Spatial Border-2 Transfer Theorems for Bounded Liquid-Vapor Transport

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Abstract

Evaporation models become useful for desert-water, porous-media, and collection problems only after the vapor-side pathway is stated with the same care as the evaporating surface. This paper develops Spatial Border-2 as a bounded vapor-side transfer layer. Border 1 supplies an interfacial source flux inherited from the companion bounded Robin-regime formulation; Border 2 records the receiving-side environment, sink, collector, pore channel, or outlet through which released vapor is transported. The main result is a stability theorem for the coupled Border-1/Border-2 parabolic transfer problem. In a declared bounded regime, differences in vapor-side observations are controlled by the Border-1 input debit, Border-2 sink or collector-law debit, coefficient perturbations, interior residuals, numerical residuals, and observation uncertainty. A one-dimensional slab/pore worked regime shows how Péclet, Border-1 supply, Border-2 Biot, and collection Damkohler-type coordinates enter a concrete debit ledger. The contribution is not a universal desert-yield law. It is a reviewable mathematical layer that converts post-surface vapor motion into a bounded transfer claim with explicit hypotheses, residual channels, and evidence levels.

Keywords: liquid-vapor transport; vapor-side boundary; Robin transfer; porous transport; water harvesting; collection flux; bounded regimes; uncertainty ledger.

1 Introduction

A surface evaporation statement is incomplete for many applications. In water harvesting, porous substrates, condensation devices, and open-air desert transport, the important question is often not only whether vapor leaves a liquid-side cell, but whether the released vapor is carried, trapped, diluted, condensed, or collected after it crosses the first interface. Spatial Border-2 is the receiving-side boundary and pathway record for that second part of the process.

The earlier Border-1 formulation identifies a liquid-side cell, an interface filter, and an exchange law for the evaporating surface. The immediately preceding bounded Robin-regime paper writes the surface source as

$$J_1(t,x,c_1) = \min\{J_{\mathrm{sup}}(t,x), K_1(t,x)(c_{\mathrm{sat}}(t,x) - c_1)_+\}, \tag{1}$$

where c_1 is the vapor trace at the interface. That law answers how much vapor is admitted into the air-side or pore-side domain. This paper asks the next question: after the Border-1 flux enters the vapor domain, what bounded statement can be made about its transport to a receiving boundary, collector, pore outlet, or observation region?

The answer is a theorem-bearing Border-2 transfer object. The object contains the vapor domain, the Border-1 input, the Border-2 sink or collector law, the vapor transport coefficients, the observation map, and a debit ledger. The main theorem says that any supported observation-level statement must pay for six named debits: Border-1 flux uncertainty, Border-2 law uncertainty, coefficient/model perturbation, interior residual, numerical representation error, and observation uncertainty. This is the mathematical core of the bounded transport statement.

Relation to the companion evaporation papers. The companion evaporation paper introduces the local liquid-side population and boundary-filtered escape mechanism. The bounded Robin-regime paper turns the first boundary law into a stable transport-inference statement. The present article extends that construction to a two-border vapor-side pathway. Border 1 is the evaporating input; Border 2 is the receiving-side condition. The new content is the transfer theorem and the worked regime showing how the output observation depends on both borders.

Scope. The paper makes conservative claims. It does not assert a universal device yield, a universal desert-water law, or a validation claim without data. It proves a conditional stability and transfer estimate inside a declared parabolic regime. Device yield, climate collection, and field deployment claims require additional geometry, calibration, environmental sampling, and engineering acceptance evidence.

2 Spatial Border-2 liquid-vapor transfer object

Let $\Omega_v \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded vapor-side or pore-side domain. Its boundary contains a Border-1 interface Γ_1, where vapor enters from the liquid/interface law, and a Border-2 receiving boundary Γ_2, where the domain exchanges with an outlet, collector, ambient layer, adsorbent, condenser, or observation surface. Other boundary pieces are collected in $\Gamma_N \cup \Gamma_D$ and are declared as no-flux, outflow, or prescribed state.

