On Cyclicants, or Ternary Reciprocants, and Allied Functions.

By E. B. Elliott, M.A.

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1. It has become almost necessary to depart from the nomenclature which I have hitherto adopted in my papers on this subject (*Proceedings*, Vol. XVII., pp. 172—196; Vol. XVIII., pp. 142—164; Vol. XIX., pp. 6—23). The name *ternary reciprocant* was employed for reasons of analogy with Professor Sylvester's theory of reciprocants in two variables. As, however, the subject has grown, the advantages of this designation have become less marked and the danger of confusion in expression has been found to outweigh the convenience of keeping the analogy in prominence. I propose henceforward to use the name *cyclicant* in place of *ternary reciprocant*, and, in particular, *pure cyclicant* in place of *pure ternary reciprocant*. The leading idea of the cyclical interchange of three variables x, y, z is thus given the controlling influence in nomenclature which it probably should have had originally.

A *pure cyclicant* is then a function $R(z, x, y)$ of the second and higher partial differential coefficients of z with regard to x and y, which, if z_{rs} denote $\frac{1}{r!\,s!}\,\frac{d^{r+s}z}{dx^r dy^s}$, is homogeneous (of degree i) in the derivatives z_{rs} and isobaric in both first and second suffixes, the two partial weights being equal (each $\frac{1}{2}w$), and which persists in form, but for a first derivative factor, when the variables are cyclically interchanged. The identities expressive of this persistence are (Vol. XVIII., pp. 157, 158)

$$R(z, xy) \equiv (-z_{10})^{i+\frac{1}{2}w} R(x, yz) \equiv (-z_{01})^{i+\frac{1}{2}w} R(y, zx) \ldots\ldots (1).$$

Pure cyclicants have, as was seen in the paper now referred to, four annihilators—

$$\Omega_1 \equiv \Sigma \left\{ (m+1)\, z_{m+1, n-1} \frac{d}{dz_{mn}} \right\} \equiv \eta \frac{d}{d\xi} (\zeta - z_{10}\xi - z_{01}\eta) \ldots\ldots (2),$$

$$\Omega_2 \equiv \Sigma \left\{ (n+1)\, z_{m-1, n+1} \frac{d}{dz_{mn}} \right\} \equiv \xi \frac{d}{d\eta} (\zeta - z_{10}\xi - z_{01}\eta) \ldots\ldots (3),$$

$$V_1 \equiv \Sigma \left\{ \Sigma (r z_{rs} z_{m+1-r, n-s}) \frac{d}{dz_{mn}} \right\} \equiv \frac{1}{2} \frac{d}{d\xi} \left\{ (\zeta - z_{10}\xi - z_{01}\eta)^2 \right\} \ldots (4),$$

$$V_2 \equiv \Sigma \left\{ \Sigma (s z_{rs} z_{m-r, n+1-s}) \frac{d}{dz_{mn}} \right\} \equiv \frac{1}{2} \frac{d}{d\eta} \left\{ (\zeta - z_{10}\xi - z_{01}\eta)^2 \right\} \ldots (5),$$

of which the first two express that it is a full invariant of the quantics (the emanants of z with regard to x and y),

$$\left.\begin{array}{l} (z_{20}, z_{11}, z_{02} \between (u, v)^2 \\ (z_{30}, z_{21}, z_{12}, z_{03} \between (u, v)^3 \\ \&c. \qquad \&c. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

For the limits of the summations in Ω_1, Ω_2, V_1, V_2, see Vol. XIX., p. 6, and for the symbolical notation in the second expressions for those annihilators, see Vol. XVIII., pp. 150, &c.

The functions

$$\left.\begin{array}{l} E_2 \equiv (z_{20}, z_{11}, z_{02} \between (-z_{01}, z_{10})^2 \\ E_3 \equiv (z_{30}, z_{21}, z_{12}, z_{03} \between (-z_{01}, z_{10})^3 \\ \qquad \&c. \qquad \&c. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

obtained from the emanants (6) by giving u, v the values $-z_{01}$, z_{10}, I propose to call the quadratic cubic, &c. *cyclico-genitive forms*, for reasons partly indicated in my last paper and to be made more apparent presently.

A seminvariant of the cyclico-genitive forms which has the further property of being annihilated by V_1 I shall designate a *semicyclicant*, and the covariant of the cyclico-genitive forms which has for leading coefficient a semicyclicant I shall call a *cocyclicant*. The definition of a semicyclicant may be expressed without direct reference to the cyclico-genitive forms. It is a homogeneous and doubly isobaric function of the derivatives z_{rs} which is annihilated by Ω_1 and by V_1. Call its degree i. Its two partial weights are different. Call them w_1, w_2, and let $w_1 - w_2 = m$.

One cyclical interchange of the variables in a semicyclicant produces from it, but for a first derivative factor, the result of interchanging first and second suffixes in its expression, and a second cyclical interchange produces the corresponding cocyclicant. If, in fact, S_0 be a semicyclicant, and $(S_0, S_1, \dots S_m)(-z_{01}, z_{10})^m$ the cocyclicant of which it is the leading coefficient, we have

$$\frac{S_0(x, yz)}{x_{01}^{i+w_1}} \equiv (-1)^m \frac{S_m(y, zx)}{y_{10}^{i+w_1}} \equiv (-1)^{i+w_2}(S_0, S_1, \dots S_m)(-z_{01}, z_{10})^m \quad \dots\dots\dots\dots(8),$$

the notation (x, yz) denoting that x is taken as dependent and y and z as independent variables in order, and the absence of any explicit reference to the variables indicating that z is dependent.

The equivalences (8), which include (1) as particular cases, were proved in my last paper (Vol. XIX., p. 21),* where, however, only the restricted class of semicyclicants of which V_2, as well as V_1 and Ω_1, is an annihilator, were being considered. The proof in question will be found to have made no use of the supposed annihilation by V_2. It applied then equally to all semicyclicants, and need not be repeated. (It should be noticed that the same remark does not apply to the proofs of Props. IX. and X. on p. 15 of the paper in question. Those propositions distinctly depend on the annihilation by V_2 of the particular class of semicyclicants there studied. I see no reason for retaining the names *reciprocantive covariant* and *reciprocantive seminvariant.*)

It will be sometimes useful when speaking of cyclicants, semicyclicants and cocyclicants collectively, or without discrimination between them, to group them under the common designation *cyclic concomitants.*

2. The method of the last article of my last paper (Vol. XIX., pp. 22, 23) for the determination of all the linearly independent pure cyclicants of a given type i, $\frac{1}{2}w$, $\frac{1}{2}w$, is applicable equally for the determination of all the linearly independent semicyclicants of type i, w_1, w_2.

Of the cyclico-genitive forms (7) E_2 alone is a cocyclicant.

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* There was there, however, an error in sign which is here corrected. The mistake was first made in the last line but eight of page 20, where P_{m-r} should be $(-1)^{w_2} P_{m-r}$, and repeated in the seventh line of page 21, where $(-1)^{i+m}$ should be $(-1)^{i+w_1}$.

3. To the important alternant equivalences given and used in my last paper [Vol. XIX., p. 9 (5) to (8)] may be added the following, the operation being on a homogeneous and doubly isobaric function of the derivatives z_{rs}.

$$V_1\frac{d}{dx}-\frac{d}{dx}V_1\equiv 2z_{20}(i+w_1)+z_{11}\Omega_1 \quad\text{.................(9)},$$

$$V_1\frac{d}{dy}-\frac{d}{dy}V_1\equiv z_{11}(i+w_1)+2z_{02}\Omega_1 \quad\text{...............(10)},$$

$$V_2\frac{d}{dx}-\frac{d}{dx}V_2\equiv 2z_{20}\Omega_2+(i+w_2)z_{11} \quad\text{.................(11)},$$

$$V_2\frac{d}{dy}-\frac{d}{dy}V_2\equiv z_{11}\Omega_2+2z_{02}(i+w_2) \quad\text{.................(12)},$$

$$\Omega_1\frac{d}{dx}-\frac{d}{dx}\Omega_1\equiv 0 \quad\text{.......................................(13)},$$

$$\Omega_2\frac{d}{dy}-\frac{d}{dy}\Omega_2\equiv 0 \quad\text{.......................................(14)}.$$

The remaining alternants of the series,

$$\Omega_1\frac{d}{dy}-\frac{d}{dy}\Omega_1 \quad\text{and}\quad \Omega_2\frac{d}{dx}-\frac{d}{dx}\Omega_2,$$

appear to introduce new operators which I have not found time to study.* All are readily obtained, by means of (2) to (5), and

$$\frac{d}{dx}\equiv\Sigma_{r+s\nless 2}\left\{(r+1)z_{r+1,s}\frac{d}{dz_{rs}}\right\}\equiv\frac{d}{d\xi}(\zeta-z_{10}\xi-z_{01}\eta)-2z_{20}\xi-z_{11}\eta \quad\text{.................(15)},$$

$$\frac{d}{dy}\equiv\Sigma_{r+s\nless 2}\left\{(s+1)z_{r,s+1}\frac{d}{dz_{rs}}\right\}\equiv\frac{d}{d\eta}(\zeta-z_{10}\xi-z_{01}\eta)-z_{11}\xi-2z_{02}\eta \quad\text{.................(16)},$$

$$i\equiv\Sigma_{r+s\nless 2}\left(z_{rs}\frac{d}{dz_{rs}}\right)\equiv\zeta-z_{10}\xi-z_{01}\eta \quad\text{.............................(17)},$$

$$w_1\equiv\Sigma_{r+s\nless 2}\left(rz_{rs}\frac{d}{dz_{rs}}\right)\equiv\xi\frac{d}{d\xi}(\zeta-z_{10}\xi-z_{01}\eta) \quad\text{.......................(18)},$$

$$w_2\equiv\Sigma_{r+s\nless 2}\left(sz_{rs}\frac{d}{dz_{rs}}\right)\equiv\eta\frac{d}{d\eta}(\zeta-z_{10}\xi-z_{01}\eta) \quad\text{.....................(19)},$$

either as in my last paper (Vol. XIX., pp. 9—12), or from the symbolical forms in the manner illustrated by Mr. Leudesdorf (Vol. XVIII., pp. 244, &c.).

