

On a Problem in Annuities, and on Arbogast's Method of Development. By PROFESSOR DE MORGAN.

IN the investigation of a little curiosity in the matter of annuities, I had occasion to want four or five terms of $\log(a + bx + cx^2 + \dots)$, and, by Arbogast's rules I accordingly wrote down the following, without a stroke of the pen more than is here given:—

$$\begin{aligned} \log a + \frac{b}{a} \cdot x + \left(\frac{c}{a} - \frac{1}{2} \frac{b^2}{a^2} \right) x^2 + \left(\frac{d}{a} - \frac{bc}{a^2} + \frac{1}{3} \frac{b^3}{a^3} \right) x^3 \\ + \left(\frac{e}{a} - \frac{bd}{a^2} - \frac{1}{2} \frac{c^2}{a^2} + \frac{b^2c}{a^3} - \frac{1}{4} \frac{b^4}{a^4} \right) x^4 \\ + \left(\frac{f}{a} - \frac{be}{a^2} - \frac{cd}{a^2} + \frac{b^2d}{a^3} + \frac{bc^2}{a^3} - \frac{b^3c}{a^4} + \frac{1}{5} \frac{b^5}{a^5} \right) x^5 + \dots \end{aligned}$$

This I might have continued to any number of terms for little beyond the trouble of writing. The method is easy to demonstrate and easy to practice; and it struck me that I might, in a few pages, give an account of it sufficient for use and detached from anything else. I therefore append it to the problem.

I have never met with the following little matter, and think it may be offered to the *Journal* for preservation in what an old dedicatior called the “storehouse of pretty conceits.”

When an annuity is payable at equal intervals in each year, every one sees—or thinks he sees—that the present value is greater than that of a yearly annuity: for, he says, the sooner you are to get money the more is it now worth. He is right: but the supposition being, as usual, that interest is payable as often as the

annuity, it is hardly safe to decide a race between $(1+r)^k$ and $\left(1 + \frac{r}{m}\right)^{mk}$ by *à priori* arguments.

Let us call the present value of all the payments made in the k th year by the name of the k th collection. Thus in quarterly payments the fifth collection is the sum of the present values of the 17th, 18th, 19th, and 20th payments. Now it is not true that *every* collection is more valuable than the single payment at the end of the year: when a certain time has elapsed, the collections are severally worth less than their single payments. A perpetual annuity is the same however often payment and interest may be subdivided; hence any two annuities, of differently distributed payments, must show a turning point, before which the collections of one exceed in present value, and after which they fall short of, the collections of the other.

Now the curiosity is that the year which contains the turning point does not depend on the number of subdivisions. At $\frac{r}{m}$ per pound per m th of a year, m being the number of payments in a year, and $\pounds \frac{1}{m}$ each payment, the turn has just been made at the end of the $\left(\frac{1}{r} + 1\right)$ th year, or else the year in which that fraction falls. Thus, at 10 per cent., when $\frac{1}{r}$ is also 10, we see this in the following table:—

Year	Five payments		Two payments.		One payment.
1	·9427	>	·9297	>	·9091
9	·4269	>	·4259	>	·4241
10	·3867	>	·3863	>	·3855
11	·3502	<	·3504	<	·3505
12	·3172	<	·3178	<	·3186

The greater the distribution of payments, the greater the value of the year's collection, until 10 years have passed, and then inversely.

The demonstration is as follows:—Equate the value of the k th yearly payment to that of the k th collection of m payments in one year. This gives

$$\frac{1}{(1+r)^k} = \frac{\left(1 + \frac{r}{m}\right)^m - 1}{r \left(1 + \frac{r}{m}\right)^{mk}}$$

or

$$k = \frac{\log \left\{ \left(1 + \frac{r}{m}\right)^m - 1 \right\} - \log r}{\log \left(1 + \frac{r}{m}\right)^m - \log(1+r)}.$$

Common development gives

$$k = \frac{1}{r} + \frac{3m+1}{4m} + \frac{5(m-1)}{12m^2}r + \dots$$

and this is always less than $\frac{1}{r} + 1$ for every value of r less than unity.

I have separately calculated this series one term further when m is infinite, or the annuity *gradual* (payable *momently*, as they say). And I thus find

$$k = \frac{1}{r} + \frac{3}{4} + 0.7 + \frac{541}{4320}r^2 + \dots$$

I now proceed to an account of Arbogast's process. This may be subdivided into a lower part, easily learnt by common algebra, and a higher part, requiring knowledge of the rules of the differential calculus.

A number of successive letters, a, b, c, d, \dots are given, each of which is called the *derivative* of the preceding. The symbol of the derivative may be the letter D : thus, $b = Da$, $c = Db$, &c.; $c = D^2a$, $d = Dc = D^2b = D^3a$, &c.