Definition 1 (Spatial Border-2 transfer object). *A Spatial Border-2 bounded liquid-vapor transfer object is a tuple*

$$\mathcal{LV} = (\Omega_v, \Gamma_1, \Gamma_2, T, c, D, W, J_1, B_2, \Theta_2, \mathcal{O}, \mathcal{R}, E, \mathcal{Q}),$$

where c is the vapor concentration or declared concentration-equivalent humidity variable, D is the vapor diffusivity or effective diffusivity, W is the declared vapor-side advective velocity, J_1 is the Border-1 source flux, B_2 is the signed Border-2 receiving law, Θ_2 is the bounded coefficient and geometry envelope, $\mathcal{O}$ is the observation map, $\mathcal{R}$ is the residual ledger, E is the evidence level, and $\mathcal{Q}$ is the permitted claim class.

The default vapor-side equation is

$$\partial_t c + W \cdot \nabla c - \nabla \cdot (D \nabla c) = s + r_\Omega \quad \text{in } \Omega_v \times (0, T), \tag{2}$$

with Border-1 input

$$-D\nabla c \cdot n = J_1 + r_1 \quad \text{on } \Gamma_1 \times (0, T), \tag{3}$$

where n is the outward normal of Ω_v. On Border 2 we use a signed monotone receiving law

$$D\nabla c \cdot n + B_2(t, x, c) = g_2 + r_2 \quad \text{on } \Gamma_2 \times (0, T). \tag{4}$$

For a linear collector, condenser, or ambient exchange law one may take

$$B_2(t, x, c) = \kappa_2(t, x)(c - c_{\text{rec}}(t, x)), \tag{5}$$

with $\kappa_2 \geq 0$. More nonlinear sink laws are allowed if they are monotone and Lipschitz on the declared trace corridor.

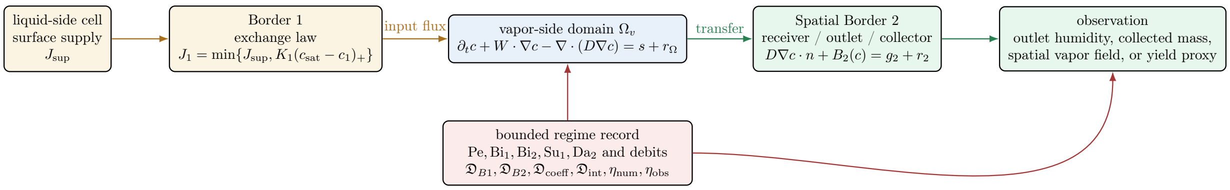

Figure 1: Spatial Border-2 turns the post-surface vapor path into a bounded transfer object. Border 1 supplies vapor; Border 2 records the receiving-side law and the observation pathway.

3 Bounded regime assumptions

The theorem uses the following standard parabolic envelope. The assumptions are intentionally explicit because they identify exactly what a reader, simulator, or experiment must declare.

Assumption 1 (Bounded Border-2 regime). *The transfer object satisfies the following conditions.*

(A1) Ω_v *is a bounded Lipschitz domain, and* $\partial\Omega_v = \Gamma_1 \cup \Gamma_2 \cup \Gamma_N \cup \Gamma_D$ *up to sets of surface measure zero. The trace convention and outward normal are fixed.*

(A2) $D \in L^\infty(\Omega_v \times (0,T))$ *with* $0 < D_- \le D \le D_+ < \infty$. *The velocity field* W *is bounded in a class for which the advection form is controlled by diffusion plus an* L^2 *term.*

(A3) The Border-1 input $J_1 \in L^2(0,T;H^{-1/2}(\Gamma_1))$ *is either the exact min-law* (1) *inherited from the previous paper or a declared approximation with a recorded debit.*

(A4) The Border-2 law $B_2(t,x,a)$ *is monotone nondecreasing in* a*, Lipschitz on the declared trace corridor, and satisfies* $B_2(t,x,0) \in L^2(\Gamma_2 \times (0,T))$. *For the linear law* (5), $0 \le \kappa_2 \le \kappa_{2,+}$.

(A5) The residuals r_Ω, r_1, r_2 *are square integrable in the dual spaces appearing in the weak formulation. Numerical and observation errors are recorded separately.*

Let

$$X_T = L^\infty(0,T;L^2(\Omega_v)) \cap L^2(0,T;H^1(\Omega_v)).$$

A weak solution is a function $c \in X_T$, with $\partial_t c \in L^2(0,T;H^{-1}(\Omega_v))$ after residuals are included, satisfying the variational form of (2)–(4). Traces on Γ_1 and Γ_2 are understood through the H^1-trace map, and boundary laws are paired in the natural $H^{-1/2}, H^{1/2}$ duality. The stability theorem compares two such weak solutions inside the declared envelope. Existence is either part of the regime record or follows from the standard bounded parabolic theory for coercive diffusion and monotone/Lipschitz Robin data; the paper does not assert existence for uncalibrated empirical closures, discontinuous switching laws, or data outside the declared corridor.