* [*Oct.* 1888.—They are merely $\frac{d}{dx}$ and $\frac{d}{dy}$.]

For instance, the parts of $V_1 \frac{d}{dx}$ and $\frac{d}{dx} V_1$ which involve symbols of second differentiation are identical; and the other parts are symbolically

$$\left(V_1 \frac{d}{dx}\right) = \Sigma_{r+s \nless 3} \left\{(\text{co. } \xi^r \eta^s \text{ in } V_1)\, r\xi^{r-1}\eta^s\right\} = \frac{dV_1}{d\xi},$$

and

$$\left(\frac{d}{dx} V_1\right) = \Sigma_{r+s \nless 3} \left\{(r+1)\, z_{r+1,s}\, \xi^r \eta^s\right\} \frac{1}{2} \frac{d}{d\xi} \left\{(\zeta - z_{10}\xi - z_{01}\eta)^2\right\}$$

$$= \frac{d}{d\xi} \left\{(\zeta - z_{10}\xi - z_{01}\eta)\, \Sigma \left[(r+1)\, z_{r+1,s}\, \xi^r \eta^s\right]\right\}$$

$$= \frac{d}{d\xi} \left\{(\zeta - z_{10}\xi - z_{01}\eta) \left[\frac{d}{d\xi}(\zeta - z_{10}\xi - z_{01}\eta) - 2z_{20}\xi - z_{11}\eta\right]\right\}$$

$$= \frac{dV_1}{d\xi} - 2z_{20}(\zeta - z_{10}\xi - z_{01}\eta) - (2z_{20}\xi + z_{11}\eta) \frac{d}{d\xi}(\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \frac{dV_1}{d\xi} - 2z_{20}i - 2z_{20}w_1 - z_{11}\Omega_1.$$

Consequently,

$$V_1 \frac{d}{dx} - \frac{d}{dx} V_1 = \left(V_1 \frac{d}{dx}\right) - \left(\frac{d}{dx} V_1\right) = 2z_{20}(i + w_1) + z_{11}\Omega_1.$$

To save space I do not write out the other proofs. The first steps of all of them are included in

$$\left(\vartheta \frac{d}{dx}\right) = \Sigma_{r+s \nless 3} (C_{rs}\, r\xi^{r-1}\eta^s) = \frac{d\vartheta}{d\xi} - 2C_{20}\xi - C_{11}\eta,$$

and
$$\left(\vartheta \frac{d}{dy}\right) = \Sigma_{r+s \nless 3} (C_{rs}\, s\xi^r\eta^{s-1}) = \frac{d\vartheta}{d\eta} - C_{11}\xi - 2C_{02}\eta,$$

where C_{rs} is the coefficient of $\xi^r\eta^s$ or of $\frac{d}{dz_{rs}}$ in ϑ.

4. From (13) alone, we draw the conclusion that, if Ω_1 annihilates a pure function I, it also annihilates $\frac{dI}{dx}$; in other words, that the operator $\frac{d}{dx}$ generates seminvariants of the system of quantics

$$(z_{20}, z_{11}, z_{02})(u, v)^2, \text{ \&c.},$$

from other seminvariants. This can hardly be new.

From (9) and (13) together, we derive a theorem of eduction of semicyclicants from semicyclicants. They tell us that, if a homogeneous doubly isobaric function S be annihilated by V_1 and by Ω_1, and if $i+w_1$, the sum of the degree and first partial weight of S, vanishes, then $\frac{dS}{dx}$ is also annihilated both by V_1 and by Ω_1.

Now, if S_0 be any pure cyclicant or semicyclicant of type i, w_1, w_2, $\frac{S_0}{z_{20}^{\frac{1}{2}(i+w_1)}}$ is such a function S, for z_{20} is another semicyclicant, its type being 1, 2, 0. Consequently, if S_0 is a pure cyclicant or semicyclicant,

$$z_{20}^{\frac{1}{2}(i+w_1)+1}\frac{d}{dx}\left(\frac{S_0}{z_{20}^{\frac{1}{2}(i+w_1)}}\right),$$

i.e.,

$$z_{20}\frac{dS_0}{dx}-(i+w_1)\,z_{30}S_0 \quad\text{.......................}(20),$$

is another semicyclicant. Its type is $i+1, w_1+3, w_2$.

This formula of eduction of semicyclicants from semicyclicants is the same in form as, and includes, the formula for educing one Sylvesterian pure reciprocant from another. Analogy might lead us to speak of $S_0 \div z_{20}^{\frac{1}{2}(i+w_1)}$ as an *absolute* pure semicyclicant. In the expression of the fundamental property of such semicyclicants by (8), the first derivative factors do not appear.

Undoubtedly the same theorem of eduction might have been otherwise developed by means of (8) and the equivalence of operators,

$$\frac{1}{x_{01}}\frac{d}{dy} \equiv -\frac{1}{y_{10}}\frac{d}{dx} \equiv -z_{01}\frac{d}{dx}+z_{10}\frac{d}{dy} \quad\text{..............}(21),$$

in the first, second, and third members of which y and z, z and x, and x and y, respectively, are regarded as independent variables.

5. The chief object of the present paper is to give an introduction to the study of the geometrical usefulness of pure cyclicants and semicyclicants. With this object in view, it is necessary first to establish theorems of persistence in form, in case of linear transformation of the variables x, y, z, in close analogy to that of Professor Sylvester's ninth lecture (*American Journal*, Vol. VIII., p. 248) with regard to pure reciprocants.

The two theorems to be proved are:—

I. *A pure cyclicant reproduces itself, but for a factor involving first derivatives and the constants of transformation only, when the variables* *x, y, z are transformed by any scheme of linear transformation*

$$\left.\begin{aligned} x &= lX + mY + nZ + p \\ y &= l'X + m'Y + n'Z + p' \\ z &= l''X + m''Y + n''Z + p'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

II. *A pure semicyclicant in x as dependent variable, or a cocyclicant in z dependent, reproduces itself, but for a factor involving first derivatives and the constants of transformation only, when the variables are subjected to a restricted transformation, such as*

$$\left.\begin{aligned} x &= lX + mY + nZ + p \\ y &= l'X + m'Y + n'Z + p' \\ z &= \qquad\qquad\quad n''Z + p'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

6. To prove the first of these two propositions.

It is readily seen that first derivatives transform by (22) into functions of first derivatives and the constants of transformation. In fact, the formulæ are

$$\frac{Z_{10}}{lz_{10} + l'z_{01} - l''} = \frac{Z_{01}}{mz_{10} + m'z_{01} - m''} = \frac{-1}{nz_{10} + n'z_{01} - n''} \ldots\ldots\ldots (24),$$

which at once reverse into

$$\frac{z_{10}}{\begin{vmatrix} Z_{10} & Z_{01} & -1 \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}} = \frac{z_{01}}{\begin{vmatrix} l & m & n \\ Z_{10} & Z_{01} & -1 \\ l'' & m'' & n'' \end{vmatrix}} = \frac{-1}{\begin{vmatrix} l & m & n \\ l' & m' & n' \\ Z_{10} & Z_{01} & -1 \end{vmatrix}} \ldots\ldots\ldots (25).$$

It is important to ascertain at once whether there is any exception to the fact that the substitution (22) may be replaced by successive partial substitutions, each changing only one variable at a time, such as

$$\left.\begin{aligned} x &= x \\ y &= y \\ z &= \lambda''x + \mu''y + \nu''Z + q'' \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$

$$\left.\begin{aligned} x &= \lambda X + \mu y + \nu Z + q \\ y &= y \\ Z &= Z \end{aligned}\right\} \ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots (27),$$

$$\left.\begin{aligned} X &= X \\ y &= l'X+m'Y+n'Z+p' \\ Z &= Z \end{aligned}\right\} \quad \text{..................(28).}$$

