

I. COVARIANT MEMORY-HISTORY FORMALISM

This section fixes the covariant memory-history sector used throughout the paper. We adopt the auxiliary-field formulation of Ref. [?] and use the same variational conventions. The spacetime is a four-dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$ with Levi-Civita connection ∇_μ , metric signature $(-+++)$, and covariant d'Alembertian

$$\square = g^{\mu\nu} \nabla_\mu \nabla_\nu. \quad (1)$$

All field variations are smooth and compactly supported, so that boundary terms vanish in integrations by parts.

The memory-history sector contains

$$\Delta_{\mu\nu}, \quad C, \quad \chi, \quad (2)$$

where

$$\Delta_{\mu\nu} = \Delta_{\nu\mu}. \quad (3)$$

The scalar C is the curvature-history field, χ is its conjugate auxiliary field, and $\Delta_{\mu\nu}$ is the memory tensor. The curvature source is the Kretschmann scalar

$$\mathcal{K} = R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma}. \quad (4)$$

The memory-history action is

$$S_{\text{mh}} = S_\Delta + S_C, \quad (5)$$

with

$$S_\Delta = \int_{\mathcal{M}} d^4x \sqrt{-g} \mathcal{L}_\Delta, \quad (6)$$

where

$$\mathcal{L}_\Delta = -\frac{1}{2} \Delta_{\mu\nu} F_C(\square \Delta^{\mu\nu}) - V(\Delta, C) + \eta G(C) \Delta_{\mu\nu} R^{\mu\nu}. \quad (7)$$

The auxiliary history sector is

$$S_C = \int_{\mathcal{M}} d^4x \sqrt{-g} \mathcal{L}_C, \quad (8)$$

where

$$\mathcal{L}_C = -\nabla_\mu \chi \nabla^\mu C - m_c^2 \chi C + \ell_* \chi \mathcal{K}. \quad (9)$$

Here $V(\Delta, C)$ is a scalar potential, $G(C)$ is a scalar function, η is a coupling constant, m_c is the infrared mass scale of the history field, and ℓ_* is the curvature-history length scale.

The state-dependent form factor is

$$F_C = \exp(A_C), \quad (10)$$

where the operator A_C acts on a rank-two tensor $X_{\mu\nu}$ by

$$(A_C X)_{\mu\nu} = \mu(C) \square X_{\mu\nu}, \quad \mu(C) = M_{\text{eff}}^{-2}(C). \quad (11)$$

For nonconstant C , multiplication by $\mu(C)$ does not commute with \square . Therefore the C -variation of F_C is defined by Duhamel's formula,

$$\delta_C F_C = \int_0^1 ds e^{sA_C} (\delta_C A_C) e^{(1-s)A_C}, \quad (12)$$

with

$$((\delta_C A_C) X)_{\mu\nu} = \mu'(C) \delta C \square X_{\mu\nu}. \quad (13)$$

The source generated by the C -dependence of S_Δ is defined by

$$\Sigma_\Delta = -\frac{1}{\sqrt{-g}} \frac{\delta S_\Delta}{\delta C}. \quad (14)$$

This sign convention is used throughout the paper.

Varying S_C with respect to χ gives

$$\begin{aligned} \delta_\chi S_C &= \int_{\mathcal{M}} d^4x \sqrt{-g} [-\nabla_\mu(\delta\chi)\nabla^\mu C - m_c^2(\delta\chi)C + \ell_*(\delta\chi)\mathcal{K}] \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \delta\chi (\square C - m_c^2 C + \ell_*\mathcal{K}). \end{aligned} \quad (15)$$

Stationarity with respect to arbitrary $\delta\chi$ gives

$$(\square - m_c^2)C = -\ell_*\mathcal{K}. \quad (16)$$

Varying S_C with respect to C gives

$$\begin{aligned} \delta_C S_C &= \int_{\mathcal{M}} d^4x \sqrt{-g} [-\nabla_\mu \chi \nabla^\mu (\delta C) - m_c^2 \chi \delta C] \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \delta C (\square \chi - m_c^2 \chi). \end{aligned} \quad (17)$$

Using Eq. (14), the total C -variation of S_{mh} is

$$\delta_C S_{\text{mh}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \delta C (\square \chi - m_c^2 \chi - \Sigma_\Delta). \quad (18)$$

Therefore

$$(\square - m_c^2)\chi = \Sigma_\Delta. \quad (19)$$

The memory-tensor equation is

$$E_\Delta^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_\Delta}{\delta \Delta_{\mu\nu}} = 0. \quad (20)$$

This equation contains the nonlocal kinetic term, the derivative of the potential with respect to $\Delta_{\mu\nu}$, and the curvature coupling proportional to $G(C)R^{\mu\nu}$.

The stress-energy tensors are defined by

$$T_{\mu\nu}^{(\Delta)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\Delta}{\delta g^{\mu\nu}}, \quad (21)$$

and

$$T_{\mu\nu}^{(C)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_C}{\delta g^{\mu\nu}}. \quad (22)$$

The total memory-history stress tensor is

$$T_{\mu\nu}^{\text{mem}} = T_{\mu\nu}^{(\Delta)} + T_{\mu\nu}^{(C)}. \quad (23)$$

The diffeomorphism Noether identity for S_Δ gives

$$\begin{aligned} \nabla^\mu T_{\mu\nu}^{(\Delta)} &= E_C^{(\Delta)} \nabla_\nu C + E_\Delta^{\alpha\beta} \nabla_\nu \Delta_{\alpha\beta} \\ &\quad - 2\nabla_\alpha \left(E_\Delta^{\alpha\beta} \Delta_{\nu\beta} \right), \end{aligned} \quad (24)$$

where

$$E_C^{(\Delta)} = \frac{1}{\sqrt{-g}} \frac{\delta S_\Delta}{\delta C} = -\Sigma_\Delta. \quad (25)$$

On solutions of the memory-tensor equation $E_{\Delta}^{\alpha\beta} = 0$, Eq. (24) reduces to

$$\nabla^{\mu} T_{\mu\nu}^{(\Delta)} = -\Sigma_{\Delta} \nabla_{\nu} C. \quad (26)$$

The diffeomorphism Noether identity for S_C is

$$\nabla^{\mu} T_{\mu\nu}^{(C)} = E_C^{(C)} \nabla_{\nu} C + E_{\chi}^{(C)} \nabla_{\nu} \chi, \quad (27)$$

where

$$E_C^{(C)} = \frac{1}{\sqrt{-g}} \frac{\delta S_C}{\delta C} = \square \chi - m_c^2 \chi, \quad (28)$$

and

$$E_{\chi}^{(C)} = \frac{1}{\sqrt{-g}} \frac{\delta S_C}{\delta \chi} = \square C - m_c^2 C + \ell_* \mathcal{K}. \quad (29)$$

