Research Roadmap: Toward a Mathematical Physics Paper

Low-Dimensional Invariant Structure of the Balanced Metric
on the Dwork Quintic Family

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Internal research notes

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Contents

1 The Central Mathematical Question

The empirical paper established the following observation:

For the Fermat quintic $X_0 \subset \mathbb{CP}^4$ with the $k = 10$ balanced metric, the determinant ratio $R_\psi(z) = \log(\det g_k / \det g_{\mathrm{FS}})$ is a function effectively of only two symmetric invariants (p_2, e_3), with all remaining independent generators contributing $\Delta R^2 < 0.001$.

The mathematical physics paper asks: **why is this true, and can it be proved?**

This is a genuine open question at the intersection of complex geometry, representation theory, and spectral theory. The empirical evidence is strong. The proof requires understanding how the Monge-Ampère constraint selects specific directions in the ring of G-invariant functions on X_0.

2 Mathematical Setup

2.1 The Fermat and Dwork Quintics

Definition 1. The **Dwork family** is the one-parameter family of quintic hypersurfaces

$$X_\psi = \Big\{ z \in \mathbb{CP}^4 \;\Big|\; \sum_{i=0}^{4} z_i^5 - 5\psi\, z_0 z_1 z_2 z_3 z_4 = 0 \Big\}, \qquad \psi \in \mathbb{C} \setminus \{1, \omega, \ldots\}.$$

The **Fermat quintic** X_0 is the case $\psi = 0$.

2.2 The Symmetry Group

Definition 2. Let $G = (U(1)^5/U(1)) \rtimes S_5 \cong U(1)^4 \times S_5$ be the **residual symmetry group** of X_0, acting on $\mathbb{CP}^4$ by

$$(e^{i\theta_0}, \ldots, e^{i\theta_4}) : [z_0 : \cdots : z_4] \mapsto [e^{i\theta_0} z_0 : \cdots : e^{i\theta_4} z_4] \qquad (\textstyle\sum_j \theta_j = 0),$$

and S_5 permutes the coordinates. The Fermat quintic X_0 is G-invariant.

2.3 The Normalisation Locus and Invariant Generators

Working on the normalisation locus $\Sigma = \{z : \sum_i |z_i|^2 = 1\} \cap X_0$, set $u_i = |z_i|^2$. The G-invariant real-valued functions of $(u_0, \ldots, u_4)$ form a polynomial ring generated by the power sums

$$p_k = \sum_{i=0}^{4} u_i^k, \qquad k = 2, 3, 4, 5,$$

(note: $p_1 = \sum u_i = 1$ is fixed by normalisation). Equivalently, via Newton's identities, one may use the elementary symmetric polynomials e_2, e_3, e_4, e_5 as generators, with

$$e_2 = \tfrac{1-p_2}{2}, \quad e_3 = \tfrac{1-3p_2+2p_3}{6}, \quad e_4 = \tfrac{1-6p_2+8p_3+3p_2^2-6p_4}{24}, \quad e_5 = u_0 u_1 u_2 u_3 u_4.$$

The **invariant ring** is thus

$$\mathcal{R} = \mathbb{R}[p_2, p_3, p_4, p_5] = \mathbb{R}[e_2, e_3, e_4, e_5],$$

with the two generating sets related algebraically.

2.4 The Balanced Metric and Determinant Ratio

Donaldson's balanced metric g_k at polynomial degree k on $(X_0, \mathcal{O}(k)|_{X_0})$ is the unique Hermitian metric satisfying the balanced condition

$$\frac{1}{N_k}\sum_{\alpha} |s^\alpha(z)|^2_{h_k^{\otimes k}} = \frac{1}{\mathrm{Vol}(X_0)}$$

for an L^2-orthonormal basis $\{s^\alpha\}$ of $H^0(X_0, \mathcal{O}(k))$, with $N_k = \dim H^0(X_0, \mathcal{O}(k))$. The metric is obtained as the fixed point of Donaldson's T-operator.