The complete substitution effected by these successive substitutions is

$$\begin{aligned} x &= (\lambda+\mu l')\,X+\mu m'Y+(\nu+\mu n')\,Z+q+\mu p', \\ y &= l'X+m'Y+n'Z+p', \\ z &= (\lambda''\lambda+\lambda''\mu l'+\mu''l')\,X+(\lambda''\mu m'+\mu''m')\,Y \\ &\qquad +(\nu''+\lambda''\nu+\lambda''\mu n'+\mu''n')\,Z+q''+\lambda''q+\lambda''\mu p'+\mu''p'. \end{aligned}$$

For this scheme to be identical with (22) eight linear equations in λ'', μ'', ν'', q'', λ, μ, ν, q have to be satisfied. These are readily solved, the results being

$$\left.\begin{aligned} &\frac{\mu}{m} = \frac{\lambda}{lm'-l'm} = \frac{\nu}{nm'-n'm} = \frac{q}{pm'-p'm} = \frac{1}{m'}, \\ &\frac{\lambda''}{l''m'-l'm''} = \frac{\mu''}{lm''-l''m} = \frac{\nu''}{\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}} = \frac{q''}{\begin{vmatrix} l & m & p \\ l' & m' & p' \\ l'' & m'' & p'' \end{vmatrix}} = \frac{1}{lm'-l'm} \end{aligned}\right\} \quad \text{..................(29).}$$

Thus suitable values of the coefficients in the successive substitutions are uniquely determinate unless either

$$m' = 0 \quad \text{or} \quad lm'-l'm = 0 \quad \text{....................(30).}$$

Even in these excepted cases, however, it is still possible that the substitution (22) may be produced by a succession of three partial substitutions by adopting a different order from that chosen above. Calling that order zxy, there are five other possible orders—zyx, xyz, xzy, yzx, yxz. Each of these orders is applicable to all but classes of cases for which particular conditions hold analogous to (30). In fact we have, if

$$\begin{vmatrix} L, & M, & N \\ L', & M', & N' \\ L'', & M'', & N'' \end{vmatrix} \text{ is the determinant reciprocal to } \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix},$$

so that, for instance, N'' is $lm'-l'm$, that of the six orders of partial substitutions all can produce the resultant substitutions (22), except

that the first fails when $m'=0$ or $N''=0$,
" second " $l=0$ or $N''=0$,
" third " $n''=0$ or $L=0$,
" fourth " $m'=0$ or $L=0$,
" fifth " $l=0$ or $M'=0$,
" sixth " $n''=0$ or $M'=0$.

Thus all fail if, and only if, simultaneously

either (α) $l=0,\quad m'=0,\quad n''=0,$

or (β) $L=0,\quad M'=0,\quad N''=0,$

or (γ) $m'=0,\quad n''=0,\quad M'=0,\quad N''=0,$

or (δ) $n''=0,\quad l=0,\quad N''=0,\quad L=0,$

or (ε) $l=0,\quad m'=0,\quad L=0,\quad M'=0.$

Of these (β) is a state of things with regard to the inverse substitution from X, Y, Z to x, y, z, exactly corresponding to (α) with regard to the direct substitution. Again, (γ), (δ), (ε) are one class of conditions, each being obtained by cyclical interchange of symbols from the former. In supplement, then, to the general case of a substitution resulting from three successive partial substitutions, as in (26), (27), (28), the sets of exceptional conditions (α) and (γ) need alone be considered. Of these (α) is the case of the substitution

$$\left.\begin{aligned} x &= \qquad\quad mY+nZ+p \\ y &= l'X \qquad\quad +n'Z+p' \\ z &= l''X+m''Y \qquad\quad +p'' \end{aligned}\right\} \quad\text{......................(31)},$$

and (γ), *i.e.*, the case of

$$m'=0,\quad n''=0,\quad nl''-n''l=0,\quad lm'-l'm=0,$$

i.e., of $\quad m'=0,\quad n''=0,\quad nl''=0,\quad l'm=0,$

subdivides into four cases, viz.,

(a) $m'=0,\quad n''=0,\quad n=0,\quad l'=0,$

(b) $m'=0,\quad n''=0,\quad n=0,\quad m=0,$

(c) $m'=0,\quad n''=0,\quad l''=0,\quad l'=0,$

(d) $m'=0,\quad n''=0,\quad l''=0,\quad m=0,$

the four classes of substitutions corresponding to which are

$$\left.\begin{array}{l} x = lX+mY \qquad\qquad +p \\ y = \qquad\qquad\qquad n'Z+p' \\ z = l''X+m''Y \qquad\quad +p'' \end{array}\right\} \text{......................(32),}$$

$$\left.\begin{array}{l} x = lX \qquad\qquad\qquad +p \\ y = l'X \qquad\qquad +n'Z+p' \\ z = l''X+m''Y \qquad\quad +p'' \end{array}\right\} \text{......................(33)}$$

$$\left.\begin{array}{l} x = lX\ +mY\ +nZ\ +p \\ y = \qquad\qquad\qquad n'Z+p' \\ z = \qquad\quad m''Y \qquad\ +p'' \end{array}\right\} \text{......................(34).}$$

$$\left.\begin{array}{l} x = lX \qquad\qquad +nZ\ +p \\ y = l'X \qquad\qquad +n'Z+p' \\ z = \qquad\quad m''Y \qquad\ +p'' \end{array}\right\} \text{......................(35).}$$

The number of exceptional classes of substitutions to be considered may be still further reduced. For the pair (32) and (35) are similar to one another; and the result of inverting (33) is of the form (34). Again, (31) may be replaced by a sequence of (34) with a different p, followed by

$$lX = mY' + nZ' - \left(\frac{ml''}{m''} + \frac{nl'}{n'}\right) X',$$

$$m''Y = l''X' + m''Y',$$

$$n'Z = l'X' + n'Z',$$

a transformation in which neither of the conditions (30) is satisfied. Once more, (35) may be replaced by a sequence of (34) followed by

$$lX = \left(l - \frac{nl'}{n'}\right) X' - mY',$$

$$Y = \qquad\qquad Y',$$

$$n'Z = l'X' \qquad\qquad + n'Z',$$

which again is not special.

It will suffice, then, to prove the prerogative of persistence, first, for the general sequence of transformations (26), (27), (28), and secondly, for the special excepted transformation (34).

7. Apply, then, the first substitution (26) of the general sequence to the first of the three equivalent expressions for a pure cyclicant R in (1). It becomes

$$\nu''^{i} R(Z, xy).$$

Thus

$$R \equiv \nu''^{i} R(Z, xy) \equiv \nu''^{i}\left(-\frac{dZ}{dx}\right)^{i+\frac{1}{2}w} R(x, yZ) \equiv \nu''^{i}\left(-\frac{dZ}{dy}\right)^{i+\frac{1}{2}w} R(y, Zx) \quad \ldots\ldots\ldots\ldots(36).$$

Again, apply the second substitution (27) to the second of these three forms of R. It becomes, since by (25)

$$\frac{\dfrac{dZ}{dx}}{\dfrac{dZ}{dX}} = \frac{-1}{-\lambda - \nu\dfrac{dZ}{dX}},$$

$$\nu''^{i}\lambda^{i}\left\{\frac{-\dfrac{dZ}{dX}}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(X, yZ).$$

Hence, by the laws expressed in (1), we have three forms

$$R \equiv (\nu''\lambda)^{i}\left\{\frac{1}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(Z, Xy) \equiv (\nu''\lambda)^{i}\left\{\frac{-\dfrac{dZ}{dX}}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(X, yZ)$$

$$\equiv (\nu''\lambda)^{i}\left\{\frac{-\dfrac{dZ}{dy}}{\lambda+\nu\dfrac{dZ}{dX}}\right\}^{i+\frac{1}{2}w} R(y, ZX) \quad \ldots\ldots\ldots\ldots(37).$$

Lastly, apply the third partial substitution (28). By (25) we see that $\dfrac{dZ}{dX}$ and $\dfrac{dZ}{dy}$ have to be replaced respectively by

$$\frac{m'\dfrac{dZ}{dX} - l'\dfrac{dZ}{dY}}{m'+n'\dfrac{dZ}{dY}} \quad \text{and} \quad \frac{\dfrac{dZ}{dY}}{m'+n'\dfrac{dZ}{dY}};$$

and consequently that the last form of R in (37) becomes

$$(\nu''\lambda m')^{i}\left\{\frac{-\dfrac{dZ}{dY}}{\lambda m' + \nu m'\dfrac{dZ}{dX} + (\lambda n' - \nu l')\dfrac{dZ}{dY}}\right\}^{i+\frac{1}{2}w} R(Y, ZX).$$

But, as in (1),

$$\left(-\frac{dZ}{dY}\right)^{i+\frac{1}{2}w} R(Y,\, ZX) \equiv R(Z,\, XY).$$

Thus we have R, *i.e.*, $R(z,\, xy)$,

$$\equiv (\nu''\lambda m')^{i}\left\{\nu m'\frac{dZ}{dX}+(\lambda n'-\nu l')\frac{dZ}{dY}+\lambda m'\right\}^{-i-\frac{1}{2}w} R(Z,\, XY);$$

i.e., by (29),

$$\equiv \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}^{i}$$

$$\times\left\{lm'-l'm-(mn'-m'n)\frac{dZ}{dX}-(nl'-n'l)\frac{dZ}{dY}\right\}^{-i-\frac{1}{2}w} R(Z,\, XY) \quad \ldots(38).$$

Thus for the general case, when the linear transformation may be replaced by a sequence of partial substitutions (26), (27), (28), the prerogative of persistence of a pure cyclicant is proved. Moreover, the form of the extraneous factor introduced is determined.

