p-12)+3E(p-16)+3E(p-20)\\&-4E(p-24)-\dots\dots\dots\dots\dots\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\&=\\&\chi(p)\\&+2\chi(p-4)+2\chi(p-8)\\&+3\chi(p-12)+3\chi(p-16)+3\chi(p-20)\\&+4\chi(p-24)-\dots\dots\dots\dots\dots\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

(ii.)

$$\begin{aligned}&E(p)\\&-2E(p-8)-2E(p-16)\\&+3E(p-24)+3E(p-32)+3E(p-40)\\&-4E(p-48)-\dots\dots\dots\dots\dots\dots\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\&=\\&(-)^{m}\{\chi(p)\\&+2\chi(p-8)+2\chi(p-16)\\&+3\chi(p-24)+3\chi(p-32)+3\chi(p-40)\\&+4\chi(p-28)+\dots\dots\dots\dots\dots\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\},\end{aligned}$$

the series being continued so long as the arguments remain positive.

The Relations of the Intersections of a Circle with a Triangle. *By* Mr. H. M. Taylor.

[*Read Feb. 14th,* 1884.]

If, in a given triangle ABC, a triangle $\alpha\beta\gamma$ be inscribed, and the circumscribed circle of the triangle $\alpha\beta\gamma$ cut the sides BC, CA, AB again respectively in α', β', γ', then, as will be proved, if the triangle $\alpha\beta\gamma$ remains constant in shape (see Fig. 1)—

(1) The angles of the triangle $\alpha'\beta'\gamma'$ are determinate.