Every function of these letters has its derivative. When it is a product of powers, this process of derivation is—

Multiply a power by its exponent, diminish that exponent by a unit, and introduce the next letter once more; and if such introduction increase an exponent already existing, divide by the exponent so increased.

But this rule is to be applied only as follows: to the last letter in all cases; to the last but one *only when the last and last but one are consecutive*, a and b , b and c , &c.; to any before the last but one, *never*.

Thus, $a^3b^4c^7$ gives as its derivative $a^3b^4.7c^6d$, from the last letter, c ; and $a^3.4b^3c.c^7$ divided by $7+1$ or 8 , from the last but one, b , since b and c are consecutive. But $a^3b^4d^7$ gives nothing but $a^3b^4.7d^6e$, since b and d are not consecutive. The following example will give further practice:—

$$D^0b^5 = b^5$$

$$D^1b^5 = 5b^4c$$

$$D^2b^5 = 5b^4d + 10b^3c^2$$

$$D^3b^5 = 5b^4e + 20b^3cd + 10b^2c^3$$

$$D^4b^5 = 5b^4f + 20b^3ce + 10b^3d^2 + 30b^2c^2d + 5b^2c^4$$

$$D^5b^5 = 5b^4g + 20b^3cf + 20b^3de + 30b^2c^2e + 30b^2cd^2 + 20bc^3d + c^5.$$

Observe that if the series of letters come to an end, all terms which would require a letter which is not forthcoming must be erased. Let the letters end at c , which amounts to saying that the whole series is $a, b, c, 0, 0, 0, \&c.$

$$\begin{aligned} Da^7 &= 7a^6b, & D^2a^7 &= 7a^6c + 21a^5b^2 \\ D^3a^7 &= (7a^6.0=0) + 42a^5bc + 35a^4b^3 \\ D^4a^7 &= (42a^5b.0=0) + 21a^5c^2 + 105a^4b^2c + 35a^3b^4. \end{aligned}$$

We can now write down $(a+b+c+\dots)^n$ and $(a+bx+cx^2+\dots)^n$, when n is integer; for

$$(a+b+c+\dots)^n = a^n + Da^n + D^2a^n + D^3a^n + \dots;$$

and we also have

$$(a+bx+cx^2+dx^3+\dots)^n = a^n + Da^n.x + D^2a^n.x^2 + D^3a^n.x^3 + \dots$$

Thus we can now write down any integer power, developed, without any writing except the result: as

$$\begin{aligned} (a+bx+cx^2)^4 &= a^4 + 4a^3bx + (4a^3c + 6a^2b^2)x^2 \\ &\quad + (12a^2bc + 4ab^3)x^3 + (6a^2c^2 + 12ab^2c + b^4)x^4 \\ &\quad + (12abc^2 + 4b^3c)x^5 + (4ac^3 + 6b^2c^2)x^6 + 4bc^3x^7 + c^4x^8. \end{aligned}$$

When we form the derivatives of any function of a , the treatment of a , when it is the last letter, or the last but one before b , is simple differentiation, as it has been already in the case of a power of a . And the rule of the last or last but one is to be preserved as before.

Thus we have for derivatives of ϕa as follows:—

$$\begin{aligned} D \phi a &= \phi' a.b \\ D^2 \phi a &= \phi' a.c + \frac{\phi'' a}{2}.b^2 \\ D^3 \phi a &= \phi' a.d + \frac{\phi'' a}{2}.2bc + \frac{\phi''' a}{2.3}.b^3 \\ D^4 \phi a &= \phi' a.d + \frac{\phi'' a}{2}(2bd+c^2) + \frac{\phi''' a}{2.3}.3b^2c + \frac{\phi^{iv} a}{2.3.4}.b^4. \end{aligned}$$

A little attention will show the following law:—

$$D^n \phi a = \phi' a.D^{n-1}b + \frac{\phi'' a}{2}.D^{n-2}b^2 + \frac{\phi''' a}{2.3}.D^{n-3}b^3 + \dots + \frac{\phi^{(n)} a}{2.3\dots n}.b^n.$$

And the following is the method of developing a function of any polynomial:—

$$\phi(a+bx+cx^2+\dots) = \phi a + D\phi a.x + D^2\phi a.x^2 + D^3\phi a.x^3 + \dots$$

For example—

$$\begin{aligned}
 (a+bx+cx^2+\dots) &= a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}bx + \left(\frac{1}{2}a^{-\frac{1}{2}}c - \frac{1}{2}\cdot\frac{1}{2}a^{-\frac{3}{2}}\frac{b^2}{2}\right)x^2 \\
 &\quad + \left(\frac{1}{2}a^{-\frac{1}{2}}d - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}a^{-\frac{3}{2}}\cdot bc + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}a^{-\frac{5}{2}}\frac{b^3}{2\cdot 3}\right)x^3 \\
 &\quad + \dots
 \end{aligned}$$

