For the case of two spheres, which has already been treated from the point of view of calculation in a great variety of ways, the necessary formulæ for representing explicitly the result just obtained may be conveniently written as follows.

Taking $$x^2+y^2+z^2+2x\coth\alpha+1=0,$$
$$x^2+y^2+z^2-2x\coth\beta+1=0$$

as the equations to the two spheres, the system of points proceeding from $(x, y, z)$ by an even number of inversions all lie in a plane through the axis of $x$, and, when $\rho$ is used for $\sqrt{y^2+z^2}$, are given by

$$x_n=\frac{(x^2+\rho^2+1)\sinh n\theta\cosh n\theta+x\cosh 2n\theta}{\sinh^2 n\theta\left(\{x+\coth n\theta\}^2+\rho^2\right)},$$

$$\rho_n=\frac{\rho}{\sinh^2 n\theta\left(\{x+\coth n\theta\}^2+\rho^2\right)},$$

$$\mu_n=\frac{\rho_n}{\rho},$$

where $$\theta=\alpha+\beta.$$

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*Waves in Canals*. *By* H. M. Macdonald. Read January 11th, 1894. Received February 9th, 1894.

It has been usual to assume that the velocity potential of the fluid motion which consists of a train of progressive waves propagated along a canal of uniform cross-section can be represented by an expression of the form $f(y, z)\cos(mx-nt)$, the notation being the same as in Basset's *Hydrodynamics*, Vol. II., Art. 392. The wave motion which has a velocity potential of this form must be such that the crests of the waves are always in planes perpendicular to the length of the canal, the particles of fluid describing ellipses whose planes are perpendicular to the cross-section. In what follows it is proposed to investigate in what cases it is possible to propagate a train of such waves of any given wave length along a canal whose sides are planes equally inclined to the vertical.

1. Taking the origin at the lowest point of any cross-section of the canal, the coordinate planes being as above, $\phi$ the velocity potential is assumed to be of the form

$$\phi = f(y, z) \cos(mx - nt) \ldots\ldots (1);$$

then

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - m^2 f = 0 \ldots\ldots (2)$$

throughout the fluid,

$$l \frac{\partial f}{\partial z} - f = 0 \ldots\ldots (3)$$

at the free surface, where $z = h$ the depth of the canal, and $l = g/n^2$.

The solution of (2) may be written

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right) f_0 + \frac{\sin\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)}{\sqrt{\frac{\partial^2}{\partial z^2} - m^2}} f_0',$$

where $f_0$ is the value of $f$, and $f_0'$ of $\frac{\partial f}{\partial y}$, when $y = 0$.

It is clear that the two parts of this solution must satisfy (3) independently, (3) being true for all values of $y$, when $z = h$, and therefore correspond to different systems of waves. The first part $\cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right) f_0$ corresponds to waves produced by disturbances symmetrical with respect to the plane of symmetry of the canal, and such that the particles of fluid originally in that plane always remain in it. The other part $\frac{\sin\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)}{\sqrt{\frac{\partial^2}{\partial z^2} - m^2}} f_0'$ corresponds to waves produced by certain asymmetrical disturbances which are such that the particles of fluid originally in the plane of symmetry of the canal oscillate in straight lines perpendicular to it. The waves discussed by Green, *Cambridge Phil. Trans.*, 1838, belong to the first class, and for these only will the expression $\sqrt{\frac{gA}{b}}$ for the velocity of propagation of long waves there found be true; the velocity of propagation of long waves of the second class would be indefinitely great, that is, very long waves of this type could not be generated.