4 Border laws and debit decomposition

Two elementary facts make the transfer estimate usable. The first keeps the Border-1 input inherited from the previous article bounded and Lipschitz. The second isolates the Border-2 debit.

Lemma 1 (Border-1 input debit). *Let* $J_1(a) = \min\{J_{\rm sup}, K_1(c_{\rm sat} - a)_+\}$. *On a trace corridor* $|a| \le c_+$, $0 \le c_{\rm sat} \le c_{{\rm sat},+}$, *and* $0 \le K_1 \le K_{1,+}$, *the flux satisfies*

$$0 \le J_1(a) \le J_{\rm sup}, \qquad |J_1(a) - J_1(b)| \le K_1|a-b|.$$

If a second parameter triple is used, then

$$|J_1(a) - \widetilde{J}_1(a)| \le |J_{\sup} - \widetilde{J}_{\sup}| + (c_{\mathrm{sat},+} + c_+)|K_1 - \widetilde{K}_1| + \widetilde{K}_1|c_{\mathrm{sat}} - \widetilde{c}_{\mathrm{sat}}|.$$

The right side is the pointwise Border-1 parameter debit.

Proof. The positive-part map is one-Lipschitz and nonincreasing, and taking the pointwise minimum with a nonnegative constant preserves boundedness and the Lipschitz constant. The parameter estimate follows by adding and subtracting the two capped clearing terms and using $|\min\{A, x\} - \min\{\widetilde{A}, \widetilde{x}\}| \le |A - \widetilde{A}| + |x - \widetilde{x}|$. □

For two Border-2 laws, write the difference at the traces $c, \widetilde{c}$ as

$$B_2(c) - \widetilde{B}_2(\widetilde{c}) = \underbrace{B_2(c) - B_2(\widetilde{c})}_{\text{monotone trace term}} + \underbrace{B_2(\widetilde{c}) - \widetilde{B}_2(\widetilde{c})}_{\text{frozen Border-2 law debit}}. \tag{6}$$

The first term has the useful sign in the energy estimate. The second term is the cost of changing the receiver law, collector coefficient, ambient concentration, material state, or boundary closure.

For the linear receiver law $B_2(c) = \kappa_2(c - c_{\mathrm{rec}})$, the frozen debit is bounded by

$$|B_2(a) - \widetilde{B}_2(a)| \le |\kappa_2 - \widetilde{\kappa}_2|(|a| + |c_{\mathrm{rec}}|) + \widetilde{\kappa}_2|c_{\mathrm{rec}} - \widetilde{c}_{\mathrm{rec}}|. \tag{7}$$

This is the receiving-side analog of the Border-1 parameter debit.

5 Main theorem: bounded Border-2 transfer stability

Let c and $\widetilde{c}$ be two weak solutions in the same declared geometry class. Denote their two sets of data by $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, and set $z = c - \widetilde{c}$.

Define the Border-1 debit by freezing the trace-dependent law at the comparison trace,

$$\mathfrak{D}_{B1} = \left\| J_1(\widetilde{c}|_{\Gamma_1}) - \widetilde{J}_1(\widetilde{c}|_{\Gamma_1}) + r_1 - \widetilde{r}_1 \right\|_{L^2(0,T;H^{-1/2}(\Gamma_1))}.$$

If J_1 is prescribed as an external flux rather than as the min-law (1), this reduces to $\left\| J_1 - \widetilde{J}_1 + r_1 - \widetilde{r}_1 \right\|$ in the same space. The remaining trace-dependent part $J_1(c|_{\Gamma_1}) - J_1(\widetilde{c}|_{\Gamma_1})$ is absorbed into the stability constant by the Lipschitz estimate in the Border-1 input lemma and the trace inequality. the Border-2 frozen-law debit

$$\mathfrak{D}_{B2} = \left\| B_2(\widetilde{c}) - \widetilde{B}_2(\widetilde{c}) + r_2 - \widetilde{r}_2 \right\|_{L^2(0,T;H^{-1/2}(\Gamma_2))},$$

and collect coefficient and interior forcing differences in

$$\mathfrak{D}_{\mathrm{coeff}} = \left\| (D - \widetilde{D})\nabla \widetilde{c} \right\|_{L^2(0,T;L^2(\Omega_v))} + \left\| (W - \widetilde{W}) \cdot \nabla \widetilde{c} \right\|_{L^2(0,T;H^{-1}(\Omega_v))},$$

$$\mathfrak{D}_{\mathrm{int}} = \| s - \widetilde{s} + r_\Omega - \widetilde{r}_\Omega \|_{L^2(0,T;H^{-1}(\Omega_v))}.$$