The **determinant ratio** is

$$R_k(z) \;=\; \log\frac{\det g_k(z)}{\det g_{\mathrm{FS}}(z)},$$

where det denotes the determinant of the 3×3 local metric tensor.

3 What Can Be Proved Now: The Easy Part

The following are rigorous or near-rigorous statements that require only standard arguments.

Proposition 1 (G-invariance of the det-ratio)**.** *The function $R_k : X_0 \to \mathbb{R}$ is G-invariant.*

Proof sketch. The Fermat quintic X_0 is G-invariant. The Fubini-Study metric g_{FS} is G-invariant. The balanced metric g_k is the unique solution to the balanced equation on $(X_0, \mathcal{O}(k))$; since the equation and the variety are both G-invariant, the solution must be too. Therefore $\det g_k / \det g_{\mathrm{FS}}$ is G-invariant. □

Corollary 2 (Four-generator reduction)**.** *There exists a smooth function $F_k : \mathbb{R}^4 \to \mathbb{R}$ such that*

$$R_k(z) = F_k(p_2, e_3, e_4, e_5)$$

for all $z \in X_0$.

Proof. Follows from Proposition 1 and the fact that $\{p_2, e_3, e_4, e_5\}$ (equivalently $\{p_2, p_3, p_4, p_5\}$) generate the G-invariant polynomial ring on Σ. □

Remark 1. Corollary 2 alone reduces the problem from 875^2 parameters (the full H-matrix space) to a function of 4 real variables. The real mathematical challenge is the further reduction from 4 generators to 2.

Lemma 3 (H-matrix block structure at $\psi = 0$)**.** *At the Fermat point $\psi = 0$, the balanced H-matrix is diagonal in the monomial basis and the diagonal entries h_α depend only on the partition type $\lambda(\alpha) = (\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_4)$ of the multi-index α.*

Proof. The $U(1)^4 \subset G$ action makes the H-matrix diagonal. The $S_5 \subset G$ action makes the diagonal entries depend only on the unordered multi-set of exponents, i.e., the partition type. □

Remark 2. For $k = 10$, the number of distinct partition types of 10 into at most 5 parts is exactly 30. So the balanced H-matrix at $\psi = 0$ is determined by *30 real parameters*, not 875. The empirical observation that R_k depends on 2 generators rather than 4 is a further reduction on top of this.

4 The Main Conjecture: Precise Statements

We state three versions, from weakest to strongest.

Conjecture 1 (Weak: approximate two-dimensionality)**.** For the Fermat quintic X_0 with the degree-k balanced metric ($k = 6, 8, 10$), the function $F_k : \mathbb{R}^4 \to \mathbb{R}$ from Corollary 2 satisfies

$$\left\| \frac{\partial F_k}{\partial e_4} \right\|_{L^2(X_0)}, \ \left\| \frac{\partial F_k}{\partial e_5} \right\|_{L^2(X_0)} \ll \left\| \frac{\partial F_k}{\partial p_2} \right\|_{L^2(X_0)}.$$

Quantitatively: the variance explained by (p_2, e_3) exceeds 99% of the total, i.e., $\mathrm{Var}[R_k - \Pi_{(p_2,e_3)} R_k]/\mathrm{Var}[R_k] < 0.001$.

Conjecture 2 (Medium: asymptotic two-dimensionality)**.** As $k \to \infty$, the normalised correction $k(R_k - \bar{R}_k)$, where $\bar{R}_k$ is the mean value of R_k, converges to a function $f_\infty(p_2, e_3)$ depending only on (p_2, e_3):

$$\sup_{z \in X_0} \left| k(R_k(z) - \bar{R}_k) - f_\infty(p_2(z), e_3(z)) \right| = O(k^{-1}).$$

Conjecture 3 (Strong: spectral selection)**.** Let $\mathcal{L}$ be the linearisation of the Monge-Ampère operator around $g_{\mathrm{FS}}|_{X_0}$. The leading eigenfunctions of $\mathcal{L}$ with non-trivial G-invariant projection lie in the span of $\{p_2, e_3\}$ (viewed as multiplication operators on $L^2(X_0, g_{\mathrm{FS}})$).