2. When the canal is formed by two planes equally inclined to the vertical plane, the conditions to be satisfied at the fixed boundary are

$$\frac{\partial f}{\partial y}\cos\alpha \pm \frac{\partial f}{\partial y}\sin\alpha = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

when $$y\cos\alpha \pm z\sin\alpha = 0,$$

$2\alpha$ being the angle which the sides of the canal make with one another. Considering waves of the first class

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right) f_0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

putting $$z = r\cos\theta,\quad y = r\sin\theta,$$

$f$ satisfies $$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} - m^2 f = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)',$$

subject to $$\left.\begin{aligned}\frac{\partial f}{\partial\theta} &= 0\\ \theta &= \pm\alpha\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)'.$$
when

Therefore $$f = \Sigma A_\mu J_\mu(m\iota z)\cos\mu\theta,$$

where $$\mu = s\pi/\alpha,$$

$s$ being a positive integer; hence

$$f_0 = \Sigma A_\mu J_\mu(m\iota z),$$

and $$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\Sigma A_\mu J_\mu(m\iota z).$$

Let $\pi/\alpha = 2k$ an even integer, then

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\int_0^\pi \cosh(mz\sin\chi)\sum_0^\infty B_s\cos 2ks\chi\, d\chi;$$

writing

$$F(\chi) \equiv \{ml\sin\chi\sinh(mh\sin\chi) - \cosh(mh\sin\chi)\}\sum_0^\infty B_s\cos 2ks\chi,$$

to satisfy (3), it is necessary that

$$\int_0^\pi \left(l\frac{\partial}{\partial z} - 1\right)\cosh(mz\sin\chi)\cosh(my\cos\chi)\sum_0^\infty B_s\cos 2ks\chi\, d\chi$$

should vanish for all values of $y$, when $z = h$, that is, observing that

$$\left(l\frac{\partial}{\partial z} - 1\right)\cosh(mz\sin\chi) = ml\sin\chi\sinh(mh\sin\chi) - \cosh(mh\sin\chi),$$

when $z = h$,
$$\int_0^\pi F(\chi)\,d\chi = 0,$$
$$\int_0^\pi F(\chi)\cos^2\chi\,d\chi = 0,\quad \int_0^\pi F(\chi)\cos^4\chi\,d\chi = 0, \text{ \&c.};$$
or
$$\int_0^\pi F(\chi)\,d\chi = 0,\quad \int_0^\pi F(\chi)\cos 2\chi\,d\chi = 0,$$
$$\int_0^\pi F(\chi)\cos 4\chi\,d\chi = 0, \text{ \&c.}$$

Now $F(\chi)$ can be expressed as a series involving cosines of even multiples of $\chi$ only; therefore $F(\chi)\equiv 0$ for all values of $\chi$ between 0 and $\pi$. That this may be true, and all values of $m$ (*i.e.*, every wave length) be possible, $\sum_0^\infty B_s \cos 2ks\chi$ must vanish for all values of $\chi$ between 0 and $\pi$, except such as make
$$ml\sin\chi\sinh(mh\sin\chi)-\cosh(mh\sin\chi)$$
vanish, $m$ and $l$ remaining the same; if $\chi_1$ is such a critical value of $\chi$, the only other possible one is $\pi-\chi_1$, for $\sin\chi$ must be the same for all such values. Let
$$\sum_0^\infty B_s\cos 2ks\chi = G(\chi),$$
where $G(\chi)$ vanishes, except where $\chi=\chi_1$ or $\pi-\chi_1$, then
$$\frac{\pi B_s}{2} = \int_0^\pi G(\chi)\cos 2ks\chi\,d\chi = \cos 2ks\chi_1 \int_0^\pi G(\chi)\,d\chi;$$
therefore
$$G(\chi) = C\,(1+2\cos 2k\chi_1\cos 2k\chi + \dots + 2\cos 2ks\chi_1\cos 2ks\chi + \dots),$$
$$2G(\chi) = \mathrm{Lt}_{\zeta=1}\, C\left\{\frac{1-\zeta^2}{1-2\zeta\cos 2k(\chi+\chi_1)+\zeta^2} + \frac{1-\zeta^2}{1-2\zeta\cos 2k(\chi-\chi_1)+\zeta^2}\right\}.$$

Substituting this expression for $\sum_0^\infty B_s\cos 2ks\chi$ in the expression for $f$, and performing the integration, it becomes
$$f = C'\cos\left(y\sqrt{\frac{\partial^2}{\partial z^2}-m^2}\right)\cosh(mz\sin\chi_1);$$
now,
$$\cosh(mz\sin\chi_1) = J_0(miz) + 2J_2(miz)\cos 2\chi_1 + 2J_4(miz)\cos 4\chi_1 + \text{\&c.};$$
and therefore $\chi_1$ must be such that $\cos 2p\chi_1$ vanishes, $p$ being an

integer, except when $p = 2ks$; this condition can only be satisfied when $k = 1$ or $k = 2$, and then $2k\chi_1 = \pi$.