Equations (16) and (19) imply

$$E_C^{(C)} = \Sigma_{\Delta}, \quad E_{\chi}^{(C)} = 0. \quad (30)$$

Substituting Eq. (30) into Eq. (27) gives

$$\nabla^{\mu} T_{\mu\nu}^{(C)} = \Sigma_{\Delta} \nabla_{\nu} C. \quad (31)$$

Adding Eqs. (26) and (31) gives

$$\nabla^{\mu} T_{\mu\nu}^{\text{mem}} = \nabla^{\mu} \left(T_{\mu\nu}^{(\Delta)} + T_{\mu\nu}^{(C)} \right) = 0. \quad (32)$$

For every field configuration satisfying

$$E_{\Delta}^{\mu\nu} = 0, \quad (\square - m_c^2) C = -\ell_* \mathcal{K}, \quad (\square - m_c^2) \chi = \Sigma_{\Delta}, \quad (33)$$

the memory-history stress tensor satisfies

$$\nabla^{\mu} T_{\mu\nu}^{\text{mem}} = 0. \quad (34)$$

When this sector is coupled to the Einstein-Hilbert action and to a matter or radiation sector with stress tensor $T_{\mu\nu}^{\text{ext}}$, the metric field equation is

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu}^{\text{ext}} + T_{\mu\nu}^{\text{mem}} \right). \quad (35)$$

The following sections apply Eqs. (16), (20), and (32) to black-hole geometries.

II. BLACK-HOLE GEOMETRIC BACKGROUND

This section fixes the black-hole geometry on which the memory-history equations of Sec. I are evaluated. The reference spacetime is the Schwarzschild manifold

$$\mathcal{M}_{\text{Schw}} = \mathbb{R}_t \times (0, \infty)_r \times S^2, \quad (36)$$

with line element

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad f(r) = 1 - \frac{2G_N M}{r}. \quad (37)$$

Here $M > 0$, $G_N > 0$, and

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \quad (38)$$

The horizon radius is

$$r_H = 2G_N M. \quad (39)$$

For the metric in Eq. (37), the Ricci tensor and Ricci scalar vanish:

$$R_{\mu\nu} = 0, \quad R = 0. \quad (40)$$

The curvature source in the history-field equation is therefore determined by the Riemann tensor. In an orthonormal frame, the independent nonzero curvature components may be written as

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \frac{2G_N M}{r^3}, \quad (41)$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}} = -\frac{G_N M}{r^3}, \quad (42)$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} = -\frac{G_N M}{r^3}, \quad (43)$$

and

$$R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} = \frac{2G_N M}{r^3}. \quad (44)$$

The quadratic contraction of the Riemann tensor is

$$\mathcal{K}_{\text{Schw}}(r) = R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} = \frac{48G_N^2 M^2}{r^6}. \quad (45)$$

Thus

$$\mathcal{K}_{\text{Schw}}(r) > 0 \quad \text{for every } r > 0, \quad (46)$$

and

$$\lim_{r \rightarrow 0^+} \mathcal{K}_{\text{Schw}}(r) = +\infty, \quad \lim_{r \rightarrow \infty} \mathcal{K}_{\text{Schw}}(r) = 0. \quad (47)$$

The curvature-history equation from Sec. I is

$$(\square - m_c^2)C = -\ell_* \mathcal{K}. \quad (48)$$

On the Schwarzschild reference geometry this equation becomes

$$(\square_{\text{Schw}} - m_c^2)C = -\frac{48\ell_* G_N^2 M^2}{r^6}. \quad (49)$$

The static spherically symmetric reduction is obtained by taking

$$C = C(r). \quad (50)$$

For such a scalar, the Schwarzschild d'Alembertian is

$$\square_{\text{Schw}} C = \frac{1}{r^2} \frac{d}{dr} \left(r^2 f(r) \frac{dC}{dr} \right). \quad (51)$$

Therefore Eq. (49) reduces to

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 f(r) \frac{dC}{dr} \right) - m_c^2 C = -\frac{48\ell_* G_N^2 M^2}{r^6}. \quad (52)$$

Let $\mathcal{K}_{\text{crit}}$ be a positive curvature threshold. The curvature-defined core region is

$$\mathcal{R}_{\mathcal{K}} = \{p \in \mathcal{M}_{\text{Schw}} : \mathcal{K}_{\text{Schw}}(p) \geq \mathcal{K}_{\text{crit}}\}. \quad (53)$$

Using Eq. (45), the boundary of this region is fixed by

$$\frac{48G_N^2 M^2}{r_{\mathcal{K}}^6} = \mathcal{K}_{\text{crit}}. \quad (54)$$

Hence

$$r_{\mathcal{K}} = \left(\frac{48G_N^2 M^2}{\mathcal{K}_{\text{crit}}} \right)^{1/6}. \quad (55)$$

Since $\mathcal{K}_{\text{Schw}}(r)$ is strictly decreasing for $r > 0$, Eq. (53) is equivalently

$$\mathcal{R}_{\mathcal{K}} = \{(t, r, \theta, \varphi) \in \mathcal{M}_{\text{Schw}} : 0 < r \leq r_{\mathcal{K}}\}. \quad (56)$$

Let C_{crit} be a positive history-field threshold. The history-defined core region is

$$\mathcal{R}_C = \{p \in \mathcal{M}_{\text{Schw}} : C(p) \geq C_{\text{crit}}\}. \quad (57)$$

This definition is invariant because C is a scalar.

The core domain used in the following sections is the intersection

$$\mathcal{R}_{\text{core}} = \mathcal{R}_{\mathcal{K}} \cap \mathcal{R}_C. \quad (58)$$

Equivalently,

$$\mathcal{R}_{\text{core}} = \{p \in \mathcal{M}_{\text{Schw}} : \mathcal{K}_{\text{Schw}}(p) \geq \mathcal{K}_{\text{crit}} \text{ and } C(p) \geq C_{\text{crit}}\}. \quad (59)$$

For comparison with core-modified static geometries, the areal-radius form

$$ds^2 = -e^{2\Psi(r)} F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2, \quad F(r) = 1 - \frac{2G_N m(r)}{r}, \quad (60)$$

is also fixed. The Schwarzschild metric is recovered by

$$\Psi(r) = 0, \quad m(r) = M. \quad (61)$$

For metrics of the form Eq. (60), the invariant definitions of the curvature and history regions are

$$\mathcal{R}_{\mathcal{K}} = \{p \in \mathcal{M} : \mathcal{K}(p) \geq \mathcal{K}_{\text{crit}}\}, \quad (62)$$

and

$$\mathcal{R}_C = \{p \in \mathcal{M} : C(p) \geq C_{\text{crit}}\}. \quad (63)$$

The coordinate r is the areal radius in the spherically symmetric chart; the core conditions are expressed by scalar inequalities.

