to me that the theorem in regard to the enneadic centre subsists for a system of 9 points such as referred to in the statement; but that if by possibility the statement be too general, the theorem must, at all events, subsist for a more special system of 9 points; and that there certainly exist systems of 10 points, such that each 9 of the points have as an enneadic centre the tenth point. [I have since ascertained that if a quartic surface with 10 nodes has a single node (3, 3), the surface is a symmetroid; whence, by what precedes, the remaining nine nodes are each of them (3, 3). Added 25th March.]

124. I notice, as a subject of investigation, the following system of correspondence; viz., given any 8 points in space: then to every point in space corresponds a line through this point: viz., the ninth line of the ennead obtained by joining the point with the 8 given points respectively, and to each line in space a point or points on the line: viz., the point or points for each of which the line is the ninth line of the ennead obtained by joining the point with the eight given points respectively.

Dr. Hirst next entered into an explanation of his paper "On the Polar Correlation of two Planes, and its connection with their Quadric Correspondence."

Profs. Cayley, Smith, Mr. Cotterill, and Dr. Hirst took part in the discussions on the papers.

The following presents were made to the Library:—

"On the determination of the Orbit of a Planet from three Observations." By A. Cayley, F.R.S.: from the author.

"Géométrie Supérieure." By M. Chasles: from the President.

Jan. 12th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. R. B. Hayward, M.A., late Fellow of St. John's College, and Mathematical Master at Harrow, and the Rev. J. Wolstenholme, M.A., Fellow of Christ's College, were proposed for election.

Mr. Walker gave an account of the following paper:—

On Systems of Tangents to Plane Cubic and Quartic Curves.

1. By a "System of Tangents" I understand all the tangents which can be drawn from a point $(x_1 y_1 z_1)$ to a proper plane curve (U). According to Dr. Salmon's notation, the equation to the system of tangents drawn to a cubic curve is obtained by equating with zero the discriminant of the binary cubic in λ, μ, (Higher Plane Curves, Ed. 1, p. 68,)

$$U\lambda^3 + \Delta_1\lambda^2\mu + \Delta\lambda\mu^2 + U_1\mu^3,$$

where Δ_1 stands for $x_1\frac{dU}{dx} + y_1\frac{dU}{dy} + z_1\frac{dU}{dz}$, and Δ for

$$x_1^2\frac{d^2U}{dx^2} + y_1^2\frac{d^2U}{dy^2} + z_1^2\frac{d^2U}{dz^2} + 2y_1z_1\frac{d^2U}{dydz} + 2z_1x_1\frac{d^2U}{dzdx} + 2x_1y_1\frac{d^2U}{dxdy};$$

while U_1 is the result of substituting x_1, y_1, z_1 for x, y, z respectively in U. It will be convenient, in what follows, to use some symbol to represent the *operator* $x_1\frac{d}{dx} + \ldots$ apart from its subject. Supposing then

$$D_1 = x_1\frac{d}{dx} + y_1\frac{d}{dy} + z_1\frac{d}{dz},$$

the cubic above in λ, μ would be

$$U\lambda^3 + D_1U\lambda^2\mu + \tfrac{1}{2}D_1^2U\lambda\mu^2 + U_1\mu^3.$$

2. As all the curves, which I shall have occasion to notice in this short paper, will be expressed in terms of $U, D_1U, \ldots U_1$, I have, at the kind suggestion of Prof. Cayley, substituted for these cumbrous symbols the single letters a, b, c, d with numerical coefficients; viz., in the case of cubic curves,

$$U = a,$$
$$D_1U = 3b,$$
$$\tfrac{1}{2}D_1^2U = 3c,$$
$$U_1 = d;$$

so that a is the given cubic function of xyz, b a quadric, c a linear function, and d a constant. The cubic in λ, μ will now take the familiar form

$$a\lambda^3 + 3b\lambda^2\mu + 3c\lambda\mu^2 + d\mu^3,$$

the discriminant of which is equal to

$$(ad-bc)^2 - 4(ac-b^2)(bd-c^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

or, after multiplication by d^2,

$$(ad^2-3bcd+2c^3)^2 + 4(bd-c^2)^3 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The geometrical interpretation of this discriminant has, as far as I am aware, only been noticed hitherto when it is arranged in the form

$$(ad^2-6bcd+4c^3)\,a + (4bd-3c^2)\,b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