Of these two cases $k = 1$ or $\pi/a = 2$ does not belong to a canal, and the only cases for which all wave lengths are possible is $k = 2$ or $\pi/a = 4$. In this case

$$f = C' \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right) \cosh\left(\frac{mz}{\sqrt{2}}\right),$$

that is, $$\phi = C' \cosh\frac{my}{\sqrt{2}} \cosh\frac{mz}{\sqrt{2}} \cos(mx - nt).$$

The wave velocity of propagation is found from the equation

$$\frac{ml}{\sqrt{2}} \sinh\left(\frac{mh}{\sqrt{2}}\right) - \cosh\left(\frac{mh}{\sqrt{2}}\right) = 0,$$

whence $$V^2 = \frac{g\lambda}{2\pi\sqrt{2}} \tan\left(\frac{2\pi h}{\lambda\sqrt{2}}\right),*$$

where $\lambda$ is the wave length.

This result was obtained by Kelland, *Edin. Trans.*, Vol. xv., but it should be mentioned that the analysis from which it was obtained as a particular case is faulty.

A velocity potential can be found for $\pi/a$ any even integer greater than four by giving a sufficient number of proper critical values to the series $\sum_0^\infty B_s \cos 2ks\chi$ to satisfy the conditions at the fixed boundaries, and this leads to conditions which $m$ must satisfy.

If $\pi/a = 6$,

$$\phi = C'\left(\cosh mz + 2\cosh\frac{mz}{2}\cosh\frac{my\sqrt{3}}{2}\right)\cos(mx - nt),$$

where $m$ must satisfy

$$m \coth mh = 2m \coth\frac{mh}{2}.$$

If $\pi/a = 8$,

$$\phi = C'\left\{\cosh\left(mz\sin\frac{\pi}{8}\right)\cosh\left(my\cos\frac{\pi}{8}\right) + \cosh\left(mz\sin\frac{3\pi}{8}\right)\cosh\left(my\cos\frac{3\pi}{8}\right)\right\}\cos(mx - nt),$$

* This expression for $V^2$ differs by a numerical factor from that given by Greenhill, *Amer. Jour. of Math.*, Vol. IX., and by Basset, *Hydrodynamics*, Vol. II., Art. 392; but it will be observed that in their notation $n^2 = g/2l$, not $g/l$.

where $m$ must satisfy

$$m \coth\left(mh \sin\frac{\pi}{8}\right)\Big/\sin\frac{\pi}{8} = m\coth\left(mh\sin\frac{3\pi}{8}\right)\Big/\sin\frac{3\pi}{8}.$$

If $\pi/a = 10$,

$$\phi = C'\left\{\cosh mz + 2\cosh\left(mz\sin\frac{\pi}{10}\right)\cosh\left(my\cos\frac{\pi}{10}\right)\right.$$
$$\left. + 2\cosh\left(mz\sin\frac{3\pi}{10}\right)\cosh\left(my\cos\frac{3\pi}{10}\right)\right\}\cos(mx-nt),$$

where $m$ must satisfy

$$m\coth mh = m\coth\left(mh\sin\frac{3\pi}{10}\right)\Big/\sin\frac{3\pi}{10} = m\coth\left(mh\sin\frac{\pi}{10}\right)\Big/\sin\frac{\pi}{10},$$

and so on for any value of $\pi/a$ an even integer.

It will be observed that the values of $m$ which satisfy these conditions are complex quantities; hence when $\pi/a$ is an even integer greater than four a train of waves with their crests in planes perpendicular to the length of the canal is impossible for any wave length. It also follows that a motion whose velocity potential is of the form $f(yz)\,e^{-p(x-ct)}$ is impossible.