III. WEYL-RICCI DECOMPOSITION OF THE CURVATURE-HISTORY SOURCE

The curvature-history equation fixed in Sec. I is

$$(\square - m_c^2)C = -\ell_* \mathcal{K}, \quad \mathcal{K} = R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma}. \quad (64)$$

The scalar source \mathcal{K} admits an invariant decomposition into Weyl and Ricci contributions in four dimensions.

Let $W_{\alpha\beta\rho\sigma}$ denote the Weyl tensor. The four-dimensional Riemann decomposition is

$$R_{\alpha\beta\rho\sigma} = W_{\alpha\beta\rho\sigma} + \frac{1}{2} (g_{\alpha\rho} R_{\sigma\beta} - g_{\alpha\sigma} R_{\rho\beta} - g_{\beta\rho} R_{\sigma\alpha} + g_{\beta\sigma} R_{\rho\alpha}) - \frac{R}{6} (g_{\alpha\rho} g_{\sigma\beta} - g_{\alpha\sigma} g_{\rho\beta}). \quad (65)$$

The Weyl tensor is trace-free,

$$g^{\alpha\rho}W_{\alpha\beta\rho\sigma} = 0. \quad (66)$$

Contracting Eq. (65) with itself and using Eq. (66) gives

$$R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} = W_{\alpha\beta\rho\sigma}W^{\alpha\beta\rho\sigma} + 2R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2. \quad (67)$$

Define

$$\mathcal{W} = W_{\alpha\beta\rho\sigma}W^{\alpha\beta\rho\sigma}, \quad \mathcal{R}_2 = R_{\alpha\beta}R^{\alpha\beta}. \quad (68)$$

Then

$$\mathcal{K} = \mathcal{W} + 2\mathcal{R}_2 - \frac{1}{3}R^2. \quad (69)$$

Substitution into Eq. (64) yields

$$(\square - m_c^2)C = -\ell_* \left(\mathcal{W} + 2\mathcal{R}_2 - \frac{1}{3}R^2 \right). \quad (70)$$

For the Schwarzschild reference geometry of Sec. II,

$$R_{\mu\nu} = 0, \quad R = 0. \quad (71)$$

Therefore

$$\mathcal{R}_2 = 0, \quad R^2 = 0, \quad (72)$$

and Eq. (69) reduces to

$$\mathcal{K}_{\text{Schw}} = \mathcal{W}_{\text{Schw}}. \quad (73)$$

Using Eq. (45), one obtains

$$\mathcal{W}_{\text{Schw}}(r) = W_{\alpha\beta\rho\sigma}W^{\alpha\beta\rho\sigma} = \frac{48G_N^2M^2}{r^6}. \quad (74)$$

The history-field equation on the Schwarzschild reference geometry is therefore

$$(\square_{\text{Schw}} - m_c^2)C = -\ell_* \mathcal{W}_{\text{Schw}} = -\frac{48\ell_*G_N^2M^2}{r^6}. \quad (75)$$

The source in Eq. (75) is entirely Weyl. It is positive before multiplication by $-\ell_*$ and satisfies

$$\mathcal{W}_{\text{Schw}}(r) > 0 \quad \text{for all } r > 0. \quad (76)$$

Its endpoint behavior is

$$\lim_{r \rightarrow 0^+} \mathcal{W}_{\text{Schw}}(r) = +\infty, \quad \lim_{r \rightarrow \infty} \mathcal{W}_{\text{Schw}}(r) = 0. \quad (77)$$

The curvature-threshold region introduced in Sec. II is, in the Schwarzschild reference geometry,

$$\mathcal{R}_{\mathcal{K}} = \{p \in \mathcal{M}_{\text{Schw}} : \mathcal{K}_{\text{Schw}}(p) \geq \mathcal{K}_{\text{crit}}\}. \quad (78)$$

Since $\mathcal{K}_{\text{Schw}} = \mathcal{W}_{\text{Schw}}$, the same region is

$$\mathcal{R}_{\mathcal{K}} = \{p \in \mathcal{M}_{\text{Schw}} : \mathcal{W}_{\text{Schw}}(p) \geq \mathcal{K}_{\text{crit}}\}. \quad (79)$$

In Schwarzschild coordinates this is

$$\mathcal{R}_{\mathcal{K}} = \{(t, r, \theta, \varphi) \in \mathcal{M}_{\text{Schw}} : 0 < r \leq r_{\mathcal{K}}\}, \quad (80)$$

where

$$r_{\mathcal{K}} = \left(\frac{48G_N^2M^2}{\mathcal{K}_{\text{crit}}} \right)^{1/6}. \quad (81)$$

Equations (70) and (75) give the curvature content of the source in the history-field equation: in a general four-dimensional geometry it contains the Weyl invariant, the Ricci-square invariant, and the scalar-curvature term; in the Ricci-flat Schwarzschild geometry it is exactly the quadratic Weyl invariant.

IV. CURVATURE-HISTORY ACTIVATION THRESHOLD

The curvature-history scalar satisfies

$$(\square - m_c^2)C = -\ell_*\mathcal{K}. \quad (82)$$

In the Schwarzschild reference geometry, Eqs. (49) and (45) give

$$(\square_{\text{Schw}} - m_c^2)C = -\frac{48\ell_*G_N^2M^2}{r^6}. \quad (83)$$

In the static spherically symmetric sector, Eq. (52) is

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 f(r) \frac{dC}{dr} \right) - m_c^2 C = -\frac{48\ell_*G_N^2M^2}{r^6}, \quad f(r) = 1 - \frac{2G_N M}{r}. \quad (84)$$

The leading curvature-driven local Laurent term of $C(r)$ near $r = 0$ is fixed by Eq. (84). Write

$$C(r) = Ar^{-3} + O(r^{-2}). \quad (85)$$

Then

$$\frac{dC}{dr} = -3Ar^{-4} + O(r^{-3}). \quad (86)$$

Using

$$r^2 f(r) = r^2 - 2G_N M r, \quad (87)$$

one obtains

$$r^2 f(r) \frac{dC}{dr} = -3Ar^{-2} + 6G_N M Ar^{-3} + O(r^{-1}). \quad (88)$$

Therefore

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 f(r) \frac{dC}{dr} \right) = -18G_N M Ar^{-6} + 6Ar^{-5} + O(r^{-4}). \quad (89)$$

The mass term satisfies

$$m_c^2 C = m_c^2 Ar^{-3} + O(r^{-2}), \quad (90)$$

so it does not contribute to the coefficient of r^{-6} . Matching the coefficient of r^{-6} in Eq. (84) gives

$$-18G_N M A = -48\ell_*G_N^2M^2. \quad (91)$$

Hence

$$A = \frac{8}{3}\ell_*G_N M. \quad (92)$$

The curvature-driven local behavior is therefore

$$C(r) = \frac{8}{3}\ell_*G_N M r^{-3} + O(r^{-2}). \quad (93)$$

For $\ell_* > 0$, $G_N > 0$, and $M > 0$, Eq. (93) gives

$$\lim_{r \rightarrow 0^+} C(r) = +\infty \quad (94)$$

for the curvature-driven Laurent branch.