3. From (1) it follows that the equation to the six tangents from (x, y, z) to the cubic U or a, may be thrown into the form

$$u_3^2 - 4u_4u_2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

the suffixes indicating the order of the functions in xyz; and where

$$u_4 = ac-b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$
$$u_3 = ad-bc \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$
$$u_2 = bd-c^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

Operating with $x_1\frac{d}{dx} + y_1\frac{d}{dy} + z_1\frac{d}{dz}$, or D_1,

$$D_1a = D_1U = 3b,$$
$$D_1b = \tfrac{1}{3}D_1^2U = 2c,$$
$$D_1c = \tfrac{1}{6}D_1^3U = d = U_1;$$

consequently $D_1 u_4 = aD_1 c + cD_1 a - 2bD_1 b$
$= ad + 3bc - 4bc = ad - bc = u_3.$

Again $D_1 u_3 = dD_1 a - bD_1 c - cD_1 b$
$= 3bd - bd - 2c^2 = 2(bd - c^2) = 2u_2.$

Lastly $D_1 u_2 = dD_1 b - 2cD_1 c = 2cd - 2cd = 0.$

If $a, b, c, (D_1 U), \ldots$ represent the results of substituting x_1, y_1, z_1 for x, y, z respectively in $a \ldots D_1 U \ldots$, then, as is well known,

$$a_1 = U_1 = d,$$
$$b_1 = \tfrac{1}{3}(D_1 U)_1 = U_1 = d,$$
$$c_1 = \tfrac{1}{6}(D_1^2 U)_1 = U_1 = d,$$

from which it at once appears that each of the three curves $u_4 = 0$, $u_3 = 0$, $u_2 = 0$ passes through the point $x_1 y_1 z_1$ from which the tangents are drawn.

Since $(x_1 y_1 z_1)$ is a point on u_2, and $D_1 u_2$ vanishes identically, u_2 represents a pair of right lines drawn through that point; and since $D_1 u_4 = u_3$, $D_1 u_3 = (2) u_2$, $(x_1 y_1 z_1)$ is a double point on u_4 and its first polar u_3, while u_2 is the pair of tangents to each of those curves at the double point.

4. The curves u_4, u_3 having a common double point and common tangents at that point, will meet again in six points, and the form of (4) shows that *at these six points the six tangents to* U *drawn from* $(x_1 y_1 z_1)$ *touch the quartic* u_4.

Further, the form of u_4 (5) shows that it touches U in the six points in which the latter is met by its first polar b; and, therefore, that u_4, U have the same six points of contact with their common tangents. Otherwise it appears from (6) that the nine points of intersection of U and u_3 lie on the conic b, and the line c.

The conic b meets u_4 in two other points besides those referred to above. Equation (5) shows that the line c is a double tangent to u_4 at those points; while (7) shows that at the same two points u_4 is met again by the two tangents at its double point $(x_1 y_1 z_1)$, which are also tangents to the conic b.

When the point $(x_1 y_1 z_1)$ is on U, this becomes a point in which two branches of u_4 touch, c being the tangent at that point both to U and u_4, which touch in four other points lying on the conic b.

(5.) The form (2) of the discriminant gives the equation to the six tangents from $(x_1 y_1 z_1)$ to U in the shape

$$v_3^2 + 4u_2^3 = 0,$$

where $v_3 = ad^2 - 3bcd + 2c^3,$

and, as before, $u_2 = bd - c^2;$

provided the point $x_1 y_1 z_1$ be not taken on the curve U.