When $\pi/a = 4$, and $\lambda$ is very great, $V^2 = gh/2$, agreeing with Green's result.

3. Let $\pi/a = 2k+1$ an odd integer, then

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\int_0^\pi\left\{\cosh(mz\sin\chi)\sum_0^\infty B_s\cos(2k+1)2s\chi\right.$$
$$\left. + \sinh(mz\sin\chi)\sum_0^\infty B'_{2s+1}\sin(2s+1)(2k+1)\chi\right\}d\chi;$$

writing

$$F(\chi) \equiv \{ml\sin\chi\sinh(mh\sin\chi) - \cosh(mh\sin\chi)\}\sum_0^\infty B_s\cos 2s(2k+1)\chi$$
$$+\{ml\sin\chi\cosh(mh\sin\chi) - \sinh(mh\sin\chi)\}\sum_0^\infty B'_{2s+1}\sin(2s+1)(2k+1)\chi,$$

that (3) may be satisfied it is necessary that

$$\int_0^\pi F(\chi)\,d\chi = 0, \quad \int_0^\pi F(\chi)\cos 2\chi\,d\chi = 0, \quad \&c.;$$

hence, as in the foregoing, $F(\chi)$ vanishes for all values of $\chi$ between 0 and $\pi$.

Therefore, that every wave length should be possible,

$$\sum_0^\infty B_s\cos 2s(2k+1)\chi \quad\text{and}\quad \sum_0^\infty B'_{2s+1}\sin(2s+1)(2k+1)\chi$$

must vanish for all values of $\chi$, except such as make

$$L\{ml\sin\chi\sinh(mh\sin\chi)-\cosh(mh\sin\chi)\}$$
$$+L'\{ml\sin\chi\cosh(mh\sin\chi)-\sinh(mh\sin\chi)\}$$

vanish. Let $\chi_1$, $\chi_2$, &c., be these critical values, then, remembering that

$$\cosh(mz\sin\chi)=J_0(m\iota z)+2J_2(m\iota z)\cos 2\chi+\&c.,$$
$$\sinh(mz\sin\chi)=2J_1(m\iota z)\sin\chi+2J_3(m\iota z)\sin 3\chi+\&c.,$$

the conditions to be satisfied at the fixed boundaries require that

$$L_1\cos 2\chi_1+L_2\cos 2\chi_2+\&c.=0,$$
$$L_1\cos 2(2k+2)\chi_1+L_2\cos 2(2k+2)\chi_2+\&c.=0,\ \&c.,$$
$$L_1\cos 4\chi_1+L_2\cos 4\chi_2+\&c.=0,\ \&c.;$$

from these it follows that

$$e^{2(2k+1)\chi_1\iota}=e^{2(2k+1)\chi_2\iota}=\&c.;$$

also $$L_1'\sin\chi_1+L_2'\sin\chi_2+\&c.=0,\ \&c.,$$

whence $$e^{(2k+1)\chi_1\iota}=e^{(2k+1)\chi_2\iota}=\&c.$$

Hence, if $\chi_1$ is a critical value, there are $2k$ other critical values given by

$$\chi_1+\frac{2\pi}{2k+1},\quad \chi_1+\frac{4\pi}{2k+1},\ \dots\ \chi_1+\frac{4k\pi}{2k+1};$$

substituting these in the sets

$$L_1\cos 2\chi_1+L_2\cos 2\chi_2+\dots=0,$$
$$L_1\cos 4\chi_1+L_2\cos 4\chi_2+\dots=0,$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$L_1\cos 4k\chi_1+L_2\cos 4k\chi_2+\dots=0,$$

it follows that $$L_1=L_2=\&c.,$$

so $$L_1'=L_2'=\&c.$$

Therefore

$$f=\cos\left(y\sqrt{\frac{\partial^2}{\partial z^2}-m^2}\right)\Big\{L\Big[\cosh(mz\sin\chi_1)$$
$$+\cosh\left\{mz\sin\left(\chi_1+\frac{2\pi}{2k+1}\right)\right\}+\&c.\Big]$$
$$+L'\left[\sinh(mz\sin\chi_1)+\&c.\right]\Big\}\quad\dots(A);$$