Let $C_{\text{crit}} > 0$ be the history-field threshold. The activated history region is defined by

$$\mathcal{R}_C = \{p \in \mathcal{M} : C(p) \geq C_{\text{crit}}\}. \quad (95)$$

This is an invariant definition because C is a scalar. In the Schwarzschild static radial sector, Eq. (94) implies that for every finite $C_{\text{crit}} > 0$ there is an interval

$$0 < r < r_C \quad (96)$$

on which

$$C(r) > C_{\text{crit}}. \quad (97)$$

The threshold radius in the leading curvature-driven approximation is obtained by solving

$$\frac{8}{3} \ell_* G_N M r_C^{-3} = C_{\text{crit}}. \quad (98)$$

Thus

$$r_C = \left(\frac{8 \ell_* G_N M}{3 C_{\text{crit}}} \right)^{1/3}. \quad (99)$$

The curvature-threshold region is

$$\mathcal{R}_{\mathcal{K}} = \{p \in \mathcal{M} : \mathcal{K}(p) \geq \mathcal{K}_{\text{crit}}\}. \quad (100)$$

For Schwarzschild,

$$\mathcal{K}_{\text{Schw}}(r) = \frac{48 G_N^2 M^2}{r^6}. \quad (101)$$

The radius of the curvature-threshold region is therefore

$$r_{\mathcal{K}} = \left(\frac{48 G_N^2 M^2}{\mathcal{K}_{\text{crit}}} \right)^{1/6}. \quad (102)$$

The core activation domain is the invariant intersection

$$\mathcal{R}_{\text{core}} = \mathcal{R}_{\mathcal{K}} \cap \mathcal{R}_C. \quad (103)$$

In the Schwarzschild static radial sector, this is

$$\mathcal{R}_{\text{core}} = \{(t, r, \theta, \varphi) \in \mathcal{M}_{\text{Schw}} : 0 < r \leq r_{\text{core}}\}, \quad (104)$$

with

$$r_{\text{core}} = \min \{r_{\mathcal{K}}, r_C\}. \quad (105)$$

A smooth activation scalar is defined by

$$\Theta_C(C) = \frac{1}{2} \left[1 + \tanh \left(\frac{C - C_{\text{crit}}}{\sigma_C} \right) \right], \quad \sigma_C > 0. \quad (106)$$

For every real value of C ,

$$-1 < \tanh \left(\frac{C - C_{\text{crit}}}{\sigma_C} \right) < 1. \quad (107)$$

Therefore

$$0 < \Theta_C(C) < 1. \quad (108)$$

Its derivative is

$$\frac{d\Theta_C}{dC} = \frac{1}{2\sigma_C} \frac{1}{\cosh^2 \left(\frac{C - C_{\text{crit}}}{\sigma_C} \right)}. \quad (109)$$

Since $\sigma_C > 0$ and $\cosh^2 x > 0$ for all real x ,

$$\frac{d\Theta_C}{dC} > 0. \quad (110)$$

Thus Θ_C is a strictly increasing scalar function of the curvature-history field. At the threshold,

$$\Theta_C(C_{\text{crit}}) = \frac{1}{2}. \quad (111)$$

The limits are

$$\lim_{C \rightarrow -\infty} \Theta_C(C) = 0, \quad \lim_{C \rightarrow +\infty} \Theta_C(C) = 1. \quad (112)$$

The activation function assigns to every spacetime point the scalar value

$$\Theta_C(p) = \Theta_C(C(p)). \quad (113)$$

Combining Eqs. (94) and (112) gives

$$\lim_{r \rightarrow 0^+} \Theta_C(C(r)) = 1 \quad (114)$$

for the curvature-driven Schwarzschild radial branch. Therefore the history-field threshold construction defines a scalar activation domain and a smooth scalar activation profile. The memory-potential construction in the next section uses $\Theta_C(C)$ as the scalar switch controlling the nonvanishing memory phase.

V. MEMORY POTENTIAL AND NONVANISHING CORE PHASE

The scalar activation function fixed in Sec. IV is

$$\Theta_C(C) = \frac{1}{2} \left[1 + \tanh \left(\frac{C - C_{\text{crit}}}{\sigma_C} \right) \right], \quad \sigma_C > 0. \quad (115)$$

It satisfies

$$0 < \Theta_C(C) < 1 \quad (116)$$

for finite C , and on the history-defined core region

$$\mathcal{R}_C = \{p \in \mathcal{M} : C(p) \geq C_{\text{crit}}\}, \quad (117)$$

one has

$$\Theta_C(C(p)) \geq \frac{1}{2}. \quad (118)$$

Define the scalar memory amplitude

$$I_\Delta = \Delta_{\mu\nu} \Delta^{\mu\nu}. \quad (119)$$

The activated memory scale is

$$\Delta_0^2(C) = \Delta_*^2 \Theta_C(C), \quad \Delta_* > 0. \quad (120)$$

Equations (116) and (120) give

$$0 < \Delta_0^2(C) < \Delta_*^2 \quad (121)$$

for finite C . On \mathcal{R}_C , Eq. (118) gives the lower bound

$$\Delta_0^2(C(p)) \geq \frac{\Delta_*^2}{2}, \quad p \in \mathcal{R}_C. \quad (122)$$

The memory potential used in this paper is

$$V_{\text{act}}(\Delta, C) = \frac{\lambda}{4} (I_{\Delta} - \Delta_0^2(C))^2, \quad \lambda > 0. \quad (123)$$

Since $\lambda > 0$, Eq. (123) implies

$$V_{\text{act}}(\Delta, C) \geq 0. \quad (124)$$

The equality condition is

$$V_{\text{act}}(\Delta, C) = 0 \iff I_{\Delta} = \Delta_0^2(C). \quad (125)$$

Thus the pointwise minimum set of the activated memory potential is

$$\mathcal{N}_{\text{mem}}(C) = \{\Delta_{\mu\nu} : \Delta_{\mu\nu}\Delta^{\mu\nu} = \Delta_0^2(C)\}. \quad (126)$$