Of course $(x_1 y_1 z_1)$ is a point on v_3. Moreover,

$$D_1 v_3 = d^2 D_1 a - 3bd D_1 c - 3cd D\, b + bc^2 D_1 c;$$

but $D_1 a = 3b$, $D_1 b = 2c$, $D_1 c = d$, whence

$$D_1 v_3 \equiv 0,$$

so that v_3 represents three right lines through the point $(x_1 y_1 z_1)$ meeting U on the conic $3bd-2c^2$ and the line c. Being drawn from a double point on u_3, u_4, they meet the former curve again in three points only,—viz., those common to u_3, U, and c,—a fact verified by the identity

$$du_3 \equiv v_3 - 2cu_2\,;$$

while they meet u_4 in six points, other than $x_1 y_1 z_1$.

Since $$cv_3 \equiv d^2(ac-b^2)+(bd-c^2)(bd-2c^2),$$

i.e., $$\equiv d^2 u_4 + u_2(bd-2c^2),$$

the six points just mentioned lie on the conic $bd-2c^2$, which has c as its chord of double contact with b.

6. In the next place let U represent a proper plane quartic curve; then the equation to the twelve tangents from $(x_1 y_1 z_1)$ will be obtained by equating with zero the discriminant of

$$(a, b, c, d, e \between \lambda, \mu)^4,$$

where

$$a = \mathrm{U},$$
$$4b = \mathrm{D}_1\mathrm{U},$$
$$6c = \tfrac{1}{2}\mathrm{D}_1^2\,\mathrm{U},$$
$$4d = \tfrac{1}{6}\mathrm{D}_1^3\,\mathrm{U},$$
$$e = \mathrm{U}_1\,;$$

D_1, as before, standing for the operator $x_1 \frac{d}{dx} + y_1 \frac{d}{dy} + z_1 \frac{d}{dz}$, and U_1 for the result of substituting x_1 for x, y_1 for y, and z_1 for z, in U. Of course b will be a cubic, c a quadratic, and d a linear function of xyz, when a or U is a quartic.

If the standard form of the discriminant of $(a, b, c, d, e \between \lambda, \mu)^4$, viz. $I^3 - 27J^2$,

where $$\mathrm{I} = ae - 4bd + 3c^2,$$
$$\mathrm{J} = ace + 2bcd - ad^2 - b^2e - c^3,$$

be transformed into $\mathrm{IK} - 3\mathrm{J}'^2$, by writing

$$\mathrm{J}' = 3\mathrm{J} + c\mathrm{I} = 4ace + 2bcd - 3ad^2 - 3b^2e,$$
$$\mathrm{K} = \mathrm{I}_8^2 + 3c(6\mathrm{J} + c\mathrm{I}),$$

then the equation to the twelve tangents from $(x_1 y_1 z_1)$ to U may be exhibited under either of the forms

$$u_4 u_8 - 3u_6^2 = 0 \quad \ldots\ldots (8),$$

or $$u_4^3 - 27v_6^2 = 0 \quad \ldots\ldots (9),$$

where $$u_4 = ae - 4bd + 3c^2 \quad \ldots\ldots (10),$$
$$v_6 = dce + 2bcd - ad^2 - b^2e - c^3 \quad \ldots\ldots (11),$$
$$u_6 = 3v_6 + cu_4 \quad \ldots\ldots (12),$$
$$u_8' = u_4^2 + 3c(6v_6 + cu_4) \quad \ldots\ldots (13),$$

the suffixes indicating the order of the functions in x, y, z.

7. If a_1, b_1, c_1, d_1 indicate the results of the substitution of x_1 for x, y_1 for y, and z_1 for z, in $a \dots d$ respectively; then, since

$$a = \mathrm{U}, \qquad a_1 = \mathrm{U}_1 = e;$$

since $$4b = \mathrm{D}_1\mathrm{U}, \qquad b_1 = \mathrm{U}_1 = e;$$

since $$6c = \tfrac{1}{2}\mathrm{D}_1^2\,\mathrm{U}, \quad c_1 = \mathrm{U}_1 = e;$$

and since $$4d_1 = \tfrac{1}{6}\mathrm{D}_1^3\,\mathrm{U}, \quad d_1 = \mathrm{U}_1 = e.$$

It is easily verified, therefore, that u_4, v_6 each pass through $(x_1 y_1 z_1)$.