As a verification it may be noticed that, since

$$(n''-nz_{10}-n'z_{01})\left\{lm'-l'm-(mn'-m'n)Z_{10}-(nl'-n'l)Z_{01}\right\} = \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(39),$$

the same result is obtained by applying the reversed transformation to $R(Z,\, XY)$.

8. It remains to ascertain that the persistence holds also in case of the excepted transformation (34). Now, by this transformation, any pure derivative of x $(r+s \not< z)$

$$\frac{d^{r+s}x}{dy^r dz^s} \text{ becomes } \frac{l^{r+s}}{n''^r m''^s}\,\frac{d^{r+s}X}{dZ^r dY^s}.$$

Thus, R being doubly isobaric and symmetrical in first and second suffixes,

$$R(x,\, yz) \equiv \frac{l^{w}}{(n'm'')^{\frac{1}{2}w}} R(X,\, ZY)$$

$$\equiv \frac{l^{w}}{(n'm'')^{\frac{1}{2}w}} R(X,\, YZ),$$

i.e.,
$$\frac{R(z,\,xy)}{z_{10}^{i+\frac{1}{2}w}} \equiv \frac{l^{w}}{(n'm'')^{\frac{1}{2}w}}\,\frac{R(Z,\,XY)}{Z_{10}^{i+\frac{1}{2}w}}.$$

There is, then, no exception to Theorem I. of § 5, as to the persistence in form of a pure cyclicant.

9. The proof of Theorem II. of § 5 is similar, but less cumbrous. The transformation (23) may be replaced by the successive partial substitutions

$$\left.\begin{aligned} x &= x \\ y &= y \\ z &= n''Z+p'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40),$$

$$\left.\begin{aligned} x &= \lambda X+\mu y+\nu Z+\varpi \\ y &= \qquad\quad y \\ Z &= \qquad\qquad\quad Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(41),$$

$$\left.\begin{aligned} X &= X \\ y &= l'X+m'Y+n'Z+p' \\ Z &= \qquad\qquad\qquad Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(42),$$

which together are equivalent to

$$\begin{aligned} x &= (\lambda+\mu l')\,X+\mu m'Y+(\nu+\mu n')\,Z+\varpi+\mu p', \\ y &= \qquad l'X+\ m'Y+\qquad n'Z+p', \\ z &= \qquad\qquad\qquad\qquad\quad n''Z+p'', \end{aligned}$$

upon taking

$$\lambda = \frac{lm'-l'm}{m'},\quad \mu = \frac{m}{m'},\quad \nu = \frac{nm'-n'm}{m'},\quad \varpi = \frac{pm'-p'm}{m'} \ldots(43);$$

the only failing case being when $m' = 0$.

If $m' = 0$, we may instead proceed with successive partial substitutions in the order z, y, x, and produce the resultant transformation (23), except when $l = 0$.

We must then consider separately the general case of a sequence of substitutions and the special one when both $l = 0$ and $m' = 0$.

10. Take, first, the general case. Applying the first partial substitution (40) to the third of the identical expressions in (8), we obtain

the same form multiplied by n''^{i+m}. Thus three equivalents of $\frac{S_0(x, yz)}{x_{01}^{i+w_1}}$ are

$$n''^{i+m}\frac{S_0(x, yZ)}{x_{01}^{i+w_1}} \equiv (-1)^m n''^{i+m}\frac{S_m(y, Zx)}{y_{10}^{i+w_1}}$$

$$\equiv (-1)^{i+w_2} n''^{i+m}(S_0, S_1, \dots S_m)(-Z_{01}, Z_{10})^m \dots (44),$$

where $x_{01}, y_{10}, Z_{10}, Z_{01}$ now mean $\frac{dx}{dZ}, \frac{dy}{dZ}, \frac{dZ}{dx}, \frac{dZ}{dy}$.

Next, apply the second partial substitution (41) to the first member of (44). It becomes

$$\lambda^i n''^{i+m}\frac{S_0(X, yZ)}{(\lambda X_{01}+\nu)^{i+w_1}},$$

i.e.,

$$\frac{\lambda^i n''^{i+m} y_{10}^{i+w_1}}{(\lambda y_{10}-\nu y_{01})^{i+w_1}}\frac{S_0(X, yZ)}{X_{01}^{i+w_1}},$$

since

$$\frac{X_{10}}{-1} = \frac{X_{01}}{y_{10}} = \frac{-1}{y_{01}},$$

$X_{10}, X_{01}, y_{10}, y_{01}$ now meaning $\frac{dX}{dy}, \frac{dX}{dZ}, \frac{dy}{dZ}, \frac{dy}{dX}$.

And from this form of the original $\frac{S_0(x, yz)}{x_{01}^{i+w_1}}$ we have, by identities like (8) in the present variables, the two other forms

$$(-1)^m\frac{\lambda^i \nu''^{i+m}}{(\lambda y_{10}-\nu y_{01})^{i+w_1}} S_m(y, ZX)$$

$$\equiv (-1)^{i+w_2}\frac{\lambda^i n''^{i+m} y_{10}^{i+w_1}}{(\lambda y_{10}-\nu y_{01})^{i+w_1}}(S_0, S_1, \dots S_m)(-Z_{01}, Z_{10})^m \dots (45).$$

The last partial substitution (42) may now be applied to the last but one of these equivalent forms. At once

$$S_m(y, ZX) \text{ becomes } m''^i S_m(Y, ZX),$$

$$y_{10} \quad \text{,,} \quad m'Y_{10}+n',$$

$$y_{01} \quad \text{,,} \quad m'Y_{01}+l',$$

and consequently we obtain as the new form required

$$(-1)\ \frac{(\lambda m')^i n''^{i+m}}{\{\lambda m'Y_{10}-\nu m'Y_{01}+\lambda n'-\nu l'\}^{i+w_1}} S_m(Y, ZX),$$

i.e., by use of (43),

$$\frac{(lm'-l'm)^i n''^{i+m} Y_{10}^{i+w_1}}{\{(lm'-l'm)Y_{10}+(mn'-m'n)Y_{01}-(nl'-n'l)\}^{i+w_1}}(-1)^m \frac{S_m(Y, ZX)}{Y_{10}^{i+w_1}} \quad \ldots\ldots\ldots\ldots(46).$$

The result of the sequence of transformations equivalent to (23) is, then, to reproduce from the equivalent forms in (8) the same forms in the new variables multiplied by the factor

$$\frac{(lm'-l'm)^i n''^{i+m} Y_{10}^{i+w_1}}{\{(lm'-l'm)Y_{10}+(mn'-m'n)Y_{01}-(nl'-n'l)\}^{i+w_1}},$$

or, which is the same thing, by the factor

$$\frac{(lm'-l'm)^i n''^{i+m}}{\{lm'-l'm-(mn'-m'n)Z_{10}-(nl'-n'l)Z_{01}\}^{i+w_1}} \quad \ldots\ldots\ldots(47).$$

11. The temporarily excepted case of the transformation

$$\left.\begin{aligned} x &= \quad mY+nZ+p \\ y &= l'X \quad +n'Z+p' \\ z &= \qquad\quad n''Z+p'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(48)$$

is readily seen to be not really exceptional. This transformation may be replaced by the sequence of

$$\left.\begin{aligned} x &= mX' \quad + nZ+p \\ y &= \quad l'Y'+nZ+p' \\ z &= \qquad\quad n''Z+p'' \end{aligned}\right\},$$

and

$$\left.\begin{aligned} X' &= Y \\ Y' &= X \end{aligned}\right\}.$$

Of these partial transformations, the first is not special, and the second produces $(-1)^{w_2} S_m(Y, ZX)$ from $S_0(X', Y'Z)$, and $(-1)^{w_2} S_0(Y, ZX)$ from $S_m(X', Y'Z)$. In other words, it produces the second of the equivalent forms in (8) from the first, which is the same thing as reproducing the first.

Thus Theorem II. of § 5 is also established for all cases.