The first variation of I_{Δ} with respect to $\Delta_{\mu\nu}$, at fixed metric, is

$$\delta I_{\Delta} = 2\Delta^{\mu\nu}\delta\Delta_{\mu\nu}. \quad (127)$$

Therefore

$$\delta_{\Delta} V_{\text{act}} = \lambda (I_{\Delta} - \Delta_0^2(C)) \Delta^{\mu\nu} \delta\Delta_{\mu\nu}. \quad (128)$$

The algebraic contribution of the potential to the memory-tensor equation is

$$\frac{\partial V_{\text{act}}}{\partial \Delta_{\mu\nu}} = \lambda (I_{\Delta} - \Delta_0^2(C)) \Delta^{\mu\nu}. \quad (129)$$

The stationary set of V_{act} with respect to $\Delta_{\mu\nu}$ is therefore

$$\lambda (I_{\Delta} - \Delta_0^2(C)) \Delta^{\mu\nu} = 0. \quad (130)$$

Since $\lambda > 0$, Eq. (130) gives two algebraic branches:

$$\Delta_{\mu\nu} = 0, \quad (131)$$

or

$$I_{\Delta} = \Delta_0^2(C). \quad (132)$$

At $\Delta_{\mu\nu} = 0$,

$$V_{\text{act}}(0, C) = \frac{\lambda}{4} \Delta_0^4(C). \quad (133)$$

Using Eq. (121), this value is strictly positive:

$$V_{\text{act}}(0, C) > 0. \quad (134)$$

On the branch $I_{\Delta} = \Delta_0^2(C)$, Eq. (125) gives

$$V_{\text{act}}(\Delta, C) = 0. \quad (135)$$

Equations (124), (134), and (135) show that the global minima are exactly the tensors satisfying

$$\Delta_{\mu\nu}\Delta^{\mu\nu} = \Delta_0^2(C). \quad (136)$$

On the core region \mathcal{R}_C , Eq. (122) turns Eq. (136) into

$$\Delta_{\mu\nu}\Delta^{\mu\nu} \geq \frac{\Delta_*^2}{2}. \quad (137)$$

Consequently,

$$\Delta_{\mu\nu} \neq 0 \quad \text{on} \quad \mathcal{N}_{\text{mem}}(C(p)), \quad p \in \mathcal{R}_C. \quad (138)$$

The activated core phase is the pointwise minimum set

$$\mathcal{P}_{\text{core}} = \{(p, \Delta_{\mu\nu}) : p \in \mathcal{R}_{\text{core}}, \Delta_{\mu\nu}\Delta^{\mu\nu} = \Delta_0^2(C(p))\}. \quad (139)$$

Using Eqs. (103) and (137), every element of $\mathcal{P}_{\text{core}}$ satisfies

$$\Delta_{\mu\nu}\Delta^{\mu\nu} \geq \frac{\Delta_*^2}{2} > 0. \quad (140)$$

Thus the memory tensor is nonzero throughout the activated core phase.

The C -derivative of the activated memory scale is

$$\frac{d\Delta_0^2}{dC} = \Delta_*^2 \frac{d\Theta_C}{dC}. \quad (141)$$

Using Eq. (109), this gives

$$\frac{d\Delta_0^2}{dC} = \frac{\Delta_*^2}{2\sigma_C} \frac{1}{\cosh^2\left(\frac{C-C_{\text{crit}}}{\sigma_C}\right)}. \quad (142)$$

Hence

$$\frac{d\Delta_0^2}{dC} > 0. \quad (143)$$

The potential contribution to the C -variation is

$$\frac{\partial V_{\text{act}}}{\partial C} = -\frac{\lambda}{2} (I_\Delta - \Delta_0^2(C)) \frac{d\Delta_0^2}{dC}. \quad (144)$$

On the activated core phase,

$$I_\Delta - \Delta_0^2(C) = 0, \quad (145)$$

and therefore

$$\left. \frac{\partial V_{\text{act}}}{\partial C} \right|_{\mathcal{P}_{\text{core}}} = 0. \quad (146)$$

The result of this section is the algebraic phase relation

$$p \in \mathcal{R}_{\text{core}} \quad \text{and} \quad (p, \Delta_{\mu\nu}) \in \mathcal{P}_{\text{core}} \quad \implies \quad \Delta_{\mu\nu}\Delta^{\mu\nu} \geq \frac{\Delta_*^2}{2} > 0. \quad (147)$$

This relation is obtained from the scalar activation function, the definition of the core region, and the global minimum condition of the activated memory potential.

VI. MEMORY STRESS-ENERGY IN THE ACTIVATED CORE

The activated memory potential of Sec. V fixes the nonzero invariant amplitude of the memory tensor. The stress-energy carried by the activated core is fixed by the variational definition of Sec. I. In this section the potential entering the memory-sector action is written as

$$V_{\text{mem}}(\Delta, C) = V_{\text{act}}(\Delta, C) + V_{\text{st}}(C), \quad (148)$$

where

$$V_{\text{act}}(\Delta, C) = \frac{\lambda}{4} (\Delta_{\mu\nu}\Delta^{\mu\nu} - \Delta_0^2(C))^2, \quad \lambda > 0, \quad (149)$$

and

$$V_{\text{st}}(C) = \rho_{\Delta} \Theta_C(C), \quad \rho_{\Delta} > 0. \quad (150)$$

The term V_{st} is independent of $\Delta_{\mu\nu}$. Therefore

$$\frac{\partial V_{\text{mem}}}{\partial \Delta_{\mu\nu}} = \frac{\partial V_{\text{act}}}{\partial \Delta_{\mu\nu}}. \quad (151)$$

Hence the pointwise minimum condition derived in Sec. V is unchanged:

$$\Delta_{\mu\nu} \Delta^{\mu\nu} = \Delta_0^2(C). \quad (152)$$

The stress-energy contribution generated by V_{st} follows from the action term

$$S_{\text{st}} = - \int_{\mathcal{M}} d^4x \sqrt{-g} V_{\text{st}}(C). \quad (153)$$

The metric variation is taken at fixed C . Since $V_{\text{st}}(C)$ is a scalar with no explicit dependence on $g^{\mu\nu}$, one obtains

$$\delta_g S_{\text{st}} = \frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{-g} g_{\mu\nu} V_{\text{st}}(C) \delta g^{\mu\nu}. \quad (154)$$

Using the variational definition

$$T_{\mu\nu}^{\text{st}} = - \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{st}}}{\delta g^{\mu\nu}}, \quad (155)$$