Again,
$$\mathrm{D}_1 a = \mathrm{D}_1\,\mathrm{U} = 4b \quad\text{.......................}\quad (14),$$
$$\mathrm{D}_1 b = \tfrac{1}{4}\mathrm{D}_1^2\,\mathrm{U} = 3c \quad\text{.......................}\quad (15),$$
$$\mathrm{D}_1 c = \tfrac{1}{12}\mathrm{D}_1^3\,\mathrm{U} = 2d \quad\text{.......................}\quad (16),$$
$$\mathrm{D}_1 d = \tfrac{1}{24}\mathrm{D}_1^4\,\mathrm{U} = \mathrm{U}_1 = e \quad\text{.................}\quad (17).$$

From these relations it is readily verified that
$$\mathrm{D}_1 u_4 \equiv 0 \quad\text{..............................}\quad (18),$$
$$\mathrm{D}_1 v_6 \equiv 0 \quad\text{..............................}\quad (19),$$
consequently u_4, v_6 represent systems of four and six right lines respectively drawn through $(x_1 y_1 z_1)$.

From (12), (19), (16), it follows at once that
$$\mathrm{D}_1\, u_6 = 2du_4;$$
and from (17), (18), $$\mathrm{D}_1^2\, u_6 = 2eu_4,$$
$$\mathrm{D}_1^3\, u_6 \equiv 0;$$
consequently $(x_1 y_1 z_1)$ is a quadruple point on u_6, and u_4 is the system of four tangents to u_6 at this point. Further the form of u_6, viz. $3v_6 + cu_4$, shows that the four lines u_4 meet u_6 only on points common to u_4 and v_6, *i.e.*, only at $(x_1 y_1 z_1)$, which is therefore a point of inflexion on each of the four branches of u_6 meeting at that point.

Also, (13), (16), (18), (19),
$$\mathrm{D}_1 u_8 = 6d\,(6v_6 + cu_4) + bcdu_4$$
$$= 12d\,(3v_6 + cu_4) = 12du_6 \quad\text{................}\quad (20);$$
consequently u_8 has its first polar with respect to $(x_1 y_1 z_1)$, through which it passes, the sextic u_6 and the line d. The point $(x_1 y_1 z_1)$ is therefore a quadruple inflexion point on u_8 also, and the four lines u_4 are the tangents to u_8 at that point, each touching one of its four branches in three consecutive points. The form of u_8, viz.
$$u_8 = u_4^2 + 3c\,(6v_6 + cu_4),$$
shows that the eight points, other than $(x_1 y_1 z_1)$, in which its four tangents u_4 meet it, lie on the conic c, which has octuple contact with u_8 at these points.

8. The curves u_6, u_8 having $(x_1 y_1 z_1)$ as a common quadruple-inflexion point, and common tangents at that point, will meet again in only twenty-four points, and the equation (8),
$$u_4\, u_8 - 3u_6^2 = 0,$$

one of the forms of the equation of the twelve tangents drawn from $(x_1 y_1 z_1)$ to U, shows that *at those points the same twelve lines are double tangents to* u_8.

It may be further shown that these twelve double tangents to u_8 touch this curve each once at its point of contact with U; or, in other words, that u_8 touches U at the twelve points of contact with the latter curve of their common tangents, the other eight points of intersection of u_8 and U lying on a conic.