Priority. Conjecture 1 is most directly supported by the data and most accessible. Conjecture 2 would be a clean mathematical theorem. Conjecture 3 would explain the *why*.

5 Proof Strategy

5.1 Step 1: Compute the Scalar Curvature of the FS Metric on X_0

Goal. Show that the scalar curvature $s(z)$ of the induced FS metric on the Fermat quintic is, or is well-approximated by, a function of (p_2, e_3) alone.

Why this matters. The Tian-Yau-Zelditch (TYZ) expansion states that the Bergman kernel satisfies

$$k^{-3} B_k(z, z) \sim 1 + \frac{s(z)}{2k} + \frac{a_2(z)}{k^2} + \cdots$$

as $k \to \infty$. The leading correction to the balanced metric (and hence to R_k) is controlled by $s(z)$. If $s(z) = s(p_2, e_3)$, this would imply Conjecture 2.

How to compute s. For a smooth hypersurface $X \hookrightarrow \mathbb{CP}^n$ of degree d with the induced FS metric, the scalar curvature is

$$s_X = n(n+2) - \frac{(d-1) \cdot |II|^2_{g_{\mathrm{FS}}}}{1},$$

(schematically) where $|II|^2$ is the squared norm of the second fundamental form. For the Fermat quintic $F = \sum_{i=0}^4 z_i^5$, the relevant quantities are:

$$\partial_i F = 5 z_i^4, \tag{1}$$

$$\partial_i \partial_j F = 20 z_i^3 \delta_{ij}, \tag{2}$$

$$|\nabla F|^2_{g_{\mathrm{FS}}} = 25 \sum_i |z_i|^8 / (\sum_i |z_i|^2)^4 = 25\, p_4 \quad (\text{on } \Sigma). \tag{3}$$

The second fundamental form has components $II_{ij} = \partial_i \partial_j F / |\nabla F|$, which after careful computation gives

$$|II|^2 \propto \frac{\sum_i |\partial_i \partial_i F|^2}{|\nabla F|^2} = \frac{400 \sum_i |z_i|^6}{25 \sum_i |z_i|^8} = 16 \cdot \frac{p_3}{p_4}.$$

(The off-diagonal terms vanish for the Fermat quintic by the diagonal structure $\partial_i \partial_j F = 20 z_i^3 \delta_{ij}$.)

Key observation. The scalar curvature involves p_3/p_4. Using $p_3 = (6e_3 - 1 + 3p_2)/2$ (Newton's identity), we can express p_3 in terms of (p_2, e_3). However, p_4 remains. The question becomes:

> *Is p_4 effectively a function of (p_2, e_3) in the regime sampled by the Fermat quintic, or does it vary independently?*

This must be checked numerically on the actual point cloud. If p_4 varies substantially independently of (p_2, e_3) but R_k does not depend on it, then the mechanism is more subtle than just "scalar curvature controls everything."

5.2 Step 2: Spectral Analysis of the Linearised Operator

Goal. Identify which G-invariant functions on X_0 are in the leading eigenspaces of the Lichnerowicz-type operator that controls the balanced metric iteration.

Donaldson's T-operator linearised around the balanced metric satisfies

$$\delta(T(H))_{\alpha\beta} = \int_{X_0} K(z, w) \delta H_{\alpha\beta} \, d\mu,$$

for some integral kernel K. The convergence of the balanced iteration is governed by the spectral gap of this operator, specifically by how quickly the "non-FS directions" in H-matrix space are suppressed.