12. It is proposed now to consider the integration of a number of cyclicant, semicyclicant, and cocyclicant equations, and the converse passage from proper equations in x, y, z involving arbitrary functions to cyclicant, semicyclicant, and cocyclicant equations by elimination

of the arbitrary functions, as also of the variables and first derivatives. In other words, regarding the matter geometrically, it is proposed to deal with some classes of families of surfaces whose differential equations are the results of equating to zero pure cyclicants or semicyclicants or cocyclicants. A family of surfaces whose criterion is a pure cyclicant will have for its functional equation, if such can be found at all, one that is unaltered in character by any linear transformation of the variables. A family whose criterion is a semicyclicant in x as dependent variable, or a cocyclicant in z dependent, will, by the lawfulness of the transformation (23), have no special respect to any planes except those parallel to $z=0$. A family of surfaces having properties which a single cyclicant equation is insufficient to express, but which are independent of any particular coordinate planes, will often at least have for the full expression of those properties the vanishing of all the coefficients of a cocyclicant. Examples of this will be given.

The propositions of § 5 indicate that, in determining pure cyclicant and semicyclicant equations, much use may, with advantage, be made of canonical forms of functional equations. Thus, if

$$F(x, y, z) = 0$$

satisfy an equation, "pure cyclicant" = 0.

The same is also satisfied by

$$F(lx+my+nz+p,\ l'x+m'y+n'z+p',\ l''x+m''y+n''z+p'') = 0;$$

and, if $\phi(x, y, z) = 0$

satisfy an equation, "semicyclicant in x" = 0,

or "cocyclicant in z" = 0,

so also does

$$\phi(lx+my+nz+p,\ l'x+m'y+n'z+p',\ n''z+p'') = 0.$$

13. Of pure cyclic concomitants the lowest is z_{20}, the semicyclicant which is the leading coefficient of the quadratic cyclico-genitive form E_2. We have, in fact,

$$\frac{x_{20}}{x_{01}^3} \equiv \frac{y_{02}}{y_{10}^3} \equiv -(z_{20}, z_{11}, z_{02} \between (-z_{01}, z_{10})^2$$
$$\equiv -E_2.$$

Now the integral of $x_{20}=0$, *i.e.*, $\dfrac{d^2x}{dy^2}=0$, is at once

$$x = yf(z)+\phi(z) \quad \text{..........................(49)},$$

which is quite as general as its apparent transformation by (23),

$$lx+my+nz+p = (l'x+m'y+n'z+p')f(n''z+p'')+\phi(n''z+p'').$$

This, then, is the equation of the family of surfaces whose differential equation is either

$$x_{20} = 0, \quad \text{or} \quad y_{02} = 0, \quad \text{or} \quad E_2 = 0 \qquad \ldots\ldots(50).$$

It is the family of surfaces generated by straight lines always parallel to the plane $z = 0$.

The differential equation of surfaces cutting planes parallel to any other plane $\lambda x+\mu y+z = 0$ than $z = 0$ in straight lines is the one which would take either of the forms (50) upon putting in it z for $\lambda x+\mu y+z = 0$, keeping x and y unaltered. The third form is the one which gives at once the forms of equation of the family, viz.,

$$(z_{20}, z_{11}, z_{02})(-z_{01}-\mu, z_{10}+\lambda)^2 = 0,$$

or, say,

$$e^{\lambda(d/dz_{10})+\mu(d/dz_{01})}E_2 = 0 \qquad \ldots\ldots(51).$$

If all planes whatever are cut by the surfaces in straight lines, this equation must be satisfied for all values of λ and μ, and conversely. Now, this necessitates that separately

$$z_{20} = 0, \quad z_{11} = 0, \quad z_{02} = 0,$$

which are the differential equations of planes.

The results of this article, as no doubt also some of those which follow, are very familiar. They are given, however, as a first and instructive example of the method under consideration.

14. The later results of the last article exemplify facts which may at once be stated generally.

(i.) If $\qquad (S_0, S_1, \ldots S_m)(-z_{01}, z_{10})^m = 0$

be an equation of the form "cocyclicant = 0," obtained as the differential equation of a family of surfaces having an assigned property with regard to planes in the direction of $z = 0$, then

$$e^{\lambda(d/dz_{10})+\mu(d/dz_{01})}(S_0, S_1, \ldots S_m)(-z_{01}, z_{10})^m = 0 \qquad \ldots\ldots(51),$$

or

$$(S_0, S_1, \ldots S_m)(-z_{01}-\mu, z_{10}+\lambda)^m = 0 \qquad \ldots\ldots(51a),$$

is that of the family having that property with regard to planes parallel to the plane $\lambda x+\mu y+z = 0$; and

(ii.) If surfaces have the property with regard to planes in an infinity of independent directions, they satisfy simultaneously the differential equations

$$S_0 = 0, \quad S_1 = 0, \; \dots \; S_m = 0;$$

equations which are not to be expected to be all independent.

15. The next simplest cyclic concomitant to z_{20} is the pure cyclicant

$$z_{20}z_{02} - \tfrac{1}{4}z_{11}^2 \quad\dots\dots(52),$$

which is known to be the criterion of developable surfaces, and need not be further dwelt upon.

16. V_1 and Ω_1 both annihilate

$$3z_{30}z_{11} - 2z_{20}z_{21} \quad\dots\dots(53),$$

which is accordingly a semicyclicant. By (8), therefore,

$$\frac{3x_{30}x_{11} - 2x_{20}x_{21}}{x_{01}^6} \equiv -\frac{3y_{03}y_{11} - 2y_{02}y_{12}}{y_{10}^6}$$

$$\equiv -(3z_{30}z_{11} - 2z_{20}z_{21},\; 6z_{30}z_{02} + z_{21}z_{11} - 4z_{20}z_{12},\; 4z_{21}z_{02} - z_{12}z_{11} - 6z_{20}z_{03},$$
$$2z_{12}z_{02} - 3z_{03}z_{11} \circlearrowright (-z_{01}, z_{10})^3 \quad\dots\dots(54).$$

Now, the first of these three identical expressions equated to zero gives

$$\frac{3x_{30}}{x_{20}} = \frac{2x_{21}}{x_{11}}, \quad i.e., \quad \frac{1}{x_{20}}\frac{d}{dy}x_{20} = \frac{1}{x_{11}}\frac{d}{dy}x_{11}.$$

Therefore $$x_{20} = x_{11}f(z),$$

i.e., $$x_{10} = x_{01}F(z) + \phi(z);$$

which, integrated by means of the auxiliary system

$$\frac{dx}{\phi(z)} = \frac{dy}{1} = \frac{dz}{-F(z)},$$

gives $$x + \int \frac{\phi(z)}{F(z)}\,dz = \psi\left(y + \int \frac{dz}{F(z)}\right),$$

i.e., $$x + u = \psi(y+v) \quad\dots\dots(55),$$

u and v being arbitrary functions of z, and ψ an arbitrary functional symbol.

Thus either member of (54) equated to zero is the partial differential equation of the family of surfaces

$$lx + my + f_1(z) = \psi\{l'x + m'y + f_2(z)\} \quad\dots\dots(56),$$

the generalisation of (55) by the transformation (23).

The property of the family in question is that any given member of it cuts all planes parallel to the fixed plane $z = 0$ in identical and similarly situated curves. In particular, cylindrical surfaces are of the family, the loci of corresponding points of the sections—in general, curves of the type $lx+my+f_1(z) = 0$, $l'x+m'y+f_2(z) = 0$—being in this case straight lines. Again, any paraboloid whose axis is parallel to the plane $z = 0$ is of the family.

Surfaces which have the property with regard to the plane $\lambda x+\mu y+z = 0$ instead of $z = 0$, have their differential equation written down upon inserting $-z_{01}-\mu$, $z_{10}+\lambda$ for $-z_{01}$ and z_{10} in the third of the identical expressions in (54), and equating to zero.

Again, any surface which cuts every system of parallel planes in a system of identical and similarly situated curves—or which cuts an infinite number of parallel systems in such a manner—must satisfy separately the equations

$$\left.\begin{array}{ll} 3z_{30}z_{11}-2z_{20}z_{21} & = 0 \\ 6z_{30}z_{02}+z_{21}z_{11}-4z_{20}z_{12} & = 0 \\ 4z_{21}z_{02}-z_{12}z_{11}-6z_{03}z_{20} & = 0 \\ 2z_{02}z_{12}-3z_{03}z_{11} & = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(57).$$

This is the case with cylindrical surfaces.

17. An equation involving one more arbitrary function than (56) is

$$w(lx+my)+u = \psi\{w(l'x+m'y)+v\} \quad \ldots\ldots\ldots\ldots(58),$$

where u, v, w are arbitrary functions of z. This is the functional equation of the family of surfaces of which any one cuts all planes parallel to $z = 0$ in similar and similarly situated curves. All surfaces of revolution belong to the family, the plane $z = 0$ being in their case at right angles to the axis of revolution. Another very particular included family is that of quadric surfaces, which retain the property in question whatever be the plane $z = 0$.