For all values of the variables which satisfy $U = 0$, *i.e.* $a = 0$, see (10), (11),

$$u_4 = 3c^2 - 4bd,$$
$$v_6 = 2bcd - b^2e - c^3;$$

whence, and from (12), $u_6 = (2cd - be)\, b$;

consequently the intersections of U, or a, and its first polar b lie on u_6.
Again, generally, (13), $u_8 = u_4(u_4 + 3c^2) + 18cv_6$;
so that for all values of x, y, z, which satisfy U or $a = 0$, substituting the values of u_4, v_6 just above,

$$u_8 = 2\{(3c^2 - 4bd)(3c^2 - 2bd) + 18bc^2d - 9b^2ce - 9c^4\}$$
$$= 2(8d^2 - 9ce)\, b^2;$$

consequently, at the twelve points in which U meets its first polar b, it touches u_8; the remaining eight points of intersection of U with u_8 lying on the conic $8d^2 - 9ce$.

9. The four tangents u_4 to u_8 at the quadruple-inflexion point $(x_1 y_1 z_1)$ count as twenty-four of the fifty-six tangents which can, in general, be drawn from a point to u_8. Of the other thirty-two, twenty-four, viz. the twelve double tangents, have been accounted for. In order to find the remaining eight, I have thought it worth while to investigate independently the equation to the thirty-six tangents which can be drawn from $(x_1 y_1 z_1)$ considered as a quadruple point only, as may be done without difficulty. It is necessary to form the discriminant of the quartic

$$u_8\lambda^4 + D_1 u_8 \lambda^3\mu + \tfrac{1}{2}D_1^2 u_8 \lambda^2\mu^2 + \tfrac{1}{6}D_1^3 u_8 \lambda\mu^3 + \tfrac{1}{24}D_1^4 u_8 \mu^4 \;\ldots\ldots\; (21).$$

Now $\qquad u_8 = u_4^2 + 3c(6v_6 + cu_4)$,

and, (20), $\qquad D_1 u_8 = 12d(3v_6 + cu_4)$;

whence, (16)—(19), $\tfrac{1}{2}D_1^2 u_8 = 6\{e(3v_6 + cu_4) + 2d^2u_4\}$,

$$\tfrac{1}{4}D_1^3 u_8 = 12deu_4,$$
$$\tfrac{1}{24}D_1^4 u_8 = 3e^2u_4.$$

Comparing the quartic (21) in λ, μ with $(A, B, C, D, E \between \lambda, \mu)^4$,

$$A = u^2 + 3c(6v + cu),$$
$$B = 3d(3v + cu),$$
$$C = e(3v + cu) + 2d^2u,$$
$$D = 3deu,$$
$$E = 3e^2u,$$

the suffixes to v_6 and u_4 being dropped for shortness. Hence

$$AE-4BD+3C^2 = 3\{e^2(u^3-27v^2)+4(ceu-d^2u+3ev)^2\},$$

and
$$ACE+2BCD-AD^2-B^2E-C^3 =$$
$$\{3e^2(u^3-27v^2)+8(ceu-d^2u+3ev)^2\}(ceu-d^2u+3ev).$$

10. If for shortness we write

$$l = e^2(u_4^3-27v_6^2),$$
$$m = (ce-d^2)u_4+3ev_6,$$

the equation to the thirty-six tangents from $(x_1 y_1 z_1)$ to u_8 will therefore be
$$(l+4m^2)^3-(3l+8m^2)^2m^2 = 0,$$
which reduces identically to $l^2(l+3m^2) = 0$.
Restoring their values for l, m, this equation becomes identically

$$u_4(u_4^3-27v_6^2)^2[e^2\{u_4^2+3c(6v_6+cu_4)\}-3d^2\{2e(3v_6+cu_4)-d^2u_4\}] = 0;$$

i.e., (13), (12), $u_4(u_4^3-27v_6^2)^2\{e^2u_8-3d^2(2eu_6-d^2u_4)\} = 0$ ...... (22).