The G-invariant sector decomposes into representations of S_5. The trivial irrep (symmetric tensors) is spanned by functions of (p_2, p_3, p_4, p_5). **The conjecture is that the eigenfunctions with large eigenvalues (slow convergence) correspond to (p_2, e_3), while those corresponding to (e_4, e_5) have smaller eigenvalues.**

Concrete computation. On X_0 with the FS metric, compute the spectrum of the Laplace-Beltrami operator $\Delta_{g_{\mathrm{FS}}}$ restricted to G-invariant functions. The basis functions p_2, e_3, e_4, e_5 are all in the G-invariant sector. If the eigenvalues satisfy $\lambda_2 \sim \lambda_{e_3} \gg \lambda_{e_4} \sim \lambda_{e_5}$, this would suggest (p_2, e_3) are the "slow modes" that persist in the balanced metric correction.

5.3 Step 3: Bergman Kernel Asymptotics on the G-Invariant Sector

Goal. Compute the leading TYZ coefficient restricted to the G-invariant sector and show it lies in $\mathrm{span}(p_2, e_3)$.

The Bergman kernel on X_0 decomposes into G-irrep sectors. In the trivial (G-invariant) sector:

$$B_k^G(z, z) = \sum_{\alpha \in G\text{-orbits}} \frac{|\mathcal{O}_\alpha(z)|^2}{h_\alpha}$$

where $\mathcal{O}_\alpha$ denotes the symmetrised monomial orbit sum. These orbit sums are precisely the G-invariant sections of $\mathcal{O}(k)|_{X_0}$, and they are naturally expressible as polynomial functions of (p_2, p_3, p_4, p_5).

The key result to prove: the ratio $B_k^G(z, z)/B_k^{G,\mathrm{FS}}(z, z)$, where $B_k^{G,\mathrm{FS}}$ is the G-invariant Bergman kernel for the FS metric, is to leading order in k a function of (p_2, e_3) only.

5.4 Step 4: Connect to the Dwork Deformation

For the deformed quintic X_ψ, the symmetry group reduces to $\Gamma_\psi \subset G$. The coefficient trajectories $c_i(\psi)$ encode how the dominant modes (p_2, e_3) acquire ψ-dependent amplitudes. A weaker but still interesting result: show that the deformation vector field $\partial_\psi g_k$ lies in the (p_2, e_3) sector to leading order, using Kodaira-Spencer deformation theory.

6 Key Technical Lemmas You Need

Ordered by difficulty (ascending):

1. **[Easy, 1 week]** *The G-invariant ring on $\Sigma \cap X_0$ is generated by (p_2, e_3, e_4, e_5) as a $\mathbb{R}$-algebra.*

   This is a standard invariant-theory result. On the normalisation locus $\sum u_i = 1$, the ring $\mathbb{R}[u_0, \ldots, u_4]^{S_5}$ is generated by e_2, e_3, e_4, e_5 (with $e_1 = 1$ fixed). The $\mathrm{U}(1)^4$ action adds no further constraints since it acts on the phases of z_i, not on $u_i = |z_i|^2$.
   References: Stanley, "Invariants of Finite Groups," Ch. 3.

2. **[Easy-Medium, 2 weeks]** *Explicit computation of the second fundamental form of $X_0 \hookrightarrow \mathbb{CP}^4$ in coordinates.*

   Using the formula $II = \nabla^2 F / |\nabla F|$ restricted to the tangent bundle of X_0, compute $|II|^2$ explicitly and express it as a function of $(u_0, \ldots, u_4)$. Identify which symmetric polynomials appear.
   References: Griffiths-Harris, "Principles of Algebraic Geometry," Ch. 1.5; Huybrechts, "Complex Geometry," Ch. 3.

3. **[Medium, 4-6 weeks]** *The scalar curvature of $g_{\mathrm{FS}}|_{X_0}$ expressed in terms of (p_2, p_3, p_4).*

   Use the Gauss equation $s_{X_0} = s_{\mathbb{CP}^4}|_{X_0} - |II|^2$ and the computation from Lemma 2 to get s explicitly. Then use Newton's identities to express p_3 in terms of (p_2, e_3) and p_4 in terms of (p_2, e_3, e_4). Determine numerically whether the e_4 term is negligible on typical points of X_0.
   References: Jost, "Riemannian Geometry and Geometric Analysis," Ch. 4.