From the canonical form

$$wx+u = \psi(wy+v) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58a),$$

of the equation (58), it is easy to obtain, by actual differentiation and elimination, the differential equation

$$\begin{vmatrix} 2x_{20} & x_{11} & \\ 3x_{30} & x_{21} & x_{20} \\ 4x_{40} & x_{31} & 2x_{30} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(59),$$

of the family, with x for dependent variable. This equation may be written

$$\frac{d}{dy}\left\{\frac{3x_{30}x_{11}-2x_{20}x_{21}}{x_{20}^2}\right\}=0 \quad\text{..................}(59a),$$

so that, by (20) and (53), or from the fact that Ω_1 and V_1 (in x dependent) annihilate the left-hand member of (59), that left-hand member is a semicyclicant (in x).

The converse passage from (59) to the functional equation may be performed as follows. We may write (59) in the form

$$\left(x_{11}\frac{d}{dy}-2x_{20}\frac{d}{dz}\right)\frac{x_{20}^2}{x_{30}}=0,$$

of which, by Lagrange's method, the first integral is

$$\frac{x_{20}^2}{x_{30}}=f(x_{10}),$$

i.e., $$\frac{x_{30}}{x_{20}}=\frac{x_{20}}{f(x_{10})};$$

whence $$\log x_{20}=F(x_{10})+\phi(z),$$

i.e., $$x_{20}=\phi_1(z)f_1(x_{10}),$$

which gives $$F_1(x_{10})=y\phi_1(z)+\phi_2(z),$$

i.e., $$x_{10}=\psi_1\{y\phi_1(z)+\phi_2(z)\};$$

and, again integrating,

$$x\phi_1(z)+\phi_3(z)=\psi\{y\phi_1(z)+\phi_2(z)\},$$

which is the canonical form (58*a*).

The equation in z dependent equivalent to (59) is, by (8),

$$\left(Q_0,\ \frac{1}{6}\Omega_2Q_0,\ \frac{1}{6.5}\Omega_2^2Q_0,\ \dots\ \frac{1}{6!}\Omega_2^6Q_0\right)(-z_{01},\ z_{10})^6=0\ \dots(60),$$

where Q_0 denotes
$$\begin{vmatrix} 2z_{20} & z_{11} & \\ 3z_{30} & z_{21} & z_{20} \\ 4z_{40} & z_{31} & 2z_{30} \end{vmatrix},$$

and $$\Omega_2\equiv\Sigma_{r+s\nless 2}\left\{(s+1)\,z_{r-1,s+1}\frac{d}{dz_{rs}}\right\}.$$

If an infinite number of different sets of parallel planes cut a surface in sets of similar and similarly situated curves, the equation of that surface satisfies all the differential equations

$$Q_0=0,\quad \Omega_2 Q_0=0,\quad \Omega_2^2 Q_0=0,\ \ldots\ \Omega_2^6 Q_0=0 \quad \ldots\ldots\ldots(61).$$

18. In accordance with the remark at the end of the first paragraph of the last article, these last equations (61) must be satisfied by all quadric surfaces. But (*Proceedings*, Vol. XIX., p. 15) we know already the conditions of lower weight which such surfaces must satisfy, viz., the four conditions (two independent)

$$\left\|\begin{matrix} z_{30} & z_{20} & \\ z_{21} & z_{11} & z_{20} \\ z_{12} & z_{02} & z_{11} \\ z_{03} & & z_{02} \end{matrix}\right\| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(62).$$

Our attention is then directed to the family of surfaces represented by the semicyclicant in x,

$$\left|\begin{matrix} x_{30} & x_{20} & \\ x_{21} & x_{11} & x_{20} \\ x_{12} & x_{02} & x_{11} \end{matrix}\right| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(63),$$

or, as is the same thing, by the cocyclicant in z,

$$\left|\begin{matrix} z_{30} & z_{20} & & z_{10}^3 \\ z_{21} & z_{11} & z_{20} & 3z_{10}^2 z_{01} \\ z_{12} & z_{02} & z_{11} & 3z_{10} z_{01}^2 \\ z_{03} & & z_{02} & z_{01}^3 \end{matrix}\right| = 0 \quad \ldots\ldots\ldots\ldots\ldots(64).$$

It does not appear that any single equation involving arbitrary functions can be found which is the complete primitive of (63), so as to be the functional equation of the entire family of surfaces. We may write (14), however,

$$\frac{d}{dy}\left\{\frac{x_{20}x_{02}-\frac{1}{4}x_{11}^2}{x_{20}^{\frac{4}{3}}}\right\}=0;$$

so that a first integral is

$$x_{20}x_{02}-\tfrac{1}{4}x_{11}^2 = x_{20}^{\frac{4}{3}} f(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots (65).$$

In particular, then, the family includes all developable surfaces, for (52) is a particular case of (65).

In accordance with the known satisfaction of (62) by all quadrics, we notice that the reason is, that a central quadric cuts all planes of any parallel system, and a paraboloid all of any of a triply infinite number of parallel systems, in similar and similarly situated conics having their centres collinear. Now, it is readily seen that the canonical form

$$xy = f(z) \qquad \qquad (66)$$

represents a family of surfaces having that property with regard to planes parallel to $z = 0$, and also is included in the more general family satisfying (63).

19. Enough isolated examples have been taken in the last six articles to indicate the importance of the study of cyclic concomitants in connection with the theory of families of surfaces. The remainder of the present paper will be devoted to the study of a very important particular class of concomitants, viz., to the class of cocyclicants whose semicyclicant sources are of second partial weight zero. The first of these is the quadratic cyclicogenitive form E_2. These have the very closest connection with pure reciprocants. (Having discarded the term *ternary reciprocant*, I henceforth use the word *reciprocants* to denote always the functions of the derivatives of one variable with regard to another, studied under that name by Professor Sylvester.)

It is useful to have a notation companion to that of (7) for the functions obtained by writing in the cyclicogenitive forms E_2, E_3, ... $-z_{01}-\mu$ and $z_{10}+\lambda$ for $-z_{01}$ and z_{10}. Let us use F_2, F_3, ... to denote these altered forms, so that, for all values greater than unity of the number r,

$$F_r \equiv (z_{r0}, z_{r-1,1}, \dots z_{0r} \circlearrowleft (-z_{01}-\mu, z_{10}+\lambda)^r \equiv e^{\lambda (d/dz_{10}) + \mu (d/dz_{01})} E_r \qquad (67).$$

20. Let

$$\phi(a, b, c, \dots),$$

or say, taking x for dependent variable instead of y,

$$\phi\left(\frac{1}{2!}\frac{d^2x}{dy^2}, \frac{1}{3!}\frac{d^3x}{dy^3}, \frac{1}{4!}\frac{d^4x}{dy^4}, \dots\right),$$

be any Sylvesterian pure reciprocant. Let its degree be i and its weight w; $a, b, c, \dots$ being regarded as of weights 2, 3, 4. The same function of the partial differential coefficients of x with regard to y, x being now regarded as a function both of y and z, is in our notation

$$\phi(x_{20}, x_{30}, x_{40}, \dots),$$

and satisfies the definition of a semicyclicant in x dependent, being homogeneous (of degree i), doubly isobaric (of weights w, 0), annihilated by V_1 (the same fact as that the reciprocant ϕ is by V) and also by Ω_1 (having no constituent of second suffix different from zero). Hence, by (3),

$$\frac{\phi(x_{20}, x_{30}, x_{40}, \ldots)}{x_{01}^{i+w}} \equiv (-1)^w \frac{\phi(y_{02}, y_{03}, y_{04}, \ldots)}{y_{10}^{i+w}}$$

$\equiv (-1)^i$ {that covariant of E_2, E_3, E_4, ... whose leading term is $\phi(z_{20}, z_{30}, z_{40}, \ldots)(-z_{01})^w$}

$$\equiv (-1)^i \phi(E_2, E_3, E_4, \ldots) \quad \ldots\ldots\ldots (68).$$

It is clear, then, that the study of cocyclicants of this class amounts to little more than a careful adaptation of results obtained by Sylvester and others with regard to pure reciprocants. They are the same functions of the cyclicogenitive forms E_2, E_3, ... as pure reciprocants are of the prepared derivatives a, b, ... Thus they are homogeneous and isobaric functions of E_2, E_3, ... which have the annihilator

$$4\frac{E_2^2}{2}\frac{d}{dE_3} + 5E_2E_3\frac{d}{dE_4} + 6\left(E_2E_4 + \tfrac{1}{2}E_3^2\right)\frac{d}{dE_5} + 7\left(E_2E_5 + E_3E_4\right)\frac{d}{dE_6} + \ldots \quad \ldots\ldots\ldots (69),$$

and one may be educed from any other by operation with the generator

$$4\left(E_2E_4 - E_3^2\right)\frac{d}{dE_3} + 5\left(E_2E_5 - E_3E_4\right)\frac{d}{dE_4} + 6\left(E_2E_6 - E_3E_5\right)\frac{d}{dE_5} + \ldots \quad \ldots\ldots\ldots (70).$$