The first factor enters in consequence of the four lines u_4 meeting the curve each in two points consecutive, on each branch, to the quadruple point. The second factor is the representative of the twelve double tangents; viz., the tangents to U from $(x_1 y_1 z_1)$: lastly, the third factor represents the remaining eight tangents, touching u_8 at the eight points in which it is met by the line d, which appeared before as a factor in its first polar (20). These last eight tangents being drawn from a quadruple point will intersect u_8 again each in two points only lying on the sextic
$$2eu_6-d^2u_4 = w_6 = 0.$$
This sextic w_6 has—as appears from its form—sextuple contact with u_8 at the six points in which they are both met by the line d. In fact w_6 has $(x_1 y_1 z_1)$ also as a quadruple-inflexion point, u_4 being also the four tangents to it at that point. Consequently w_6 meets u_8 in twenty-four points other than $(x_1 y_1 z_1)$; viz., in the eight common to u_8, u_4 and the conic c, besides the sixteen above alluded to. This appears from the identity
$$eu_8-3cw_6 \equiv u_4\{eu_4-3c(ce-d^2)\}.$$

11. It appears from (13) that the six lines v_6 meet u_8 in twenty-four points, other than $(x_1 y_1 z_1)$, lying on the quartic
$$u_4+3c^2 = 0,$$
the other eight points common to this quartic, and u_8 being the eight common to the four lines u_4 and the conic c, which in fact are the eight points of contact with the quartic of its four double tangents u_4.

12. When the point $(x_1 y_1 z_1)$ falls on the curve U, U_1 or $e = 0$, and

$$u_4 = -4bd+3c^2,$$
$$v_6 = 2bcd-ad^2-c^3,$$
$$u_6 = (2bc-3ad)d,$$
$$u_8 = 2(8b^2-9ac)d^2.$$

In this case, then, u_8 breaks up into the square of the line d—the tangent to U at the point $(x_1 y_1 z_1)$—and the sextic

$$v_6 = 8b^2 - 9ac,$$

while, correspondingly, u_6 breaks up into the line d and the quintic

$$v_5 = 2bc - 3ad,$$

which may be verified to be the first polar of v_6. For since

$$D_1 a = 4b, \quad D_1 b = 3c, \quad D_1 c = 2d, \quad D_1 d = e,$$

$$D_1 v_6 = 16bD_1 b - 9aD_1 c - 9cD_1 a$$

$$= 48bc - 18ab - 36bc = 6(2bc - 3ad) = 6v_5.$$

Again, $$D_1 v_5 = 2bD_1 c + 2cD_1 b - 3aD_1 d - 3dD_1 a$$

$$= 4bd + bc^2 - 12bd = 2(3c^2 - 4bd) = 2u_4;$$

so that v_6, v_5 have $(x_1 y_1 z_1)$ as a common quadruple point and four common tangents at that point; consequently they meet again in ten points only. The ten tangents which can be drawn from $(x_1 y_1 z_1)$ to U or a touch v_6 at the same points, as is evident from the form of v_6; and being drawn from a quadruple point do not meet that curve again.

Prof. Cayley made a few remarks on Mr. Walker's paper, in the course of which he suggested the alteration subsequently adopted by the Author.

Mr. S. Roberts then read a paper

On the Order and Singularities of the Parallel of an Algebraical Curve.

1. A Parallel of a curve is variously defined as the envelope of a circle of constant radius which touches the given curve, or has its centre thereon; and, as the locus of the centre of a circle of constant radius which touches the given curve.

The most obvious property of a Parallel is, that among the normal distances of any point on it from the primitive curve, there is always one of given length, which I call the Modulus of the Parallel. Moreover, if at any point on the primitive we draw a normal, two points will be determined on the parallel by taking upon the normal lengths equal to the Modulus in opposite directions from the point; and the normals to the Parallel at these points coincide with the corresponding normal of the Primitive.

The parallel, therefore, has the same normals as the primitive, but each is normal at two points. The latter curve is a parallel to itself to the modulus 0, but we shall have reason to remark that as the normal distances which determine points on the parallel are measured in two directions, on opposite sides of the corresponding branch of the primitive, this, as parallel to itself, must be reckoned twice.

Since the primitive is included in the family of parallels, they may