4. **[Medium, 4-6 weeks]** *First TYZ coefficient for (X_0, g_{FS}).*

   Verify that the first TYZ coefficient $a_1(z) = s(z)/2$ restricted to G-invariant functions on X_0 is well-approximated by a function of (p_2, e_3). This is the key analytic lemma connecting the Bergman kernel asymptotics to the observed two-dimensionality.
   References: Zelditch (1998), J. Diff. Geom.; Lu (2000), J. Geom. Anal.; Fine (2012), J. Diff. Geom.

5. **[Hard, 3+ months]** *Spectral gap estimate: the Laplacian on G-invariant functions on (X_0, g_{FS}) has a spectral gap separating (p_2, e_3) eigenfunctions from (e_4, e_5) eigenfunctions.*

   This is the main technical theorem. It requires: (a) computing or estimating the first few eigenvalues of $\Delta_{g_{\mathrm{FS}}}|_{G\text{-inv}}$ on X_0, and (b) showing these correspond to the (p_2, e_3) sector. Can potentially be approached numerically first (using the balanced H-matrix data you already have) then rigorously.
   References: Braun-Brelidze-Douglas-Ovrut (2008), JHEP; Ashmore-He-Heyes-Ovrut (2023), JHEP.

6. **[Hard, 3+ months]** *The leading G-invariant correction to the balanced metric from the FS metric at large k lies in* $\mathrm{span}(p_2, e_3)$.

   This is Conjecture 2 made rigorous. Approach via the Donaldson iteration as a fixed-point problem and show that the deviation from the FS metric, in the G-invariant sector, contracts onto the (p_2, e_3) subspace.
   References: Donaldson (2001), JDG; Fine (2012), JDG; Keller (2009), arXiv.

7 The Tractable Version of the Paper

You do *not* need to prove all six lemmas to write a mathematical physics paper. A publishable paper could contain:

- **Theorem** (easy): $R_k(z)$ is G-invariant, hence a function of (p_2, e_3, e_4, e_5) (Lemmas 1).
- **Computation** (medium): The scalar curvature $s(z)$ of $g_{\text{FS}}|_{X_0}$ expressed explicitly in symmetric polynomials (Lemma 3). Identify whether e_4, e_5 appear in $s(z)$.
- **Conjecture with evidence**: The correction $k(R_k - \bar{R}_k)$ converges to a function of (p_2, e_3) alone, supported by the polynomial ablation data and k-stability table.
- **Corollary**: The symbolic formula is the leading-order expression for this limit in the class of rational functions of (p_2, e_3).

This structure is publishable in *Letters in Mathematical Physics* (with a theorem + conjecture + evidence) or *Journal of Geometry and Physics* (computation + conjecture).

8 The Ideal Target Structure

If Lemma 4 (TYZ coefficient computation) succeeds and the first TYZ coefficient is indeed a function of (p_2, e_3), then the paper has:

- **Theorem**: First TYZ coefficient on X_0 is in span(p_2, e_3). This implies $R_k \sim f(p_2, e_3) + O(k^{-2})$ for large k.
- **Symbolic formula as corollary**: The SR formula is the rational function of (p_2, e_3) that best approximates this leading correction.
- **Numerical confirmation**: The ablation tables confirm the $O(k^{-2})$ suppression of e_4, e_5 contributions.

This would be a *Communications in Mathematical Physics* paper.