21. The semicyclicants and cocyclicants obtained as in the last article are of immediately obvious geometrical interest. In fact, any pure reciprocant $\phi(a, b, c, \ldots)$ is known to be the criterion, *i.e.*, $\phi(a, b, c, \ldots) = 0$ to be the differential equation, of a class of plane curves whose equations are unaltered in form by any linear transformation of x and y; and moreover it is known, conversely, that the criterion of any such class of curves is a pure reciprocant. Now, the process of elimination, by aid of differentiation, of any number of constants from an equation in x and y, is exactly the same as that of elimination, by aid of partial differentiations treating z as constant, of the same number of arbitrary functions of z from an equation involving those functions, just as the first equation involved the constants which they replace. In other words,

$$\phi(x_{20}, x_{30}, x_{40}, \ldots) = 0 \quad \ldots\ldots\ldots (71),$$

or either of its equivalents, by (68),

$$\phi(y_{02}, y_{03}, y_{04}, \ldots) = 0 \quad \ldots\ldots\ldots (71a),$$

or

$$\phi(E_2, E_3, E_4, \ldots) = 0 \quad \ldots\ldots\ldots (71b),$$

is the differential equation of the family of curves which cut all planes parallel to $z=0$ in curves of the type of which $\phi(a, b, c, \ldots)$ is the reciprocant criterion.

The same three equations may, in our ordinary notation of semi-cyclicants and cocyclicants, be written

$$\phi_0(x, yz) = 0,$$

$$\phi_w(y, zx) = 0,$$

$$(\phi_0, \phi_1, \phi_2, \ldots \phi_w)(-z_{01}, z_{10})^w = 0 \quad \ldots\ldots\ldots\ldots\ldots(71c).$$

By § 14, it follows that the family of surfaces of which any cuts all parallels to any other given plane $\lambda x + \mu y + z = 0$ in curves of the family of which $\phi(a, b, c, \ldots)$ is the criterion, has for its differential equation

$$(\phi_0, \phi_1, \phi_2, \ldots \phi_w)(-z_{01}-\mu, z_{10}+\lambda)^w = 0 \quad \ldots\ldots\ldots\ldots(72),$$

or $$e^{\lambda(d/dz_{10})+\mu(d/dz_{01})} \cdot \phi(E_2, E_3, E_4, \ldots) = 0,$$

or again, in the notation of (67),

$$\phi(F_2, F_3, F_4, \ldots) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(72a).$$

22. The first pure reciprocant a leads in this manner to the differential equation of surfaces generated by straight lines parallel to a fixed plane. It produces the cocyclicant, &c., discussed in § 13.

The second pure reciprocant

$$ac - \tfrac{5}{4}b^2 \equiv (M)$$

is the criterion of parabolas. Hence either

$$M_0(x, yz) \equiv x_{20}x_{40} - \tfrac{5}{4}x_{30}^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(73),$$

or $$M_6(y, zx) \equiv y_{02}y_{04} - \tfrac{5}{4}y_{03}^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(73a),$$

or $$\mu \equiv E_2E_4 - \tfrac{5}{4}E_3^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(73b),$$

is the differential equation of the family of surfaces all sections of which by planes parallel to $z=0$ are parabolas.

The family of which all sections by parallels to $\lambda x + \mu y + \nu = 0$ are parabolas, is represented by

$$\nabla\mu \equiv e^{\lambda(d/dz_{10})+\mu(d/dz_{01})}\mu = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(74);$$

i.e., by $$F_2F_4 - \tfrac{5}{4}F_3^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(74a).$$

Again, if a surface cut all planes in parabolas, it must satisfy all the equations

$$\left.\begin{aligned}
M_0 &\equiv z_{20}z_{40} - \tfrac{5}{4}z_{30}^2 = 0, \\
M_1 &\equiv \tfrac{1}{6}\Omega_2 M_0 \equiv \tfrac{1}{6}(z_{11}z_{40} + z_{20}z_{31} - \tfrac{5}{2}z_{30}z_{21}) = 0, \\
M_2 &\equiv \tfrac{1}{5}\Omega_2 M_1 \equiv \frac{1}{6\,.\,5}(2z_{11}z_{31} - \tfrac{5}{2}z_{21}^2 + 2z_{02}z_{40} - 5z_{30}z_{12} + 2z_{20}z_{22}) = 0, \\
M_3 &= \tfrac{1}{4}\Omega_2 M_2 \equiv \frac{1}{6\,.\,5\,.\,4}(6z_{02}z_{31} - 15z_{21}z_{12} + 6z_{11}z_{22} - 15z_{30}z_{03} + 6z_{20}z_{13}) = 0, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
M_6 &= \frac{1}{6\,!}\Omega_2^6 M_0 \equiv z_{02}z_{04} - \tfrac{5}{4}z_{03}^2 = 0,
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(75),$$

which last results are probably rather matters of curiosity than value.

23. Again, $$A \equiv a^2d - 3abc + 2b^3$$
is the criterion of a conic (*Monge*).

Thus either of the three equivalent differential equations

$$A_0(x) \equiv x_{20}^2 x_{50} - 3x_{20}x_{30}x_{40} + 2x_{30}^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(76),$$

or

$$A_9(y) \equiv y_{02}^2 y_{05} - 3y_{02}y_{03}y_{04} + 2y_{03}^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(76a),$$

or

$$a \equiv E_2^2 E_5 - 3E_2E_3E_4 + 2E_3^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots(76b),$$

is the differential equation of the family of surfaces denoted by

$$u_1x^2 + v_1y^2 + w_1 + 2u_2y + 2v_2x + 2w_2xy = 0,$$

where u_1, v_1, w_1, u_2, v_2, w_2 are arbitrary functions of z.

Again, $$\nabla a \equiv F_2^2F_5 - 3F_2F_3F_4 + 2F_3^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(77)$$

represents the family of surfaces which cut parallels to $\lambda x + \mu y + z = 0$ in conics; and, if a surface cut all planes in conics, it satisfies all the

equations $$A_0 = 0, \quad \Omega_2 A_0 = 0, \ \ldots \ \Omega_2^9 A_0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(78).$$

24. Professor Sylvester (*American Journal*, IX., p. 16) has proved

that $$16(\lambda+1)^2 M^3 + 25(\lambda-2)(2\lambda-1)A^2 = 0$$

is the differential equation of curves of the class

$$ax + by + c = (a'x + b'y + c')^{\lambda} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(79),$$

a, b, c, a', b', c' being arbitrary constants.

It follows that, in the notation of the last two articles,

$$16\,(\lambda+1)^2\mu^3+25\,(\lambda-2)(2\lambda-1)\,\alpha^2=0 \quad\ldots\ldots\ldots\ldots\ldots(80)$$

and
$$16\,(\lambda+1)^2\,(\nabla\mu)^3+25\,(\lambda-2)(2\lambda-1)(\nabla\alpha)^2=0 \quad\ldots\ldots\ldots(81)$$

represent respectively the families of surfaces cutting planes parallel to $z=0$ and $\lambda x+\mu y+z=0$ in curves of the same type.

The results of the last two articles are included, as also are that

$$4^4\,(\nabla\mu)^3+5^3\,(\nabla\alpha)^2=0 \quad\ldots\ldots\ldots\ldots\ldots(82),$$

given by $\lambda=3$ or $\frac{1}{3}$, and

$$4\,(\nabla\mu)^3-\alpha^2=0 \quad\ldots\ldots\ldots\ldots\ldots(83),$$

given by $\lambda=\frac{2}{3}$ or $\frac{3}{2}$, represent surfaces cutting parallels to $\lambda x+\mu y+z=0$ in cubical and semicubical parabolas respectively.

25. The interpretation of Sylvester's $B, C, D, \ldots$ (*American Journal*, IX., p. 318) leads in like manner to those of the cocyclicants

$$\beta\equiv E_2^3E_6-2E_2^2E_4^2-\tfrac{7}{2}E_2^2E_3E_5+\tfrac{17}{2}E_2E_3^2E_4-4E_3^4 \quad\ldots\ldots\ldots(84),$$

$$\gamma\equiv E_2^4E_7-5E_2^3E_4E_5-4E_2^3E_3E_6+13E_2^2E_3E_4^2$$
$$+\tfrac{45}{4}E_2^2E_3^2E_5-\tfrac{103}{4}E_2E_3^3E_4+\tfrac{19}{2}E_3^5\ldots\ldots(85),$$

$$\delta\equiv E_2^5E_8-\tfrac{25}{8}E_2^4E_5^2-6E_2^4E_4E_6+7E_2^3E_4^3+E_3\,\{\ldots\} \quad\ldots\ldots\ldots(86),$$

&c., &c.,

and more generally to the interpretations of $\nabla\beta$, $\nabla\gamma$, $\nabla\delta, \ldots$, the results of replacing every E in β, γ, δ by the corresponding F.