Theorem 1 (Bounded Border-2 transfer stability)**.** *Under the bounded Border-2 regime Assumption 1, there is a constant C_T, depending only on the declared geometry, time horizon, trace constants, diffusion bounds, velocity-envelope constants, and Lipschitz bounds of the border laws, such that*

$$\begin{aligned} &\|z\|^2_{L^\infty(0,T;L^2(\Omega_v))} + D_- \|\nabla z\|^2_{L^2(0,T;L^2(\Omega_v))} + \int_0^T \int_{\Gamma_2} (B_2(c) - B_2(\widetilde{c}))(c - \widetilde{c})\, dS dt \\ &\quad \le C_T \Big(\|c_0 - \widetilde{c}_0\|^2_{L^2(\Omega_v)} + \mathfrak{D}^2_{B1} + \mathfrak{D}^2_{B2} + \mathfrak{D}^2_{\mathrm{coeff}} + \mathfrak{D}^2_{\mathrm{int}} \Big). \end{aligned} \tag{8}$$

In particular, the Border-2 monotone trace term is nonnegative. If an observation map $\mathcal{O} : X_T \to Y$ satisfies

$$\|\mathcal{O}(u) - \mathcal{O}(v)\|_Y \le L_{\mathcal{O}} \|u - v\|_{X_T},$$

then any numerical representation error η_{num} and observation uncertainty η_{obs} give

$$\|\mathcal{O}(c) - \mathcal{O}(\widetilde{c})\|_Y \le C_T^{1/2} L_{\mathcal{O}} \Big(\|c_0 - \widetilde{c}_0\|_{L^2} + \mathfrak{D}_{B1} + \mathfrak{D}_{B2} + \mathfrak{D}_{\mathrm{coeff}} + \mathfrak{D}_{\mathrm{int}} \Big) + \eta_{\mathrm{num}} + \eta_{\mathrm{obs}}. \tag{9}$$

Proof. Subtract the two weak formulations and test the difference by z. The time derivative gives $\frac{1}{2}\frac{d}{dt}\|z\|_{L^2}^2$. The diffusion part gives the coercive term $D_- \|\nabla z\|_{L^2}^2$ plus the coefficient perturbation $(D - \widetilde{D})\nabla \widetilde{c}$, which is bounded by duality and Young's inequality. The velocity terms are controlled by the declared advection envelope; differences in velocity enter $\mathfrak{D}_{\mathrm{coeff}}$.

On Γ_1, the boundary difference is paired with the trace of z. When the Border-1 source is the trace-dependent min-law, split

$$J_1(c|_{\Gamma_1}) - \widetilde{J}_1(\widetilde{c}|_{\Gamma_1}) = \big(J_1(c|_{\Gamma_1}) - J_1(\widetilde{c}|_{\Gamma_1})\big) + \big(J_1(\widetilde{c}|_{\Gamma_1}) - \widetilde{J}_1(\widetilde{c}|_{\Gamma_1})\big).$$

The first term is bounded by the Border-1 Lipschitz constant times the trace of z and is absorbed through the trace theorem and Young's inequality. The second term is the frozen Border-1 law mismatch and is exactly the $\mathfrak{D}_{B1}$ contribution. For a prescribed external flux, the same line is read directly with $J_1 - \widetilde{J}_1$. On Γ_2, split the boundary-law difference as in (6). The monotone part contributes

$$\int_{\Gamma_2} (B_2(c) - B_2(\widetilde{c}))z \, dS \ge 0.$$

The frozen law mismatch and residual difference are controlled by the trace theorem and Young's inequality, giving $\mathfrak{D}_{B2}^2$ after absorbing the small trace-energy contribution. The interior forcing difference gives $\mathfrak{D}_{\mathrm{int}}^2$ by duality. The resulting differential inequality has the form

$$\frac{d}{dt}\|z\|_{L^2}^2 + D_- \|\nabla z\|_{L^2}^2 + \text{nonnegative Border-2 term} \le C\|z\|_{L^2}^2 + C(\mathfrak{D}_{B1}^2 + \mathfrak{D}_{B2}^2 + \mathfrak{D}_{\mathrm{coeff}}^2 + \mathfrak{D}_{\mathrm{int}}^2),$$

with constants fixed by the declared regime. Integration and Gronwall's inequality give (8). The observation estimate follows from the Lipschitz property of $\mathcal{O}$ and by adding numerical and measurement uncertainty. □