in order that all wave lengths may be possible, these critical values must be such that they give only two different values to $\sin\chi$, as there is only one constant, viz., $L'/L$, at our disposal; this will be so when $k = 1$, *i.e.*, $\pi/a = 3$, and only then, and $\chi_1$ is then determined by

$$\pi - \chi_1 = \chi_1 + 2\pi/3,$$

whence $$\chi_1 = \frac{\pi}{6}.$$

Therefore the only case for which all wave lengths are possible, $\pi/a$ being an odd integer, is $\pi/a = 3$, when

$$f = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\left[L\left\{\cosh\left(mz\sin\frac{\pi}{6}\right) + \cosh\left(mz\sin\frac{5\pi}{6}\right) + \cosh\left(mz\sin\frac{3\pi}{2}\right)\right\} + L'\left\{\sinh\left(mz\sin\frac{\pi}{6}\right) + \sinh\left(mz\sin\frac{5\pi}{6}\right) + \sinh\left(mz\sin\frac{3\pi}{2}\right)\right\}\right],$$

which may be written

$$f = A\left[\cosh m(z+\beta) + 2\cosh m\left(\frac{z}{2} - \beta\right)\cosh\frac{my\sqrt{3}}{2}\right].$$

In this case $l$ is given by

$$ml = \coth m(h+\beta) = 2\coth m\left(\frac{h}{2} - \beta\right),$$

whence $$m^2l^2 - 3ml\coth\frac{3mh}{2} + 2 = 0.$$

Hence

$$\phi = A\left[ml\cosh m(z-h) + \sinh m(z-h) + 2\cosh\frac{my\sqrt{3}}{2}\left\{ml\cosh m\left(\frac{z}{2}+h\right) - \sinh m\left(\frac{z}{2}+h\right)\right\}\right]\cos(mx - nt),$$

the velocity of propagation being given by

$$g/V^2 = \frac{3m}{2}\coth\frac{3mh}{2}\left\{1 \pm \sqrt{1 - \frac{8}{9}\tanh^2\frac{3mh}{2}}\right\}.$$

The form of the velocity potential in this, the only possible case for

$\pi/\alpha$ an odd integer, is somewhat complicated, but it may be verified by the case of long waves when it leads to Green's result $V^2 = gh/2$. The other solution $V^2 = \infty$ means that no long wave corresponding to the lower sign in the above expression for the velocity of propagation would be generated.

If the upper sign is taken in the expression for $g/V^2$, $ml$ is always greater than 2, and the free surface of the wave is given by

$$\zeta = A\left\{1+2\sqrt{\frac{m^2l^2-1}{m^2l^2-4}}\cosh\left(\frac{my\sqrt{3}}{2}\right)\right\}\sin(mx-nt),$$

the cross-section of the wave at a crest being a catenary with its lowest point in the middle of the canal. If the lower sign is taken, $ml$ is always less than 1, and the free surface is given by

$$\zeta = A\left\{1-2\sqrt{\frac{1-m^2l^2}{4-m^2l^2}}\cosh\frac{my\sqrt{3}}{2}\right\}\sin(mx-nt),$$

the cross-section of the wave at a crest being a catenary with its highest point in the middle of the canal. When the wave length $\lambda$ is great, that form must be taken for which $ml > 2$, and when $\lambda$ is small that for which $ml < 1$.

The above expressions for the velocity potential and velocity of propagation in the case $\pi/\alpha = 3$ lead to Stokes' result for waves with their crests perpendicular to a beach sloping to the horizon at an angle of $\pi/6$. [In this case $ml > 2$.]

In the *Amer. Jour. of Math.*, Vol. IX., Greenhill has tried to obtain the solution of the above case by modifying the solution for standing waves across the same canal; the expression for the velocity potential so obtained can be got from the above equation (A) by putting

$$\chi_1 = \pi/12,$$

but it then contains one undetermined constant less than the number necessary to give a solution for every wave length. In the same paper the propagation of a bore along the canal is investigated, assuming

$$\phi = F.\cosh(mx-nt);$$

this expression seems objectionable on the ground that the displacement given by it could become large, and the theory is only applicable to waves where the displacement is always small.