Eq. (154) gives

$$T_{\mu\nu}^{\text{st}} = -g_{\mu\nu} V_{\text{st}}(C) = -\rho_{\Delta} \Theta_C(C) g_{\mu\nu}. \quad (156)$$

Let n^μ be a unit timelike vector at a point p :

$$g_{\mu\nu} n^\mu n^\nu = -1. \quad (157)$$

The local energy density associated with $T_{\mu\nu}^{\text{st}}$ is

$$\varepsilon_{\text{st}} = T_{\mu\nu}^{\text{st}} n^\mu n^\nu. \quad (158)$$

Substituting Eq. (156) into Eq. (158) gives

$$\varepsilon_{\text{st}} = \rho_{\Delta} \Theta_C(C). \quad (159)$$

On the history-defined activated region,

$$\mathcal{R}_C = \{p \in \mathcal{M} : C(p) \geq C_{\text{crit}}\}, \quad (160)$$

Sec. IV gives

$$\Theta_C(C(p)) \geq \frac{1}{2}. \quad (161)$$

Therefore

$$\varepsilon_{\text{st}}(p) \geq \frac{\rho_{\Delta}}{2} > 0, \quad p \in \mathcal{R}_C. \quad (162)$$

Since

$$\mathcal{R}_{\text{core}} = \mathcal{R}_{\mathcal{K}} \cap \mathcal{R}_C, \quad (163)$$

Eq. (162) holds at every point of $\mathcal{R}_{\text{core}}$:

$$\varepsilon_{\text{st}}(p) \geq \frac{\rho_{\Delta}}{2} > 0, \quad p \in \mathcal{R}_{\text{core}}. \quad (164)$$

Equation (156) also gives

$$T_{\mu\nu}^{\text{st}}(p) \neq 0, \quad p \in \mathcal{R}_{\text{core}}. \quad (165)$$

The full memory-sector stress tensor is decomposed as

$$T_{\mu\nu}^{(\Delta)} = T_{\mu\nu}^{\text{kin}} + T_{\mu\nu}^{\text{act}} + T_{\mu\nu}^{\text{curv}} + T_{\mu\nu}^{\text{st}}. \quad (166)$$

Here $T_{\mu\nu}^{\text{kin}}$ is generated by the nonlocal kinetic term, $T_{\mu\nu}^{\text{act}}$ by V_{act} , $T_{\mu\nu}^{\text{curv}}$ by the coupling $\eta G(C)\Delta_{\mu\nu}R^{\mu\nu}$, and $T_{\mu\nu}^{\text{st}}$ by V_{st} . The total memory-history stress tensor remains

$$T_{\mu\nu}^{\text{mem}} = T_{\mu\nu}^{(\Delta)} + T_{\mu\nu}^{(C)}. \quad (167)$$

The C -source contribution generated by S_{st} is

$$\Sigma_{\text{st}} = -\frac{1}{\sqrt{-g}} \frac{\delta S_{\text{st}}}{\delta C}. \quad (168)$$

Using Eq. (153), one obtains

$$\Sigma_{\text{st}} = \frac{dV_{\text{st}}}{dC} = \rho_{\Delta} \frac{d\Theta_C}{dC}. \quad (169)$$

The derivative of Θ_C is

$$\frac{d\Theta_C}{dC} = \frac{1}{2\sigma_C} \frac{1}{\cosh^2\left(\frac{C-C_{\text{crit}}}{\sigma_C}\right)}, \quad \sigma_C > 0. \quad (170)$$

Hence

$$\Sigma_{\text{st}} > 0. \quad (171)$$

The full source in the auxiliary equation is therefore

$$\Sigma_{\Delta} = \Sigma_{\text{kin}} + \Sigma_{\text{act}} + \Sigma_{\text{curv}} + \Sigma_{\text{st}}, \quad (172)$$

where each term is the C -variation of the corresponding part of S_{Δ} .

The conservation identity proved in Sec. I applies to the full memory-history action with $V_{\text{mem}} = V_{\text{act}} + V_{\text{st}}$. On solutions of

$$E_{\Delta}^{\mu\nu} = 0, \quad (\square - m_c^2)C = -\ell_*\mathcal{K}, \quad (\square - m_c^2)\chi = \Sigma_{\Delta}, \quad (173)$$

the total memory-history stress tensor satisfies

$$\nabla^{\mu} T_{\mu\nu}^{\text{mem}} = 0. \quad (174)$$

The gravitational field equation with an external matter or radiation sector is

$$G_{\mu\nu} = 8\pi G_N (T_{\mu\nu}^{\text{ext}} + T_{\mu\nu}^{\text{mem}}). \quad (175)$$

Equations (165) and (175) show that the activated core contains a nonzero variational stress-energy contribution from the memory sector. The local lower bound on the corresponding timelike energy density is Eq. (164).

VII. INFORMATION-ENCODING CRITERION

The activated core phase established in Sec. V is characterized by the nonzero invariant

$$\Delta_{\mu\nu}\Delta^{\mu\nu} = \Delta_0^2(C) = \Delta_*^2\Theta_C(C) \quad (176)$$

on $\mathcal{P}_{\text{core}}$. The memory-history stress tensor is covariantly conserved on shell by Eq. (174). This section fixes the criterion by which the activated core data encode black-hole history data.

Let $\mathfrak{S}_{\text{core}}$ denote the set of on-shell field configurations

$$s = (g_{\mu\nu}, C, \chi, \Delta_{\mu\nu}) \quad (177)$$

satisfying

$$E_{\Delta}^{\mu\nu} = 0, \quad (\square - m_c^2)C = -\ell_* \mathcal{K}, \quad (\square - m_c^2)\chi = \Sigma_{\Delta}, \quad (178)$$

and satisfying the activated-core relation

$$(p, \Delta_{\mu\nu}) \in \mathcal{P}_{\text{core}} \quad \text{for} \quad p \in \mathcal{R}_{\text{core}}. \quad (179)$$

Two configurations $s_1, s_2 \in \mathfrak{S}_{\text{core}}$ are gauge-equivalent when there exists a diffeomorphism $\phi : \mathcal{M} \rightarrow \mathcal{M}$ such that

$$s_2 = \phi^* s_1. \quad (180)$$

The corresponding gauge-equivalence class is denoted by

$$[s]. \quad (181)$$

The core memory datum of a configuration s is the restriction of the memory-history fields to the activated core domain:

$$\gamma_{\text{mem}}(s) = (C, \chi, \Delta_{\mu\nu})|_{\mathcal{R}_{\text{core}}}. \quad (182)$$

The gauge-quotient core memory space is

$$\Gamma_{\text{mem}}^{\text{core}} = \{\gamma_{\text{mem}}(s) : s \in \mathfrak{S}_{\text{core}}\} / \text{Diff}(\mathcal{R}_{\text{core}}). \quad (183)$$

