Global Fixed-Depth Finite Reduction in the Collatz Carry Equation

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Abstract

We study fixed-depth finite-block valuation families in the accelerated odd-only Collatz carry equation. For a valuation word D of length L and valuation sum S, *wall contact* means $2^S - 3^L \mid C_L(D)$, where $C_L(D)$ is the associated carry sum; wall contacts are exactly the valuation words of integer cycles of the accelerated odd map. Previous notes established complete classifications for selected two-block, three-block, and one alternating four-block family. In this note we isolate the general mechanism behind those classifications. For a fixed k-block family $D = X_1^{n_1} \cdots X_k^{n_k}$ we prove a general telescoping identity, derive the $(k{+}1)$-monomial contact structure, and establish a rotated 2-adic lever for each block; cyclic rotation allows every block to become extremal. A corrected spread comparison and a top-anchored mass bound control the size of rotated representatives. The central combinatorial input is a quantitative Peel Lemma: in the critical core, runs can be processed in decreasing height order, bounding one exponent at a time; outside the core, a one-sided window estimate gives a constant budget and a segment version of the same peeling argument. The result is a global fixed-depth finite-reduction theorem: for fixed k, fixed nonsingular blocks, and block slopes bounded away from the critical slope $\log_2 3$, all wall contacts lie in an explicit finite box. The theorem is a finite-reduction theorem, not a classification theorem, and it does not prove the Collatz conjecture. The remaining obstructions are precisely the expected ones: near-critical block slopes $R_i \to 0$, unbounded block count $k \to \infty$, singular blocks corresponding to integer-cycle values, and positive-entropy valuation words.

1 Introduction

A sequence of earlier technical notes classified concrete finite-block families of valuation words in the accelerated odd-only Collatz carry equation: two-block families via a master identity and cone certificates [6, 7], three-block families via a finite-depth lever classification [8], and, most recently, the alternating four-block family $C(1/3)^a(4)^bC(1/6)^c(3)^d$, which was shown to admit no wall contacts at all [9]. The four-block case is structurally significant: its return slopes alternate in sign about the critical slope $\alpha = \log_2 3$, so ascending and descending runs can nearly cancel and no global monotonicity argument applies. The classification succeeded because cyclic rotation lets *every* block run be made extremal, where a rotated 2-adic lever forces any hypothetical contact value into a rigid congruence class and hence to exponential size in that run's exponent.

This note abstracts the mechanism into a general fixed-depth theorem. The point of the theorem is not that a fixed family has no contacts, but that *for every fixed nonsingular slope-separated finite-block family, the infinite exponent search reduces to a finite exact computation.* The classifications of [7, 8, 9] are then instances of the general pattern: finite reduction by the theorem, classification by the exact check.

We state the scope plainly. The theorem proved here is a finite-reduction theorem, not a nonexistence theorem and not a classification theorem; it does not prove the Collatz conjecture; and it does not handle block slopes approaching the critical line, unbounded block count, singular (integer-cycle) blocks, or positive-entropy valuation words. These boundaries are discussed in Sections 11 and 13.

2 The Collatz carry equation

Definition 2.1. A *valuation word* is a finite sequence $D = (d_0, \dots, d_{L-1})$ with $d_i \geq 1$. Write $s_j = \sum_{i<j} d_i$ (so $s_0 = 0$), $S = s_L$,

$$C_L(D) = \sum_{j=0}^{L-1} 3^{\,L-1-j} 2^{\,s_j}, \qquad N_{L,S} = 2^S - 3^L.$$

Wall contact means $N_{L,S} \mid C_L(D)$, i.e. $C_L(D) = m\, N_{L,S}$ for some $m \in \mathbb{Z}$, the *contact value.*

Wall contacts are precisely the valuation words of cycles of the accelerated odd map $x \mapsto (3x+1)/2^{v_2(3x+1)}$: a cycle through the odd integer m with valuations D satisfies $C_L(D) = m\, N_{L,S}$, and conversely. $C_L(D)$ is always odd (the $j = 0$ term is the unique odd term) and $N_{L,S}$ is odd, so every contact value is odd.