The expression (A) gives a velocity potential for every case $\pi/\alpha$ an odd integer which satisfies the conditions at the fixed boundary, but the free surface condition in every case but $\pi/a = 3$ will lead to conditions which $m$ must satisfy; *e.g.*, if $\pi/a = 5$, the velocity potential is

$$\phi = \cos\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)\Big[L\left\{\cosh mz + 2\cosh\left(mz\sin\frac{\pi}{10}\right) + 2\cosh\left(mz\sin\frac{3\pi}{10}\right)\right\} + L'\left\{\sinh mz + 2\sinh\left(mz\sin\frac{\pi}{10}\right) - 2\sinh\left(mz\sin\frac{3\pi}{10}\right)\right\}\Big]\cos(mx - nt),$$

$\chi_1$ being chosen so that $m$ has to satisfy the least possible number of conditions; $m$ is given by the equation

$$\{\epsilon\epsilon'(\coth mh\epsilon + \coth mh\epsilon') + \epsilon\coth mh\epsilon - \epsilon'\coth mh\epsilon'\}(\epsilon\coth mh\epsilon - \epsilon'\coth mh\epsilon') = (\epsilon + \epsilon')^2,$$

where $\epsilon = -1 + \sin\frac{\pi}{10}, \quad \epsilon' = \sin\frac{3\pi}{10} + 1;$

the roots of this equation are complex quantities. Similarly, the other cases $\pi/a$ an odd integer may be investigated.

4. The solution
$$f = \frac{\sin\left(y\sqrt{\frac{\partial^2}{\partial z^2} - m^2}\right)}{\sqrt{\frac{\partial^2}{\partial z^2} - m^2}} f_0'$$

gives a wave motion which is possible for all wave lengths when $\pi/a = 4$, and the velocity potential then is

$$\phi = A\sinh\frac{my}{\sqrt{2}}\sinh\frac{mz}{\sqrt{2}}\cos(mx - nt);$$

the wave velocity of propagation is given by

$$V^2 = \frac{g\lambda}{2\pi\sqrt{2}}\coth\frac{2\pi h}{\lambda\sqrt{2}},$$

and, if $\lambda$ is small, $$V^2 = \frac{g\lambda}{2\pi\sqrt{2}}.$$

It can be shown, by an analysis similar to the preceding, that standing waves across a canal of triangular cross-section are only possible in the same two cases, the solutions for which were given by Kirchhoff, *Gesam. Abhand.*, Vol. II.

From the above investigation, it appears that, if a wave with a plane front is set up in a canal of triangular cross-section, it will be propagated without change of form in two cases only, viz., when the angle which the sides make with one another is either 90° or 120°; in all other cases the wave front will not remain plane for any great distance along the canal. It follows from what precedes that there is no angle which forms the limit between stability and instability, as stated in Basset's *Hydrodynamics*, Vol. II., Art. 394; a wave motion of some kind must be possible for any angle. Green's investigation above referred to requires that it should be possible to expand the velocity potential in powers of the $y$, $z$ coordinates, and that powers higher than the squares of these should be negligible; this will be true when the front of the wave is approximately plane, so that the results there arrived at would be true in the case of a wave whose front is initially plane for some distance along the canal.

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## *Thursday, February 8th,* 1894.

### A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Miss Edith Lees, Mr. F. W. Hill, M.A., City of London School, and Major Hippisley, R.E., were elected members of the Society. Miss Lees was admitted into the Society.

The President announced the death of Mr. William Racster, M.A., for many years a colleague, at Woolwich, of Prof. Sylvester. He was elected a member October 16th, 1865, and died December 30th, 1893. Also of Mr. William Paice, M.A., for twenty-two years an Assistant Master in University College School. He was a Life Governor of University College; a sub-Examiner in Mathematics, for five years, of the London University; an Assistant Examiner in Magnetism at South Kensington; and the author of a small work entitled *Energy and Motion.* On the death of the Rev. W. Stainton Moses, his colleague at the school, which event took place on the 5th September, 1892, he succeeded that gentleman as the editor of the

Spiritualist journal *Light.* He was elected a member April 11th, 1872, and died January 24th, 1894.