The encoding map is

$$\mathfrak{E} : \mathfrak{S}_{\text{core}} / \text{Diff}(\mathcal{M}) \longrightarrow \Gamma_{\text{mem}}^{\text{core}}, \quad \mathfrak{E}([s]) = [\gamma_{\text{mem}}(s)]. \quad (184)$$

This map is well defined. Indeed, if $s_2 = \phi^* s_1$, then

$$\gamma_{\text{mem}}(s_2) = \phi^* (C_1, \chi_1, \Delta_{1\mu\nu})|_{\phi^{-1}(\mathcal{R}_{\text{core}})}, \quad (185)$$

and the scalar inequalities defining $\mathcal{R}_{\text{core}}$ are preserved under pullback:

$$\mathcal{K}_{s_2}(p) = \mathcal{K}_{s_1}(\phi(p)), \quad C_{s_2}(p) = C_{s_1}(\phi(p)). \quad (186)$$

Therefore gauge-equivalent spacetime configurations determine the same element of $\Gamma_{\text{mem}}^{\text{core}}$.

A history parameter q is encoded in the activated core when there exists a gauge-invariant functional

$$\mathcal{O} : \Gamma_{\text{mem}}^{\text{core}} \longrightarrow \mathbb{R} \quad (187)$$

such that the map

$$q \longmapsto \mathcal{O}(\mathfrak{E}([s_q])) \quad (188)$$

is injective on the history family under consideration.

The Schwarzschild mass family is

$$\mathfrak{H}_{\text{Schw}} = \{M \in \mathbb{R} : M > 0\}. \quad (189)$$

For every $M \in \mathfrak{H}_{\text{Schw}}$, the curvature-driven radial branch of the history field has the leading behavior obtained in Sec. IV:

$$C_M(r) = \frac{8}{3} \ell_* G_N M r^{-3} + O(r^{-2}). \quad (190)$$

Define the core coefficient functional

$$\mathcal{A}_C([\gamma_{\text{mem}}]) = \lim_{r \rightarrow 0^+} r^3 C_M(r). \quad (191)$$

In the spherically symmetric Schwarzschild sector, r is the areal radius defined by the area $4\pi r^2$ of the symmetry two-spheres. Hence \mathcal{A}_C is invariant under diffeomorphisms preserving the spherically symmetric representation of the same geometry.

Substituting Eq. (190) into Eq. (191) gives

$$\mathcal{A}_C(\mathfrak{E}([s_M])) = \frac{8}{3}\ell_* G_N M. \quad (192)$$

For $M_1, M_2 > 0$,

$$\mathcal{A}_C(\mathfrak{E}([s_{M_1}])) = \mathcal{A}_C(\mathfrak{E}([s_{M_2}])) \quad (193)$$

implies

$$\frac{8}{3}\ell_* G_N M_1 = \frac{8}{3}\ell_* G_N M_2. \quad (194)$$

Since $\ell_* > 0$ and $G_N > 0$,

$$M_1 = M_2. \quad (195)$$

Thus the map

$$M \mapsto \mathcal{A}_C(\mathfrak{E}([s_M])) \quad (196)$$

is injective on $\mathfrak{H}_{\text{Schw}}$.

The same conclusion is obtained directly from the curvature source. For the Schwarzschild family,

$$\mathcal{K}_M(r) = \frac{48G_N^2 M^2}{r^6}. \quad (197)$$

At fixed areal radius $r > 0$, define

$$\mathcal{A}_{\mathcal{K}}(M; r) = r^6 \mathcal{K}_M(r). \quad (198)$$

Then

$$\mathcal{A}_{\mathcal{K}}(M; r) = 48G_N^2 M^2. \quad (199)$$

For $M_1, M_2 > 0$,

$$\mathcal{A}_{\mathcal{K}}(M_1; r) = \mathcal{A}_{\mathcal{K}}(M_2; r) \quad (200)$$

implies

$$M_1^2 = M_2^2. \quad (201)$$

Since both masses are positive,

$$M_1 = M_2. \quad (202)$$

The activated memory amplitude is tied to the history field by

$$I_{\Delta} = \Delta_{\mu\nu} \Delta^{\mu\nu} = \Delta_*^2 \Theta_C(C) \quad (203)$$

on $\mathcal{P}_{\text{core}}$. Since

$$\frac{d\Theta_C}{dC} > 0, \quad (204)$$

the map $C \mapsto \Theta_C(C)$ is injective. Therefore, at a fixed point where C is finite, equality of activated memory amplitudes gives

$$I_{\Delta,1} = I_{\Delta,2} \implies \Theta_C(C_1) = \Theta_C(C_2) \implies C_1 = C_2. \quad (205)$$

Thus the scalar invariant I_Δ carries the same pointwise history-field value C on the activated branch.

Combining Eqs. (192), (195), and (205), the activated core memory data separate the Schwarzschild mass family:

$$\mathfrak{E}([s_{M_1}]) = \mathfrak{E}([s_{M_2}]) \implies M_1 = M_2. \quad (206)$$

Therefore the mass parameter of the Schwarzschild history family is encoded in the gauge-quotient core memory configuration.

The encoding criterion used in the rest of the paper is the separation condition

$$\mathfrak{E}([s_1]) \neq \mathfrak{E}([s_2]). \quad (207)$$

For two on-shell configurations, Eq. (207) means that the corresponding activated core memory data are inequivalent under diffeomorphisms. In the Schwarzschild mass family, Eq. (206) establishes that distinct positive masses determine distinct gauge-quotient core memory configurations.

VIII. MEMORY CAPACITY AND REGULATED ENTROPY

The encoding map constructed in Sec. VII is

$$\mathfrak{E} : \mathfrak{S}_{\text{core}} / \text{Diff}(\mathcal{M}) \longrightarrow \Gamma_{\text{mem}}^{\text{core}}. \quad (208)$$

For the Schwarzschild mass family, Sec. VII established

$$\mathfrak{E}([s_{M_1}]) = \mathfrak{E}([s_{M_2}]) \implies M_1 = M_2, \quad M_1, M_2 > 0. \quad (209)$$

The regulated memory entropy is defined by counting distinguishable elements of the gauge-quotient memory image.