Corollary 1 (Collector-output and outlet-output claims)**.** *Suppose the declared observation is either the collected mass*

$$\mathcal{O}_{\mathrm{col}}(c) = \int_0^T \int_{\Gamma_2} \kappa_2 (c - c_{\mathrm{rec}})_+ \, dS dt$$

or a bounded linear outlet humidity observation. On a bounded trace corridor, $\mathcal{O}$ is Lipschitz in X_T. Therefore any supported collection or outlet-humidity comparison is controlled by the right side of (9). A yield claim is admissible only after the corresponding observation, calibration, and environmental debits are included.

6 Worked bounded regime: one-dimensional slab, pore, or collector

The simplest reviewable Border-2 example is a one-dimensional vapor path $0 < x < L$. Border 1 is the evaporating surface at $x = 0$. Border 2 is a collector, outlet, or receiving material at $x = L$. For the steady diffusion-led case with constant $D > 0$, no interior source, and a linear receiving law, the model is

$$-Dc''(x) = 0, \qquad -Dc'(0) = J_1, \qquad -Dc'(L) = \kappa_2(c(L) - c_{\mathrm{rec}}).$$

The solution is affine. The common flux J satisfies

$$J = J_1 = \kappa_2(c(L) - c_{\mathrm{rec}}), \qquad c(0) - c(L) = \frac{JL}{D}. \tag{10}$$

If the Border-1 supply law is capped by (1), then the realized transfer flux is limited by both the input law and the downstream receiver. In a linearized corridor around a declared state, the following algebraic expression is recorded as a linearized corridor ledger for the slab/pore calculation:

$$J_{\mathrm{path}} \approx \min\left\{ J_{\mathrm{sup}},\ K_1(c_{\mathrm{sat}} - c(0))_+,\ \frac{D\kappa_2}{D + \kappa_2 L}(c(0) - c_{\mathrm{rec}})_+ \right\}, \tag{11}$$

with a residual recorded whenever the state, geometry, or receiver law leaves the linear corridor. The factor $D\kappa_2/(D + \kappa_2 L)$ is the series resistance of interior diffusion and Border-2 exchange inside this corridor ledger.

A nondimensional ledger for this example is

$$\mathrm{Pe} = \frac{U_0 L}{D}, \qquad \mathrm{Bi}_1 = \frac{K_1 L}{D}, \qquad \mathrm{Bi}_2 = \frac{\kappa_2 L}{D}, \qquad \mathrm{Su}_1 = \frac{J_{\mathrm{sup}} L}{D c_0}, \qquad \mathrm{Da}_2 = \frac{T\kappa_2 |\Gamma_2|}{|\Omega_v|}. \tag{12}$$

Here $\mathrm{Bi}_2 \ll 1$ means the receiver is weak compared with interior diffusion, so the output is receiver-limited. $\mathrm{Bi}_2 \gg 1$ means the receiver is strong, so the interior path or Border-1 supply becomes the limiting step. Da_2 is a residence-time-to-collection coordinate: it compares the observation time to the characteristic Border-2 removal time.

Coordinate	Low-regime reading	High-regime reading
Pe	diffusion-led slab or pore	airflow/advection changes path residence time
Bi_1	weak first-border clearing	surface exchange can drive the path
Bi_2	weak collector/outlet sink	receiving boundary can rapidly remove vapor
Su_1	supply cap is restrictive	downstream path, not supply, may limit output
Da_2	short exposure or weak receiver	long exposure or strong receiver; collection observation becomes more sensitive

Table 1: Worked slab/pore regime coordinates. The table classifies the transport statement before interpreting a collection or outlet observation.