At the request of Lord Kelvin, P.R.S., and by the permission of the Council, Mr. J. J. Walker exhibited and described Lord Kelvin's model of his Tetrakaidekahedron.

This was a model (for the making of which Lord Kelvin acknowledges his obligation to Prof. Crum Brown, D.Sc., M.D., F.R.S., Prof. of Chemistry in the University of Edinburgh) of the form named "orthoidal" by the author; viz., it is a form derived by homogeneous strain from the "orthic," a surface bounded by eight regular hexagons and four squares, first described in the *Acta Mathematica*, Vol. XI., "The Division of Space with Minimum Partitional Area," a paper reproduced in the *Phil. Mag.* for 1887 (second half-year).

Lord Kelvin's own account of the surface will be found in Vol. LV., *Royal Society's Proceedings* (pp. 1–16).*

Votes of thanks were passed to Mr. Walker and to Lord Kelvin. A conversation ensued in which Messrs. S. Roberts, Forsyth, MacMahon, Cunningham, Elliott, and the President took part.

Abstracts were communicated of the following papers:—

On a Class of Groups defined by Congruences: Prof. W. Burnside.

Some Properties of the Uninodal Quartic and Quintic having a Triple Point: Mr. W. R. W. Roberts.

A cabinet likeness of Prof. Mathews was presented by him to the Album.

The following presents were made to the Library:—

Bodhanundānath Swami.—"Kalyāna Manjushā"; Calcutta, 1893.

"Nautical Almanac for 1897."

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VIII., No. 1; Manchester, 1893–4.

"Bulletin des Sciences Mathématiques," 2ème Série, Tome XVII., November and December, 1893; Paris.

"Proceedings of the Royal Society," Vol. LIV., No. 329.

"Report of the Superintendent of the U.S. Naval Observatory," to June, 1893; Washington.

"Journal of the Institute of Actuaries," No. 172, January, 1894.

"Papers read before the Mathematical and Physical Society of the University of Toronto during 1891–2," Toronto, 1892.

"Proceedings of the Physical Society of London," Vol. XII., Pt. 3; December, 1893.

"Bulletin of the New York Mathematical Society," Vol. III., No. 4.

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* *Cf.* also *Nature* for March 8th and 15th, pp. 445-8, 469-71.

"Bulletin de la Société Mathématique de France," Tome xxi., No. 9; Paris.

"Bulletin de la Société Mathématique de France," Table des 20 Premiers Volumes; Paris, 1894.

Gram, J. P.—"Essai sur la Restitution du Calcul de Léonard de Pise sur l'équation $x^3+2x^2+10x=20$," pamphlet.

Gram, J. P.—"Rapport sur quelques Calculs entrepris par M. Bertelsen et concernant les Nombres Premiers," 4to pamphlet.

Zeuthen, H. G.—"Note sur l'Histoire des Mathématiques," pamphlet.

Barrett, T. S.—"Magic Squares," second edition, 8vo; Berkhamsted, 1894.

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# *On a Class of Groups defined by Congruences.* By Prof. W. Burnside.

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## 1. *Introductory.*

Most of the groups of finite order which occur in connexion with problems of higher analysis can be defined by means of congruences. This is true, for example, of the group of the modular equation, and of the groups on which the division of the periods of the hyperelliptic functions depends. In his standard treatise (*Traité des Substitutions et des Equations Algébriques*) M. Camille Jordan has investigated at length the more important properties of the general linear group, defined by sets of congruences of the form

$$\left.\begin{array}{l} x_1' \equiv a_1x_1+b_1x_2+\ldots+c_1x_n \\ x_2' \equiv a_2x_1+b_2x_2+\ldots+c_2x_n \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n' \equiv a_nx_1+b_nx_2+\ldots+c_nx_n \end{array}\right\} \pmod{p},$$

where the coefficients are ordinary integers.