Let

$$\mathfrak{H}_{[M_-, M_+]} = \{M \in \mathbb{R} : 0 < M_- \leq M \leq M_+\}, \quad M_+ > M_-, \quad (210)$$

and define

$$\alpha_C = \frac{8}{3} \ell_* G_N. \quad (211)$$

Equation (192) gives the core memory coefficient

$$\mathcal{A}_C(M) = \alpha_C M. \quad (212)$$

The image of the mass interval under \mathcal{A}_C is the closed interval

$$I_C = [\alpha_C M_-, \alpha_C M_+]. \quad (213)$$

Its length is

$$L_C = \alpha_C (M_+ - M_-). \quad (214)$$

Let $\varepsilon_C > 0$ be the resolution in the invariant coefficient \mathcal{A}_C . A subset

$$\mathcal{D}_C = \{a_1, a_2, \dots, a_N\} \subset I_C \quad (215)$$

is ε_C -separated when

$$|a_i - a_j| \geq \varepsilon_C \quad \text{for } i \neq j. \quad (216)$$

Ordering the elements as

$$a_1 < a_2 < \dots < a_N, \quad (217)$$

Eq. (216) gives

$$a_N - a_1 = \sum_{j=1}^{N-1} (a_{j+1} - a_j) \geq (N-1)\varepsilon_C. \quad (218)$$

Since $a_1, a_N \in I_C$,

$$a_N - a_1 \leq L_C. \quad (219)$$

Combining Eqs. (218) and (219) yields

$$N \leq \frac{L_C}{\varepsilon_C} + 1. \quad (220)$$

Hence

$$N \leq \left\lfloor \frac{L_C}{\varepsilon_C} \right\rfloor + 1. \quad (221)$$

The set

$$a_j = \alpha_C M_- + (j-1)\varepsilon_C, \quad j = 1, \dots, \left\lfloor \frac{L_C}{\varepsilon_C} \right\rfloor + 1, \quad (222)$$

lies in I_C and is ε_C -separated. Therefore the maximal number of distinguishable coefficient values is

$$N_C(\varepsilon_C) = \left\lfloor \frac{L_C}{\varepsilon_C} \right\rfloor + 1 = \left\lfloor \frac{\alpha_C(M_+ - M_-)}{\varepsilon_C} \right\rfloor + 1. \quad (223)$$

The regulated entropy associated with the mass-family memory image is

$$S_C(\varepsilon_C) = \log N_C(\varepsilon_C). \quad (224)$$

The same count can be written in terms of a mass resolution

$$\varepsilon_M = \frac{\varepsilon_C}{\alpha_C}. \quad (225)$$

Substitution into Eq. (223) gives

$$N_M(\varepsilon_M) = \left\lfloor \frac{M_+ - M_-}{\varepsilon_M} \right\rfloor + 1, \quad S_M(\varepsilon_M) = \log N_M(\varepsilon_M). \quad (226)$$

Equations (223) and (226) are identical descriptions of the same finite-resolution memory count.

A surface-regulated core memory capacity is now defined on a closed two-surface \mathcal{B} with induced metric q_{AB} and area

$$A_{\mathcal{B}} = \int_{\mathcal{B}} d^2y \sqrt{q}. \quad (227)$$

Let d_{mem} be an integer satisfying

$$d_{\text{mem}} \geq 2, \quad (228)$$

and define the regulator area

$$a_{\text{mem}} = 4G_N \log d_{\text{mem}}. \quad (229)$$

The number of complete regulator cells on \mathcal{B} is

$$N_{\mathcal{B}} = \left\lfloor \frac{A_{\mathcal{B}}}{a_{\text{mem}}} \right\rfloor. \quad (230)$$

The corresponding regulated memory data space is

$$\mathcal{H}_{\text{mem}}^{\text{reg}}(\mathcal{B}) = \bigotimes_{j=1}^{N_{\mathcal{B}}} \mathbb{C}^{d_{\text{mem}}}. \quad (231)$$

Its dimension is

$$\dim \mathcal{H}_{\text{mem}}^{\text{reg}}(\mathcal{B}) = d_{\text{mem}}^{N_{\mathcal{B}}}. \quad (232)$$

The regulated surface memory entropy is

$$S_{\text{mem}}^{\text{reg}}(\mathcal{B}) = \log \dim \mathcal{H}_{\text{mem}}^{\text{reg}}(\mathcal{B}) = N_{\mathcal{B}} \log d_{\text{mem}}. \quad (233)$$

Using Eqs. (229) and (230), Eq. (233) becomes

$$S_{\text{mem}}^{\text{reg}}(\mathcal{B}) = \left\lfloor \frac{A_{\mathcal{B}}}{4G_N \log d_{\text{mem}}} \right\rfloor \log d_{\text{mem}}. \quad (234)$$

For every real x ,

$$x - 1 < \lfloor x \rfloor \leq x. \quad (235)$$

Applying Eq. (235) to

$$x = \frac{A_{\mathcal{B}}}{4G_N \log d_{\text{mem}}} \quad (236)$$

gives the exact bounds

$$\frac{A_{\mathcal{B}}}{4G_N} - \log d_{\text{mem}} < S_{\text{mem}}^{\text{reg}}(\mathcal{B}) \leq \frac{A_{\mathcal{B}}}{4G_N}. \quad (237)$$

For the Schwarzschild horizon cross-section,

$$r_H = 2G_N M, \quad A_H = 4\pi r_H^2 = 16\pi G_N^2 M^2. \quad (238)$$

Substituting $A_{\mathcal{B}} = A_H$ into Eq. (234) gives

$$S_{\text{mem}}^{\text{reg}}(H) = \left\lfloor \frac{16\pi G_N^2 M^2}{4G_N \log d_{\text{mem}}} \right\rfloor \log d_{\text{mem}}. \quad (239)$$

Equivalently,

$$S_{\text{mem}}^{\text{reg}}(H) = \left\lfloor \frac{4\pi G_N M^2}{\log d_{\text{mem}}} \right\rfloor \log d_{\text{mem}}. \quad (240)$$

The exact horizon-area bounds are

$$\frac{A_H}{4G_N} - \log d_{\text{mem}} < S_{\text{mem}}^{\text{reg}}(H) \leq \frac{A_H}{4G_N}. \quad (241)$$

The entropy quantities in this section are therefore fixed by two finite counts. The first count,

$$S_C(\varepsilon_C) = \log \left[\left\lfloor \frac{\alpha_C(M_+ - M_-)}{\varepsilon_C} \right\rfloor + 1 \right], \quad (242)$$

counts distinguishable elements of the on-shell Schwarzschild mass-family memory image at resolution ε_C . The second count,

$$S_{\text{mem}}^{\text{reg}}(\mathcal{B}) = \left\lfloor \frac{A_{\mathcal{B}}}{4G_N \log d_{\text{mem}}} \right\rfloor \log d_{\text{mem}}, \quad (243)$$

is the surface-regulated memory capacity associated with the closed surface \mathcal{B} . For the Schwarzschild horizon cross-section it satisfies the exact bound in Eq. (241).