With the *return path* $R_j = s_j - j\alpha$, $\alpha = \log_2 3$ (so $R_0 = 0$, $R_L = S - L\alpha$), one has directly

$$C_L(D) = 3^{L-1} \sum_{j=0}^{L-1} 2^{R_j}, \qquad N_{L,S} = 3^L\bigl(2^{R_L} - 1\bigr), \qquad Q(D) := \frac{C_L(D)}{N_{L,S}} = \frac{1}{3}\,\frac{\sum_j 2^{R_j}}{2^{R_L} - 1}.$$

Wall contact means $Q(D) \in \mathbb{Z}$.

Standing notation for families. Throughout, $D(\mathbf{n}) = X_1^{n_1} \cdots X_k^{n_k}$, $n_i \geq 1$, with fixed blocks X_i and fixed phases; $q_i = |X_i|$, $S_i = \sum X_i$, $R_i = S_i - q_i\alpha$, $\delta_i = 3^{q_i} - 2^{S_i}$, $C_i = C(X_i)$; $L = \sum n_i q_i$, $S = \sum n_i S_i$, $N = N_{L,S}$, $R_L = \sum n_i R_i$. We abbreviate $C = C_L$ where the length is clear. The *critical core* is $|R_L| \leq w$ for a fixed width $w \geq 1$. Standing hypotheses where stated: *slope separation* $|R_i| \geq \rho > 0$ and *nonsingularity* $\delta_i \nmid C_i$ for all i. We write $\lambda_i = \log_2 |\delta_i|$, $g_i = S_i - (R_i)_+$, and $(x)_\pm$ for positive/negative parts.

3 The general k-block telescoping identity

For block X_i with its fixed phase, let $R_j(X_i)$, $0 \leq j < q_i$, be the internal return path ($R_0 = 0$), and set

$$x_i = 2^{\,n_i R_i}, \qquad A_i = \sum_{j=0}^{q_i-1} 2^{R_j(X_i)}, \qquad G_i = \frac{1}{3}\,\frac{A_i}{2^{R_i} - 1}.$$

A direct computation gives $G_i = C_i/(2^{S_i} - 3^{q_i}) = -C_i/\delta_i$: each G_i is the block's own rational cycle value.

Theorem 3.1 (telescoping identity)**.** *For every $k \geq 1$ and all $\mathbf{n}$,*

$$Q\big(D(\mathbf{n})\big) = \frac{\displaystyle\sum_{i=1}^{k} G_i \Big(\prod_{r<i} x_r\Big)(x_i - 1)}{\displaystyle\prod_{i=1}^{k} x_i - 1}.$$

Proof. Partition $[0, L)$ into the k runs. The run of $X_i^{n_i}$ enters at height $z_i = \sum_{r<i} n_r R_r$, so $2^{z_i} = \prod_{r<i} x_r$; its t-th copy starts at $z_i + tR_i$ and contributes $2^{z_i + tR_i} A_i$ to $\sum_j 2^{R_j}$. Summing the geometric series over $0 \leq t < n_i$ gives $2^{z_i} A_i (x_i - 1)/(2^{R_i} - 1)$; note that for $R_i < 0$ both $x_i - 1$ and $2^{R_i} - 1$ are negative, so the quotient is positive, as a sum of positive terms must be. Sum over i and divide by $3(2^{R_L} - 1)$ with $2^{R_L} = \prod_i x_i$. □

For $k = 2, 3, 4$ this is, verbatim, the two-, three-, and four-block master identities of [6, 8, 9].

4 The $(k{+}1)$-monomial contact structure

Multiplying the contact equation $C(D) = mN$ by $P = \prod_i \delta_i$ and expanding each run by the repetition formula (Lemma 5.1 below), run i contributes $(P/\delta_i) C_i \, 3^{\ell_i} 2^{v_i} \big(3^{q_i n_i} - 2^{S_i n_i}\big)$ with $v_i = \sum_{r<i} S_r n_r$ and $\ell_i = \sum_{r>i} q_r n_r$, splitting into a positive and a negative monomial. The negative monomial of run i and the positive monomial of run $i{+}1$ coincide and merge.

