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LIII. *On the Lateral Deflexion and Vibration of "Clamped-Directed" Bars.* By JOHN MORROW, M.Sc. (Vict.), D. Eng. (L'pool), Lecturer in Engineering, Armstrong College (University of Durham) *.

Section I.—*Introduction and Contents.*

§ 1. THE vibrations of a bar under the terminal conditions which are probably most frequent in engineering practice, appear to have hitherto attracted but little attention. These conditions occur when one end of the bar is clamped, and the other is constrained to retain its original direction. Such conditions may be realized by having two initially parallel bars, CA and DB (fig. 1), each clamped at one end, with the otherwise free ends connected by a rigid bar AB.

Figs. 1 and 2 show two distinct ways in which such a

Fig. 1.

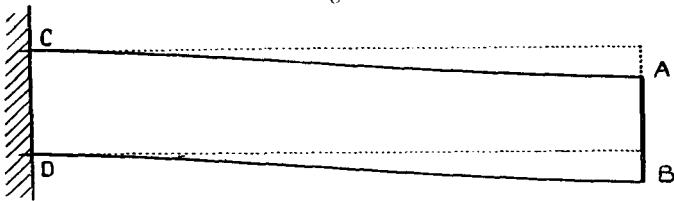
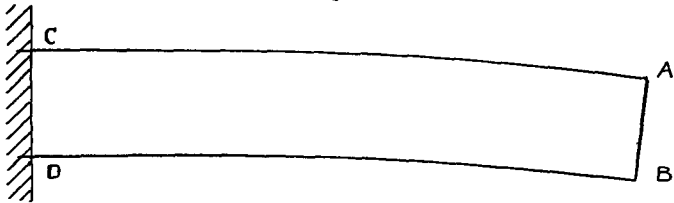


Fig. 2.



system may vibrate. In the former case we have pure lateral vibration, whereas in the latter there is a necessary accompaniment of longitudinal vibrations in each bar. The former case is the more important since in it the frequency of the fundamental type will, in general, be the lower.

I propose to refer to a bar under the end conditions of fig. 1 as a "Clamped-Directed" bar.

Although there exists a considerable literature on the theory of the lateral vibration of thin rods, the practical application of the results is extremely limited. The explanation of this is to be sought, not in any lack of enterprise on the part of

* Communicated by the Physical Society : read June 12, 1908.

the technologist, but rather in the fact that the problems that have been solved are not necessarily those of which the solutions are most required in practice.

§ 2. As an important example, in which the terminal conditions here considered occur, one might mention the case of the cylinder of a steam-engine supported on two or more mild steel standards. It will be recognized that the upper ends of the standards must be treated as "directed."

It is worthy of notice that these terminal conditions are mentioned by Lord Rayleigh in his 'Theory of Sound'* but that the directed end is at once dismissed from consideration with the remark that "there are no experimental means by which the contemplated constraint could be realized."

§ 3. *Notation and End Conditions.*—

Let y, y_z = deflexions at points x and z in the length of the bar ;

ω, I = area and moment of inertia of cross-section, the cross-dimensions being supposed small compared with the length ;

l = length of bar ;

ρ = density of the material ;

E = Young's Modulus for the material (when there is an axial pull P in the bar, E stands for $P/\omega + \text{Young's Modulus}$) ;

t = time, measured from any instant at which y is everywhere zero ;

N = number of complete vibrations per second.

The symbol \ddot{y} is written for d^2y/dt^2 ; and if y_1 is the instantaneous deflexion at some particular point in the length of the bar, we have, for simple harmonic vibrations,

$$-\ddot{y}/y = -\ddot{y}_1/y_1 = k^2 \text{ (say), } \dots \dots (1)$$

where

$$N = k/2\pi.$$

The curve assumed by the elastic central line at any instant is given in terms of y_1 . If a is the amplitude at the point in the length at which y_1 is the instantaneous deflexion, the value of y_1 is given by

$$y_1 = a \sin kt. \dots \dots (2)$$

The origin will always be taken at the clamped end. At $x=0$ we have, therefore, $y=dy/dx=0$; whilst at the directed

* See Rayleigh's 'Sound,' vol. i. p. 259, 1894 edition.

end, $x=l$, $dy/dx=0$, and, if there be no concentrated load there, $d^3y/dx^3=0$.