For this worked regime, an observation tolerance $\tau_{\mathrm{obs}} > 0$ may be assigned to outlet humidity, collected mass, or integrated vapor concentration. The normalized debit ledger is

$$\mathcal{S}_2 = \frac{C_T^{1/2} L_{\mathcal{O}}}{\tau_{\mathrm{obs}}}\big(\mathfrak{D}_{B1} + \mathfrak{D}_{B2} + \mathfrak{D}_{\mathrm{coeff}} + \mathfrak{D}_{\mathrm{int}}\big) + \frac{\eta_{\mathrm{num}} + \eta_{\mathrm{obs}}}{\tau_{\mathrm{obs}}}. \tag{13}$$

The bounded transfer claim is observation-ready when $\mathcal{S}_2 \le 1$ for the declared observation and regime. This is not a universal acceptance rule; it is a transparent way of saying that the named debits fit inside the declared tolerance budget.

7 Evidence levels and permitted claims

The theorem separates mathematical transfer control from empirical validation. Evidence levels are attached to a specific Border-2 object, not to the whole evaporation theory.

Level	Border-2 status	Permitted claim
E0	object components declared	the receiving-side pathway is named and inspectable
E1	bounded regime and scale coordinates declared	qualitative transfer interpretation inside a stated regime
E2	theorem debit ledger closed for a computation	one bounded numerical transfer statement
E3	calibrated source observation attached	one observation-level transfer statement
E4	repeated geometry, material, or ventilation stress	repeated bounded transfer statement across declared stresses
E5	device-level acceptance with uncertainty and replication	engineering or deployment claim within the tested envelope

Table 2: Border-2 evidence ladder. The theorem supports E1–E2 once the mathematical debits are bounded; observation and engineering claims require additional data.

Claim type	What this paper permits
Mechanism claim	Border-1 input and Border-2 receiving conditions can be written in one bounded liquid-vapor transfer object.
Theorem claim	In the declared parabolic envelope, observation differences are controlled by named debits as in the bounded Border-2 transfer stability theorem.
Computation claim	A numerical result is admissible when it states the object, mesh/time residual, coefficient envelope, and observation tolerance.
Observation claim	A measured transfer statement is admissible only with calibration, sensor placement, uncertainty, and environmental records.
Device-yield claim	A device claim additionally needs geometry, acceptance criteria, replicated environmental conditions, and collection measurements.

Table 3: Permitted claims for the rebuilt article. The table is included to keep the article theorem-bearing without overpromising validation or yield.

8 Discussion

Spatial Border-2 matters because it prevents a common compression in evaporation reasoning. A liquid-side escape statement is often treated as if it were already a water-harvesting or collection statement. That jump is unsafe. The vapor may remain near the surface, be cleared by ventilation, leak through a pore path, condense on a material, exit a channel, or be lost to the ambient environment. The second border is where those outcomes become mathematically and experimentally distinct.

The theorem supplies a stable accounting layer. If the Border-1 flux is uncertain, the theorem assigns that uncertainty to $\mathfrak{D}_{B1}$. If the receiving boundary coefficient, ambient concentration, or collector law is uncertain, the theorem assigns it to $\mathfrak{D}_{B2}$. If diffusivity, airflow, porous geometry, or interior source modeling changes, the debit enters $\mathfrak{D}_{\text{coeff}}$ or $\mathfrak{D}_{\text{int}}$. If the model is computed or measured, the numerical and observation errors remain visible. This is the main practical contribution: post-surface vapor statements become bounded transfer claims rather than informal extrapolations.

The worked slab example also shows why Border 2 is not cosmetic. Even with a fixed evaporating input, the realized output depends strongly on the receiving-side Biot number, exposure time, and path resistance. A weak receiver can make a strong evaporating surface look ineffective. A strong receiver can reveal that the bottleneck is the liquid-side supply or the interior path. These are physically different regimes and should not be merged into a single evaporation coefficient.

9 Conclusion

This paper rebuilds Spatial Border-2 as a theorem-bearing liquid-vapor transfer article. The receiving-side boundary is formalized as part of a bounded parabolic transfer object, coupled to the Border-1 evaporation input and equipped with a debit ledger. The main theorem proves that observation-level differences are controlled by Border-1 input error, Border-2 receiving-law error, coefficient/model perturbations, interior residuals, numerical error, and observation uncertainty. The one-dimensional slab/pore example shows how the theorem becomes a usable regime ledger for water-harvesting, porous-media, and collection problems. The resulting article is a conservative but useful bridge from surface evaporation to bounded vapor-side transport.

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