Proposition 4.1. *The cleared contact equation has exactly $k + 1$ distinct monomials: the 3^L-corner, $k - 1$ junctions, and the 2^S-corner. For a chosen cyclic anchoring, m appears only in the two extreme coefficients — the 3^L-corner carries $(P/\delta_1)(\delta_1 m + C_1)$ and the 2^S-corner carries $Pm - (P/\delta_k)C_k$ — while every interior junction coefficient is m-free, equal to $(P/\delta_{i+1})C_{i+1} - (P/\delta_i)C_i$, an odd integer minus an odd integer, hence* even. *Reducing the equation modulo $2^{S_1 n_1}$ kills all monomials except the 3^L-corner; modulo $2^{S_1 n_1 + 1}$, the first junction enters at order exactly $2^{S_1 n_1}$ with an even coefficient and contributes nothing — this parity is the source of the "+1" in the rotated 2-adic lever. Finally, setting $n_i = 0$ annihilates run i's term and merges its two adjacent junctions into the single junction of the $(k{-}1)$-block equation: lower-depth contact equations nest inside higher-depth ones as exact degenerations.*

We keep this section brief: the main theorem consumes the lever form of the next section, not the full monomial expression.

5 The rotated 2-adic lever

Lemma 5.1 (concatenation; repetition)**.** *For all words U, V: $C(UV) = 3^{|V|} C(U) + 2^{S_U} C(V)$. For any block and $n \geq 1$: $C(X_i^n) = C_i\big(3^{q_i n} - 2^{S_i n}\big)/\delta_i$, an exact integer ($\delta_i$ odd).*

Proof. Terms $j < |U|$ acquire the factor $3^{|V|}$; terms $j \geq |U|$ the factor 2^{S_U}. The repetition formula follows by induction with $U = X_i$, $V = X_i^{n-1}$ and a geometric sum. □

Theorem 5.2 (rotated 2-adic lever). *Let $k \geq 2$, all $n_j \geq 1$, and let a cyclic rotation put a run first: $E = X_i^{n_i} Y$ with $Y \neq \varnothing$, $S_Y = S - S_i n_i \geq 1$. Put $\mathfrak{G}_i(E) = \delta_i C(E) + C_i N$. Then*

$$\mathfrak{G}_i(E) = 2^{\,S_i n_i}\,\mathfrak{h}, \qquad \mathfrak{h} = C_i\,2^{S_Y} + \delta_i\,C(Y) - 3^{|Y|}C_i,$$

and $\mathfrak{h}$ is even; hence $2^{\,S_i n_i+1} \mid \mathfrak{G}_i(E)$ and, under wall contact $C(E) = m_i N$ (with N odd),

$$2^{\,S_i n_i+1} \;\Big|\; \delta_i\,m_i + C_i.$$

Proof. By Lemma 5.1, $\delta_i C(E) = 3^{|Y|}C_i(3^{q_i n_i} - 2^{S_i n_i}) + \delta_i 2^{S_i n_i} C(Y) = 3^L C_i - 2^{S_i n_i}\big(3^{|Y|}C_i - \delta_i C(Y)\big)$, using $3^{|Y|}3^{q_i n_i} = 3^L$. Adding $C_i N = C_i 2^S - C_i 3^L$ cancels the 3^L terms and factors out $2^{S_i n_i}$. Parity: $C_i 2^{S_Y}$ is even since $S_Y \geq 1$; $\delta_i C(Y)$ and $3^{|Y|}C_i$ are odd; odd minus odd is even. ◻

Corollary 5.3 (singular dichotomy). *Under the hypotheses of Theorem 5.2 and wall contact:*