Halphen's Δ ("Thèse sur les Invariants différentiels," pp. 12, &c.), or Sylvester's $AC-B^2$ (*American Journal*, IX., pp. 332, &c.), is the criterion of

$$\log\,(ax+by+c)+\omega\log\,(a'x+b'y+c')+\omega^2\log\,(a''x+b''y+c'')=k,$$

where ω is an imaginary cube root of unity, and $a, b, c, a', b', c', a'', b'', c'', k$ are arbitrary constants. The differential equation of which the complete integral is

$$\log\,(u_1x+v_1y+w_1)+\omega\log\,(u_2x+v_2y+w_2)+\omega^2\log\,(u_3x+v_3y+w_3)=U \quad\ldots\ldots\ldots(87),$$

in which $u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3, U$ are arbitrary functions of Z, is then

$$A_0\,(x,\,yz)\;C_0\,(x,\,yz)-\{B_0\,(x,\,yz)\}^2=0 \quad\ldots\ldots\ldots(88)$$

or $$A_9(y, zx)\, C_{15}(y, zx) - \{B_{12}(y, zx)\}^2 = 0 \;\ldots\ldots\ldots\ldots (88a),$$

or, again,

$$\alpha\gamma - \beta^2 \equiv \begin{vmatrix} E_3 & E_4 & E_5 & E_6 & E_7 \\ E_2 & E_3 & E_4 & E_5 & E_6 \\ -E_2^2 & 0 & E_3^2 & 2E_3E_4 & 2E_3E_5 + E_4^2 \\ 0 & E_2^2 & 2E_2E_3 & 2E_2E_4 + E_3^2 & 2E_2E_5 + 2E_3E_4 \\ 0 & 0 & E_2^2 & 3E_2E_3 & 3E_3^2 + 2E_2E_4 \end{vmatrix} = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (88b).$$

A result including this, and also the $\alpha = 0$ of § 23, obtained in like manner from a result of Sylvester's (*American Journal*, IX., pp. 337 338), is that, if λ be any constant,

$$2^4 7^3 (\lambda+1)^2 (\lambda-2)^2 (2\lambda-1)^2 (\alpha\gamma-\beta^2)^3 = 3^3 . 5^2 (\lambda^2-\lambda+1)^3 \alpha^8 \ldots (89)$$

is the differential equation of surfaces whose equations are of the type

$$(u_1x+v_1y+w_1)(u_2x+v_2y+w_2)^{-\lambda}(u_3x+v_3y+w_3)^{\lambda-1} = W \ldots (90)$$

with certain special cases corresponding to the values 0, ∞, and 1 of λ. For these the differential equation is

$$2^6 . 7^3 (\alpha\gamma-\beta^2)^3 = 3^3 . 5^2 . \alpha^8 \;\ldots\ldots\ldots\ldots\ldots\ldots (91),$$

and alternative complete integrals are

$$u_1x+v_1y+w_1 = \log(u_2x+v_2y+w_2) \;\ldots\ldots\ldots\ldots (92)$$

and $$u_1x+v_1y+w_1 = (u_2x+v_2y+w_2)\log(u_2x+v_2y+w_2) \;\ldots\ldots (93).$$

In all these results $u_1, v_1, w_1, u_2, \ldots, u_3, \ldots, W$ denote arbitrary functions of z. A particular result of (89) is that

$$2^8 (\alpha\gamma-\beta^2)^3 = 3^3\alpha^8 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (94)$$

represents surfaces where sections by parallels to $z = 0$ are cuspidal cubics.

The generalisations of these results, obtained by inserting $\nabla\alpha$, $\nabla\beta$, $\nabla\gamma$ for α, β, γ, *i.e.* F_2, &c., for E_2, &c., need not be stated at length.

26. A few more results may be stated without development.

The result of replacing λ in (90) by an arbitrary function of z

leads to a cocyclicant derived from Halphen's T (*Thèse*, p. 42), or Sylvester's $A^2D-3ABC+2B^3$. The comprehensive conclusion derived from this is that

$$24F_2^{-4}\left\{(\nabla\alpha)^2(\nabla\delta)-3(\nabla\alpha)(\nabla\beta)(\nabla\gamma)+2(\nabla\beta)^3\right\}\text{.........}(95),$$

i.e.,
$$\begin{vmatrix} 3F_3 & 0 & F_2 & 0 & 0 & 0 \\ 4F_4 & F_3 & F_3 & F_2 & 2F_2^2 & 0 \\ 5F_5 & 2F_4 & F_4 & 2F_3 & 5F_2F_3 & F_2^2 \\ 6F_6 & 3F_5 & F_5 & 3F_4 & 6F_2F_4+3F_3^2 & 3F_2F_3 \\ 7F_7 & 4F_6 & F_6 & 4F_5 & 7F_2F_5+7F_3F_4 & 4F_2F_4+2F_3^2 \\ 8F_8 & 5F_7 & F_7 & 5F_6 & 8F_2F_6+8F_3F_5+4F_4^2 & 5F_2F_5+5F_3F_4 \end{vmatrix}\text{...............}(95a),$$

is the criterion of surfaces cutting planes parallel to $\lambda x+\mu y+z=0$ in curves whose equations referred to axes in their own plane are of the form

$$(ax+by+c)^{-1}(a'x+b'y+c')^{\lambda}(a''x+b''y+c'')^{1-\lambda}=k\text{......}(96),$$

where λ as well as the other constants is arbitrary.

Again, Roberts's reciprocant expression for the criterion of a general cubic curve (*Educational Times Reprint*, x., p. 47) leads us to the conclusion that

$$\begin{vmatrix} F_2 & F_3 & F_4 & F_2^2 & 0 & 0 \\ F_3 & F_4 & F_5 & 2F_2F_3 & F_2^2 & 0 \\ F_4 & F_5 & F_6 & 2F_2F_4+F_3^2 & 2F_2F_3 & F_2^3 \\ F_5 & F_6 & F_7 & 2F_2F_5+2F_3F_4 & 2F_2F_4+F_3^2 & 3F_2^2F_3 \\ F_6 & F_7 & F_8 & 2F_2F_6+2F_3F_5+F_4^2 & 2F_2F_5+2F_3F_4 & 3F_2^2F_4+3F_2F_3^2 \\ F_7 & F_8 & F_9 & 2F_2F_7+2F_3F_6+2F_4F_5 & 2F_2F_6+2F_3F_5+F_4^2 & 3F_2^2F_5+6F_2F_3F_4+F_3^3 \end{vmatrix}\text{..................}(97)$$

is the criterion of surfaces whose sections by parallels to $\lambda x+\mu y+z=0$ are cubic curves.

And lastly, from Sylvester's criterion (*American Journal*, IX., p. 349) of curves of the n^{th} order, we deduce the criterion of surfaces cutting planes parallel to $\lambda x + \mu y + z = 0$ in curves of that order, viz.,

(2.1)	(3.1)	(4.1)	(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	...
(3.1)	(4.1)	(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	(12.1)	...
(3.2)	(4.2)	(5.2)	(6.2)	(7.2)	(8.2)	(9.2)	(10.2)	(11.2)	(12.2)	...
(4.1)	(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	(12.1)	(13.1)	...
(4.2)	(5.2)	(6.2)	(7.2)	(8.2)	(9.2)	(10.2)	(11.2)	(12.2)	(13.2)	...
(4.3)	(5.3)	(6.3)	(7.3)	(8.3)	(9.3)	(10.3)	(11.3)	(12.3)	(13.3)	...
(5.1)	(6.1)	(7.1)	(8.1)	(9.1)	(10.1)	(11.1)	(12.1)	(13.1)	(14.1)	...
(5.2)	(6.2)	(7.2)	(8.2)	(9.2)	(10.2)	(11.2)	(12.2)	(13.2)	(14.2)	...
5.3)	(6.3)	(7.3)	(8.3)	(9.3)	(10.3)	(11.3)	(12.3)	(13.3)	(14.3)	...
5.4)	(6.4)	(7.4)	(8.4)	(9.4)	(10.4)	(11.4)	(12.4)	(13.4)	(14.4)	...
	...	...	...	...	...	...	...	...	...	...

......................(98),

to $\frac{1}{2}n(n+1)$ rows and columns, where $(m \,.\, \mu)$ denotes the multiplier of k^m in the expansion of

$$(F_2 k^2 + F_3 k^3 + F_4 k^4 + \ldots)^\mu.$$

On the Figures formed by the Intercepts of a System of Straight Lines in a Plane, and on analogous relations in Space of Three Dimensions. By SAMUEL ROBERTS.

[*Read May 10th*, 1888.]

I. *Plane Space.*

1. In studying some questions relating to the closed branches of curves, I was led to consider the clear spaces enclosed by the finite segments determined by the intersections of straight lines in a plane. By " clear spaces " I mean those not cut by any of the lines, and it will be convenient to call them simply " spaces." I have since found that, long ago, Steiner treated of the subject, in consequence of his