9 Immediate Next Steps (Ordered)

1. **This week.** Numerically verify: does p_4 vary independently of (p_2, e_3) on your sampled points? Run `poly3_r2([p4], R_psi)` and `poly3_r2([p2, e3, p4], R_psi)` on your Kaggle data. If p_4 can be approximated by a function of (p_2, e_3) on X_0, then the scalar curvature story is consistent. If not, the mechanism is more subtle.
2. **Week 2-3.** Prove Lemma 1 (G-invariant ring). Write it up carefully with full algebraic proof. This is publishable as-is as a standalone lemma.
3. **Week 3-6.** Compute $|II|^2$ explicitly (Lemma 2). Use the Fermat quintic gradient $\nabla F = 5(z_0^4, \dots, z_4^4)$ and the Hessian $\nabla^2 F = 20\,\text{diag}(z_0^3, \dots, z_4^3)$. Write down $s(z)$ as a function of (p_2, p_3, p_4) explicitly.
4. **Month 2.** Read the TYZ literature (Zelditch 1998, Lu 2000). Compute or look up the second asymptotic coefficient for degree-d hypersurfaces in $\mathbb{CP}^n$. Try to apply it to the quintic case.
5. **Month 3+.** Attempt the spectral gap computation (Lemma 5). Start numerically using the Laplacian eigenvalue code from Braun et al. (2008). If you can compute the first 10 eigenvalues of $\Delta|_{G\text{-inv}}$ and confirm they correspond to p_2, e_3, that is strong evidence for Conjecture 3.

10 Reading List

Essential (read first):

1. S. K. Donaldson, "Scalar curvature and projective embeddings I," J. Diff. Geom. **59** (2001), 479–522. *[Defines the T-operator and balanced metrics; proves convergence to CY.]*

2. S. Zelditch, "Szegő kernels and a theorem of Tian," Int. Math. Res. Not. **6** (1998), 317–331. *[The foundational TYZ expansion paper.]*

3. Z. Lu, "On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch," Am. J. Math. **122** (2000), 235–273. *[Explicit computation of a_1, a_2 in the TYZ expansion.]*

Important context:

4. J. Fine, "Calabi flow and projective embeddings," J. Diff. Geom. **93** (2013), 55–99. *[Connects Bergman kernels to the balanced metric iteration rigorously.]*

5. X. Wang, "Canonical metrics on stable vector bundles," Commun. Anal. Geom. **13** (2005), 253–285. *[Good background on balanced metrics and convergence rates.]*

6. V. Braun, T. Brelidze, M. R. Douglas, B. A. Ovrut, "Eigenvalues and eigenfunctions of the scalar Laplace operator on Calabi-Yau manifolds," JHEP **2008** (2008), 120. *[Numerically computes Laplacian eigenvalues on the Fermat quintic. Gives you the spectral data you need for Lemma 5.]*

7. R. P. Stanley, "Invariants of Finite Groups and their Applications to Combinatorics," Bull. Amer. Math. Soc. **1** (1979), 475–511. *[Invariant ring theory needed for Lemma 1.]*

Background geometry:

8. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978). Chapters 1.5 (second fundamental form) and 3.3 (Calabi-Yau).

9. D. Huybrechts, *Complex Geometry: An Introduction*, Springer (2005). Chapter 3 (Kähler manifolds) and Chapter 5 (Calabi-Yau).

11 Summary of What This Paper Would Be

What it is NOT	An ML paper. Symbolic regression appears only as a computational tool in Section 4 to identify the rational formula.
What it IS	A mathematical physics paper on the G-invariant structure of the balanced metric on the Dwork quintic family.
Main object	$R_k(z) = \log(\det g_k / \det g_{\mathrm{FS}})$, the balanced metric volume-form ratio.
Main result	This function is G-invariant (theorem), lives in a 4-generator ring (theorem), and is dominated by its projection onto $\mathrm{span}(p_2, e_3)$ (conjecture + evidence, or theorem if TYZ works).
Evidence type	Algebraic proof (invariant ring) + analytic computation (scalar curvature) + numerical support (ablation, k-stability).
Target	*Letters in Mathematical Physics* (conjecture + evidence) or *Communications in Mathematical Physics* (with theorem).
Timeline	2–3 months to a Letters in Math Phys submission; 6–12 months to a CMP-level result.