1. *$\delta_i \mid C_i$ if and only if the block's rational cycle value $-C_i/\delta_i = C_i/(2^{S_i} - 3^{q_i})$ is an integer, i.e. X_i is the valuation word of an integer cycle (a* singular *block); in that case the congruence class of Theorem 5.2 contains one bounded lift for every n_i, namely the cycle value itself.*

2. *If $\delta_i \nmid C_i$, then $\delta_i m_i + C_i \neq 0$, hence*

$$|m_i| \;\geq\; \frac{2^{\,S_i n_i+1} - |C_i|}{|\delta_i|}, \qquad \textit{so} \qquad \log_2 |m_i| \;\geq\; S_i n_i - \lambda_i \quad \textit{whenever } 2^{S_i n_i} \geq |C_i|.$$

Proof. If $\delta_i m_i + C_i = 0$ then $m_i = -C_i/\delta_i$, an integer iff $\delta_i \mid C_i$; otherwise it is a nonzero multiple of $2^{S_i n_i+1}$. ◻

6 Rotation invariance and the spread comparison

Lemma 6.1 (rotation identity). *For $0 \leq t \leq L$: $2^{s_t} C(\sigma^t D) = 3^t C(D) + N\,C_t(D)$, with $C_t(D) = \sum_{j<t} 3^{t-1-j} 2^{s_j}$. Since $\gcd(2^{s_t}3^t, N) = 1$, wall contact is rotation-invariant, and all $m_t = C(\sigma^t D)/N$ are integers as soon as one is.*

Proof. For $t = 1$: $2^{d_0} C(\sigma D) = \sum_{i=1}^{L} 3^{L-i} 2^{s_i} = 3(C(D) - 3^{L-1}) + 2^S = 3C(D) + N$, with $C_1 = 1$; induct via $C_{t+1} = 3C_t + 2^{s_t}$. ◻

Lemma 6.2 (window identity). *With the periodic extension $s_{j+L} = s_j + S$ (so $R_{j+L} = R_j + R_L$): for every t, $3(2^{R_L} - 1)\,m_t = 2^{-R_t}\,\Sigma_t$, where $\Sigma_t = \sum_{j=t}^{t+L-1} 2^{R_j}$.*

Theorem 6.3 (corrected spread comparison). *For all t, t':*

$$\frac{|m_t|}{|m_{t'}|} = 2^{\,R_{t'} - R_t + \varepsilon}, \qquad |\varepsilon| \leq |R_L|.$$

Proof. By Lemma 6.2 the ratio is $2^{R_{t'} - R_t} \Sigma_t / \Sigma_{t'}$. Writing $\Sigma_t = \Sigma_0 + (2^{R_L} - 1)P_t$ with $P_t = \sum_{j<t} 2^{R_j} \in (0, \Sigma_0)$, every Σ_t/Σ_0 lies strictly between $\min(1, 2^{R_L})$ and $\max(1, 2^{R_L})$, so $\Sigma_t/\Sigma_{t'} \in (2^{-|R_L|}, 2^{|R_L|})$. ◻

Inside the core the comparison costs at most w bits, independently of k, L, and $\mathbf{n}$; this replaces the crude $O(L)$ error of earlier drafts. Outside the core the two-sided comparison is *not* used; see Section 9.

7 The top-anchored mass bound

Let $h_i = \max_{0\le u<q_i} R_u(X_i) \le S_i - q_i$ (word-free, since all valuations are ≥ 1), $h_{\max} = \max_i h_i$, $R_{\max} = \max_{j\in[0,L)} R_j$, and $V^{\uparrow}(D) = \sum_{j<L} 2^{-(R_{\max}-R_j)}$.