If we complete the operations as we go on, we find

$$\begin{aligned}
 &a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}bx + \left(\frac{1}{2}a^{-\frac{1}{2}}c - \frac{1}{8}a^{-\frac{3}{2}}b^2\right)x^2 + \left(\frac{1}{2}a^{-\frac{1}{2}}d - \frac{1}{4}a^{-\frac{3}{2}}bc + \frac{1}{16}a^{-\frac{5}{2}}b^3\right)x^3 \\
 &\quad + \left(\frac{1}{2}a^{-\frac{1}{2}}e - \frac{1}{4}a^{-\frac{3}{2}}bd - \frac{1}{8}a^{-\frac{3}{2}}c^2 + \frac{3}{16}a^{-\frac{5}{2}}b^2c - \frac{5}{128}a^{-\frac{7}{2}}b^4\right)x^4,
 \end{aligned}$$

and so on.

The demonstration of all this is as follows:—Take any power of $b+c+d+\dots$, say the 10th. If we write down every combination of *ten* letters, with or without repetition, and then make every variation of these letters, each such variation is one of the terms of the tenth power required. Thus

$$bb\ ccc\ dddd\ e\ \text{takes } \frac{\overline{10}}{\overline{2}\ \overline{3}\ \overline{4}\ \overline{1}} \text{ variations;}$$

where \overline{n} signifies $1.2.3\dots n$. And

$$\frac{\overline{10}}{\overline{2}\ \overline{3}\ \overline{4}\ \overline{1}} b^2c^3d^4e \text{ is one of the terms required.}$$

This is the common *multinomial theorem* of algebra.

Now the rule above used, employ it on what letter we please, is sure to change one term of the power into another having some one letter *thrown forward*, some one b changed into c , or some one c changed into d , &c. Try it on e in

$$\begin{aligned}
 &\frac{\overline{n}}{\overline{p}\ \overline{q}\ \overline{r}\ \overline{s}} c^pe^qf^rh^s, \text{ which will become} \\
 &\frac{\overline{n}}{\overline{p}\ \overline{q}\ \overline{r}\ \overline{s}} \cdot \frac{q}{r+1} c^pe^{q-1}f^{r+1}h^s, \text{ or } \frac{\overline{n}}{\overline{p}\ \overline{q-1}\ \overline{r+1}\ \overline{s}} c^pe^{q-1}f^{r+1}h^s,
 \end{aligned}$$

another term. And here an e is changed into f . If, then, we take b^n and use the process again and again *on every letter*, we shall be manufacturing the terms of $(b+c+d+\dots)^n$ in a very easy way. But we shall produce the same terms over and over and over again. If we confine ourselves to the last letter, and the last but one when the last two are consecutive, we shall produce each term once and once only. This I have now to prove.

Suppose we take any combination of seven, as $cc\ eee\ gg$. These letters are in lots of $2\ 3\ 2$. Throw back the first letter in the last lot, and repeat this process again and again. We cannot fail by this succession to find our way back to $bbbbbb$. As follows:—

As to $\phi(a+bx+cx^2+\dots)$, let $b+cx+dx^2+\dots$ be z : then

$$\begin{aligned}
 \phi(a+xz) &= \phi a + \phi' a . xz + \frac{\phi'' a}{2} x^2 z^2 + \frac{\phi''' a}{2.3} x^3 z^3 + \dots \\
 &= \phi a + \phi' a . x(b + Db . x + D^2 b . x^2 + \dots) + \frac{\phi'' a}{2} x^2 (b^2 + D b^2 . x + D^2 b . x^2 + \dots) \\
 &\quad + \frac{\phi''' a}{2.3} x^3 (b^3 + D b^3 . x + D^2 b^3 . x^2 + \dots) + \dots \\
 &= \phi a + \phi' a . bx + \left(\phi' a . Db + \frac{\phi'' a}{2} . b^2 \right) x^2 \\
 &\quad + \left(\phi' a . D^2 b + \frac{\phi'' a}{2} . D b^2 + \frac{\phi''' a}{2.3} . b^3 \right) x^3 \\
 &\quad + \left(\phi' a . D^3 b + \frac{\phi'' a}{2} . D^2 b^2 + \frac{\phi''' a}{2.3} D b^3 + \frac{\phi^{IV} a}{2.3.4} b^4 \right) x^4 + \dots \\
 &= \phi a + D \phi a . x + D^2 \phi a . x^2 + D^3 \phi a . x^3 + D^4 \phi a . x^4 + \dots
 \end{aligned}$$

by the mode of forming $D^4 \phi a$ already shown.

I shall be happy to solve any difficulties which any of your readers may find.