Section II.—Unloaded Massive Bar.

§ 4. In the ordinary case of a bar vibrating under the effect of its own mass only, the form of the elastic central line and the frequency of the lateral vibrations are to be determined from the well-known equation

$$\frac{d^4y}{dx^4} = \mu^4 y,$$

in which

$$\mu^4 = -\frac{\rho\omega}{EI} \frac{\ddot{y}_1}{y_1},$$

and of which the general solution is

$$y = A \sin \mu x + B \cos \mu x + C \sinh \mu x + D \cosh \mu x.$$

The end conditions give four equations whose compatibility requires that

$$\tan \mu l = -\tanh \mu l, \quad . \quad . \quad . \quad (3)$$

of which the least root is

$$\mu l = 2.36502 \dots \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The frequency of the fundamental is therefore

$$N = \frac{5.5933}{2\pi} \sqrt{\frac{EI}{\rho\omega l^4}}.$$

§ 5. It can be shown that the values of θ satisfying

$$\cos \theta \cosh \theta = 1, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

are the same as those which satisfy

$$\tan \frac{\theta}{2} = \pm \tanh \frac{\theta}{2},$$

and hence that the admissible roots of equation (3) are each one half of the alternate roots of (5). Accurate solutions of (5) have been provided by Rayleigh*, and from these the values of μl which satisfy (3) are

$$\mu_1 l = 2.365020$$

$$\mu_2 l = 5.497804$$

$$\mu_3 l = 8.639380,$$

after which

$$\mu_n l = (n - \frac{1}{4})\pi,$$

to more than six decimal places, n being an integer.

* 'Theory of Sound,' Rayleigh, vol. i. p. 278, 1894 edition.

§ 6. The values of μl given above for the lower harmonics may be readily calculated by the method explained below for $\mu_2 l$.

We know that $\mu_2 l = (2 - \frac{1}{4})\pi$ approximately. Hence

$$\tanh \mu_2 l = .999967, \quad . \quad . \quad . \quad . \quad (5a)$$

to the limits of accuracy of the tables at my disposal. By equation (3) therefore

$$\tan \mu_2 l = -.999967 \text{ approximately.}$$

$$\therefore \mu_2 l = 2\pi - .7853815 = 5.4978038,$$

which is more than sufficiently accurate*. Even when this method is used for the fundamental, the error is less than 0.01 per cent.

§ 7. To examine the curve assumed by the centre line of the bar, we obtain a further relation by putting $y = y_1$ when $x = l$; whence it can easily be shown that

$$y = \frac{1}{2}y_1 [(\sinh \mu l + \sin \mu l)(\sin \mu x - \sinh \mu x) - (\cosh \mu l - \cos \mu l)(\cos \mu x - \cosh \mu x)] \div (1 - \cos \mu l \cosh \mu l). \quad (6)$$

The values of μl to be used in this equation are those given in § 5.

§ 8. When dealing with harmonics the positions of the nodes are to be found from (6) by putting $y = 0$. We thus get, at each node,

$$\frac{\sin \mu x - \sinh \mu x}{\cos \mu x - \cosh \mu x} = \frac{\cosh \mu l - \cos \mu l}{\sinh \mu l + \sin \mu l} \quad . \quad . \quad . \quad (7)$$

The fundamental is free from nodes. For the first harmonic the value of the right-hand side of (7) is 1.000033. For higher tones it may be taken as unity.

Solving (7) by trial, we find in the case of the first harmonic that the distance of the node from the clamped

* This method depends on the fact that a slight difference between two numbers of the magnitudes involved makes no appreciable difference in their hyperbolic tangents. If h is small

$$\tanh(x+h) - \tanh x = h(1 - \tanh^2 x)$$

approximately. In the example in the text the result shows that

$$\tanh \mu_2 l = \tanh(2 - \frac{1}{4})\pi$$

to at least eight decimal places. There can be no hesitation, therefore, in accepting equation (5a). The method appears more natural than the normal one, which would be to put $\mu_2 l = (2 - \frac{1}{4})\pi + x$ in $\tan \mu_2 l$ and $\tanh \mu_2 l$ (where x is small), and to proceed by approximations.