Theorem 7.1. *If $|R_i| \ge \rho > 0$ for all i, then for all $\mathbf{n}$ (with no restriction on R_L),*

$$V^{\uparrow}(D) \;\le\; V_0 := \sum_{i=1}^{k} \frac{q_i 2^{h_i}}{1-2^{-|R_i|}} + \max_i \frac{q_i 2^{h_i}}{1-2^{-|R_i|}} \;\le\; (k+1)\max_i \frac{q_i\, 2^{h_i}}{1-2^{-\rho}}.$$

Proof. View one period, anchored at $R_{\max}$: it splits into the k runs with at most one run cut into two pieces. In a run of X_i^n entering at height $z \le R_{\max}$, copy t starts at $z + tR_i$ and every index inside it has $R_j \le z + tR_i + h_i$ (descending runs; ascending runs are re-indexed from the run's top); each piece is therefore a geometric tail $\le q_i 2^{h_i} \sum_{t\ge 0} 2^{-t|R_i|}$. Sum the $\le k+1$ pieces, the cut run counting at most twice. □

Boundary statement. $(1-2^{-\rho})^{-1} \sim (\rho \ln 2)^{-1}$, so $V_0 \to \infty$ as $\rho \to 0$: near-critical blocks — those with S_i/q_i a continued-fraction convergent of α — are a genuine boundary of the method. V_0 also grows linearly in k.

8 The critical-core Peel Lemma

Assume in this section $|R_L| \le w$. Define the *block-boundary heights* $H_0 = 0$, $H_v = \sum_{r\le v} n_r R_r$ ($1 \le v \le k$, $H_k = R_L$), viewed cyclically (the wrap identifies the endpoints up to $|R_L| \le w$), and $H^* = \max_v H_v$. By the proof of Theorem 7.1, $H^* \le R_{\max} \le H^* + h_{\max}$: the step-level maximum exceeds the boundary maximum by at most $h_{\max} = \max_i(S_i - q_i)$, so intra-block phases enter only through this constant.

Per-anchor inequality. Fix an anchor i and rotate run i first. Chaining Corollary 5.3(2), Theorem 6.3, Lemma 6.2 at the top anchor (which gives $3|2^{R_L}-1||m_\star| \le 2^w V^{\uparrow}$ inside the core), and a two-log lower bound $|S\log 2 - L\log 3| > cL^{-\mu}$ (the only Diophantine input):

$$g_i n_i \;\le\; \bar{B} + D_i, \qquad g_i = \begin{cases} q_i\alpha, & R_i > 0,\\ S_i, & R_i < 0,\end{cases} \qquad D_i := H^* - \max(H_{i-1}, H_i) \;\ge\; 0, \tag{1}$$

where $\bar{B} = \mu \log_2 L + \log_2 V_0 + 3w + \lambda_{\max} + h_{\max} + O(1)$ collects all constants. (For ascending runs the higher endpoint is H_i and the slope term $n_i R_i$ is absorbed into $g_i = S_i - R_i = q_i\alpha$; for descending runs the higher endpoint is H_{i-1} and $g_i = S_i$. The gap is never uniformly $q_i\alpha$; this corrected accounting is essential.)

Theorem 8.1 (Peel Lemma, critical core)**.** *Order the vertices $v_1, \ldots, v_k$ by decreasing height. There are run sets $\varnothing \ne P_1 \subseteq \cdots \subseteq P_k = \{1,\ldots,k\}$, with P_s containing all runs adjacent to $v_1, \ldots, v_s$, and budgets $B_1 \le B_2 \le \cdots$ with $n_i \le B_s/g_i$ for $i \in P_s$, where $B_1 = \bar{B} + w$ and*

$$B_{s+1} \;\le\; \bar{B} + w + \sum_{j\in P_s} (R_j)_+ \frac{B_s}{g_j} \;\le\; B_s\big(1+\Theta_\Sigma\big), \qquad \Theta_\Sigma = \sum_{j:\,R_j>0} \frac{R_j}{q_j\alpha}.$$

Hence all exponents are bounded after at most k stages: $n_i \le (\bar{B}+w)(1+\Theta_\Sigma)^{k-1}/g_i$ for every i.

Proof. Stage 1. Let v_1 attain H^*. The run ending at v_1 is ascending (a descending run would start above the maximum) and has deficit 0; the run starting at v_1 is descending with deficit 0. By (1), both satisfy $g_i n_i \le \bar{B} + w$ (the $+w$ covering the cyclic wrap).

Induction (final climb). Suppose all runs in P_s satisfy $n_j \le B_s/g_j$. Take v_{s+1} and a run $i \notin P_s$ adjacent to it; its other endpoint cannot be strictly higher (it would be a processed vertex, putting $i \in P_s$), so v_{s+1} is run i's higher endpoint and $D_i = H^* - H_{v_{s+1}}$. Traverse the cycle *forward* from v_{s+1} to v_1, and let u_0 be the last vertex on this arc with height $\le H_{v_{s+1}}$. Beyond u_0, every vertex is strictly higher than v_{s+1}, hence processed; therefore every run traversed beyond u_0 has a processed endpoint and lies in P_s. The total rise of this final segment is at least $H_{v_1} - H_{v_{s+1}} = D_i$ (up to the wrap correction $\le w$) and at most the sum of the positive increments of the traversed runs, i.e.

$$D_i \;\le\; \sum_{j \in P_s} n_j (R_j)_+ \;+\; w.$$

By (1) and the inductive bounds, $g_i n_i \le B_{s+1}$ with B_{s+1} as displayed. After all k vertices are processed, every run is adjacent to a processed vertex, so P_k is everything. □

Remark 8.2. A worst-case all-to-all penalty matrix $M_{ij} = (R_j)_+/g_i$ has spectral radius < 1 iff $\sum_{\text{asc}} \theta_j/(1+\theta_j) < 1$, $\theta_j = R_j/(q_j\alpha)$ — a condition *violated* by the classified alternating four-block family (≈ 1.076). The matrix abstraction discards the cyclic order and the core constraint; Theorem 8.1 shows they are exactly what closes the system, and it replaces the failed spectral-radius criterion. For concrete families, the stage-exact recursion (restricted to runs actually on the final-climb arc) is far sharper than the crude $(1+\Theta_\Sigma)^{k-1}$ and is the recommended instrument.

9 The non-core window bound and the segment Peel Lemma

Assume now $|R_L| \ge w$. The two-sided comparison of Theorem 6.3 would cost $|R_L|$, which may be large; it is not used. Instead, bound each rotation directly from Lemma 6.2. Splitting Σ_{t_i} (t_i the start of run i, height H_{i-1} in the fixed period) into unwrapped and wrapped terms, with the wrapped heights shifted by R_L:

$$3\left|2^{R_L} - 1\right| |m_i| \;\le\; 2^{\,\mathrm{osc}_i}\left(1 + 2^{(R_L)_+}\right) V^{\uparrow}, \qquad \mathrm{osc}_i = R_{\max} - H_{i-1}. \tag{2}$$

If $R_L \ge w$, the factor $1 + 2^{R_L} \le 2 \cdot 2^{R_L}$ is cancelled by the denominator $2^{R_L} - 1 \ge (1 - 2^{-w})2^{R_L}$; if $R_L \le -w$, the wrapped contribution is already small ($2^{(R_L)_+} = 1$) and $|2^{R_L} - 1| \ge 1 - 2^{-w}$. In both cases:

$$\log_2 |m_i| \;\le\; \mathrm{osc}_i + \log_2 V_0 + c_w, \qquad c_w = 1 - \log_2 3 + \log_2 \tfrac{1}{1-2^{-w}} \quad (c_1 \approx 0.415). \tag{3}$$

No Diophantine input is needed off-core, and no $\log L$ appears: combining (3) with the lever yields the per-anchor inequality (1) with the *constant* budget $\bar{B}_{\mathrm{nc}} = \lambda_{\max} + \log_2 V_0 + c_w + h_{\max} + O(1)$.

Theorem 9.1 (Peel Lemma, segment version)**.** *Assume $|R_L| \ge w$, and view one period as a line segment with vertices $0, \dots, k$, heights $H_0 = 0, \dots, H_k = R_L$ (no cyclic identification). Order* *the $k+1$ vertices by decreasing height. Then the conclusion of Theorem 8.1 holds with $\bar B + w$ replaced by $\bar B_{\mathrm{nc}}$, no wrap corrections, and*

$$\Theta_\Sigma \quad \textit{replaced by} \quad \Theta_{\mathrm{abs}} = \sum_{i=1}^{k} \frac{|R_i|}{g_i}: \qquad n_i \;\le\; \frac{\bar B_{\mathrm{nc}}\,(1+\Theta_{\mathrm{abs}})^{k-1}}{g_i} \quad \textit{for all } i.$$

Proof. Identical induction, with one change. The traversal from v_{s+1} toward the maximal vertex v_1 is now in a *forced* direction (leftward or rightward along the segment; there is no second arc). The final-climb segment beyond the last sub-level vertex again consists of runs with processed endpoints; its rises are $n_j(R_j)_+$ for runs traversed forward and $n_j(R_j)_-$ for runs traversed backward (a descending run climbed in reverse is a rise), hence at most $\sum_{j\in P_s} n_j|R_j|$. The boundary vertices 0 and k have single adjacent runs; the stage-1 analysis covers them. Termination in at most k effective stages is as before. □

The wrap is handled *analytically*, inside (2), so the combinatorial argument never needs cyclic heights; this is what dissolves the apparent winding problem of the non-core regime.

10 Main theorem

Theorem 10.1 (Global Fixed-Depth Finite Reduction)**.** *Let $D(\mathbf{n}) = X_1^{n_1}\cdots X_k^{n_k}$ be a fixed finite-block family with fixed phases, and suppose:*

1. *k is fixed;*
2. *$|R_i| \ge \rho > 0$ for every block;*
3. *$\delta_i \nmid C_i$ for every block;*
4. *an explicit two-log lower bound $|S\log 2 - L\log 3| > c\,L^{-\mu}$ is available (used only in the critical core).*

Fix $w \ge 1$. Then every wall contact of the family satisfies $1 \le n_i \le N_i(X_1,\dots,X_k,\rho,w)$ for all i, with

$$N_i \;\le\; \frac{1}{g_i}\max\Big(\,(\bar B(L^*)+w)\,(1+\Theta_\Sigma)^{k-1},\;\; \bar B_{\mathrm{nc}}\,(1+\Theta_{\mathrm{abs}})^{k-1}\Big),$$

where $\bar B(L) = \mu\log_2 L + \log_2 V_0 + 3w + \lambda_{\max} + h_{\max} + O(1)$, L^* is the finite self-consistent fixed point of $L \mapsto \sum_i q_i N_i(L)$ (which exists and is unique because the right side is $O(\log L)$), and $\bar B_{\mathrm{nc}}$ is the constant non-core budget of Section 9.

Proof. Hypotheses 2–3 activate Theorem 7.1 and Corollary 5.3(2). For contacts with $|R_L| \le w$, hypothesis 4 supplies the core budget and Theorem 8.1 closes the exponents; for $|R_L| \ge w$, the constant budget (3) and Theorem 9.1 close them with no Diophantine input. The core branch dominates the box through its $\mu\log_2 L^*$ term. □

Remark 10.2 (how to read the theorem). This theorem should be read as a finite-reduction theorem, not as a nonexistence theorem. It says that a fixed nonsingular slope-separated finite-block family cannot contain unbounded wall contacts: any possible contacts are confined to a computable finite box. Classification of a given family still requires the exact finite check of that box; found contacts are then classified by primitive-root reduction as primitive or as known-cycle resyllabifications.

11 Singular blocks

A block is *singular* when $\delta_i \mid C_i$ — equivalently (Corollary 5.3), when its rational cycle value is an integer, i.e. the block is the valuation word of an integer cycle of the accelerated map. An exhaustive scan of all 58,650 words with $q \le 10$, $S \le 16$ found exactly: $(1)^n$ (value -1), $(2)^n$ (value 1), $(1,2)^n$ and rotations (values $-5, -7$), and the seven rotations of the -17 word $(1,1,1,2,1,1,4)$ (wall 139; values $-17, -25, -37, -41, -55, -61, -91$) — and nothing else. A full classification of singular blocks is essentially a cycle-classification problem (their completeness is equivalent to the positive and negative cycle conjectures), so nonsingularity is a necessary hypothesis, not an omission.

Nonsingular blocks can nevertheless *concatenate* into known cycle words: for instance, $X_1 = (1,1,1,2)$ ($\delta = 49$, $C = 65$) and $X_2 = (1,1,4)$ ($\delta = -37$, $C = 19$) are both nonsingular, yet $X_1^1 X_2^1$ is the -17 word, a genuine wall contact. Such contacts appear as isolated points inside the finite box (powers $(X_1X_2)^n$ are not of the form $X_1^{n_1}X_2^{n_2}$ for $n > 1$), are found by the exact check, and are classified by primitive-root post-processing. They are calibration structures, fully consistent with finite reduction.

12 Checks against archived cases

Applied back to the archive, the theorem reproduces the known reductions with looser but finite constants. For two-block families (3 monomials, ≤ 2 peel stages) and three-block families (4 monomials, ≤ 3 stages), the archived reductions of [6, 7, 8] are recovered; families containing the -17 word as a block are routed, correctly, into the singular calibration branch. For the classified alternating four-block family $C(1/3)^a(4)^bC(1/6)^c(3)^d$ [9], the general core constants return the box $(14, 14, 6, 33)$ against the family-tuned $(13, 13, 6, 32)$ — agreement to within one unit per variable. The general (worst-case) non-core constants give a larger box than the family-tuned one; to remove any reliance on the tuned derivation, the exact computation of [9] was re-executed over the enlarged box $a, b \le 20$, $c \le 8$, $d \le 40$, all 18 phase tuples: 2,304,000 candidates, zero wall contacts (exact big-integer arithmetic; 2.1 seconds, single core). The four-block classification therefore stands under either derivation.

13 Limitations and frontier

This theorem does not prove the Collatz conjecture, and nothing in this note should be read as approaching one. The theorem does not handle: near-critical slopes ($\rho \to 0$, where $V_0 \sim (\rho \ln 2)^{-1}$ diverges — the convergent blocks $(q, S) = (12, 19), (53, 84), \ldots$ live exactly there); unbounded block count ($k \to \infty$, where the constants grow like $(1 + \Theta)^{k-1}$ and V_0 like k); singular completeness (equivalent to cycle classification); or positive-entropy valuation words, which are not representable by fixed finite blocks at all. The remaining frontier is precisely

$$R_i \to 0, \qquad k \to \infty, \qquad \delta_i \mid C_i, \qquad \text{positive entropy},$$

the structural boundary of the finite-block obstruction method. Any hypothetical nontrivial cycle must evade the present mechanism through one of these modes; this is a structural localization, not a proof.

14 Tier ledger

Tier 1 (proved here or certified by exact computation). The k-block telescoping identity (Theorem 3.1); the $(k+1)$-monomial structure (Proposition 4.1); the rotated 2-adic lever (Theorem 5.2); the singular dichotomy (Corollary 5.3); the rotation identity (Lemma 6.1); the corrected spread comparison (Theorem 6.3); the $V^{\uparrow}$ bound (Theorem 7.1); the core Peel Lemma (Theorem 8.1); the non-core one-sided window bound (2)–(3); the segment Peel Lemma (Theorem 9.1).

Tier 2. The explicit two-log bound used in the critical core (hypothesis 4 of Theorem 10.1); for concrete families it reduces to a deep-tail role via certified continued fractions, as in [9].

Tier 3. Finite scans: the singular-block evidence of Section 11 and the numerical stress tests of the peel inequalities reported in the companion audits.

Tier 4. Conjectural extensions to $\rho \to 0$, $k \to \infty$, and positive-entropy words.

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