456 Dr. J. Morrow on the Lateral Deflexion and
end is given by

$$x = .7166 l.$$

For higher harmonics, the i th tone has $i-1$ nodes; and of these the first node, that is the one nearest the clamped end, is at

$$\frac{x}{l} = \frac{5.0175}{4i-1};$$

the second occurs at

$$\frac{x}{l} = \frac{8.9993}{4i-1},$$

and beyond this the j th node is given with sufficient accuracy by

$$\frac{x}{l} = \frac{4j+1}{4i-1}.$$

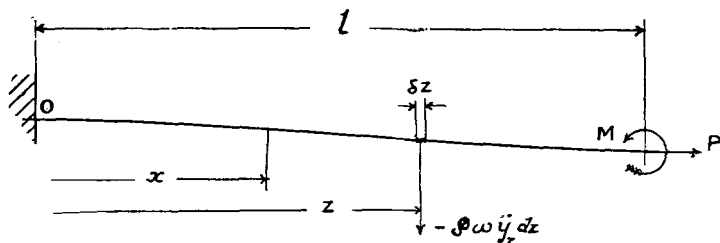
Section III.—Unloaded Massive Bar, Longitudinal Tension.

§ 9. When the bar is subjected to an axial tensile force P as shown in figure 3, the differential equation, based on the ordinary Bernoulli-Eulerian theory, may be written

$$EI \frac{d^2 y}{dx^2} + \rho \omega \int_x^l \ddot{y}_z (z-x) dz + P(y_1 - y) + M = 0. \quad (8)$$

in which M is the couple required to direct the end and y_1 is the deflexion there.

Fig. 3.



Differentiating twice with respect to x ,

$$EI \frac{d^4 y}{dx^4} + \rho \omega \ddot{y} - P \frac{d^2 y}{dx^2} = 0.$$

which is the differential equation common to all unloaded bars of constant flexural rigidity. It may be written

$$\frac{d^4 y}{dx^4} + \frac{\rho \omega}{EI} \ddot{y} - \frac{P}{EI} \frac{d^2 y}{dx^2} = 0, \quad (9)$$

for the case of stationary simple harmonic vibrations. The

general solution is then of the form

$$y = A \cosh \alpha x + B \sinh \alpha x + C \cos \beta x + D \sin \beta x,$$

in which, whatever the end conditions,

$$\left. \begin{aligned} -\frac{\ddot{y}_1}{y_1} &= \frac{\alpha^2}{\rho\omega} (EI\alpha^2 - P) \\ -\frac{\ddot{y}_1}{y_1} &= \frac{\beta^2}{\rho\omega} (EI\beta^2 + P) \end{aligned} \right\} \dots \dots \dots (10)^*$$

The terminal conditions give four equations for which it is necessary that

$$\alpha \tanh \alpha l = -\beta \tan \beta l. \quad \dots \dots \dots (11)$$

Elimination of α and β from the equations (10) and (11) would give an expression for $-\frac{\ddot{y}_1}{y_1}$ and enable the frequency to be calculated.

§ 10. The form of the centre line of the bar at any instant is then given by

$$y = y_1 (\tanh \alpha l \sinh \alpha x - \cosh \alpha x - \frac{\alpha}{\beta} \tanh \alpha l \sin \beta x + \cos \beta x) \div \left(\frac{1}{\cos \beta l} - \frac{1}{\cosh \alpha l} \right),$$

whilst the couple M at the directed end can be obtained from

$$\frac{M}{y_1} + P = -\rho\omega \frac{\ddot{y}_1}{y_1} \left(\frac{1}{\beta^2} \cosh \alpha l + \frac{1}{\alpha^2} \cos \beta l \right) \div (\cosh \alpha l - \cos \beta l).$$

§ 11. Failing an exact solution of equations (10) and (11), a useful approximate one can be obtained by limiting the longitudinal force to such values as may occur in practice.

Writing $\phi = \alpha l$, $\theta = \beta l$, equation (11) is

$$\phi \tanh \phi = -\theta \tan \theta.$$

$$\therefore \frac{d\phi}{d\theta} = -\frac{\theta + \tan \theta + \theta \tan^2 \theta}{\phi + \tanh \phi - \phi \tanh^2 \phi}.$$

Also (10) gives

$$\theta^2 = \phi^2 - \frac{Pl^2}{EI} \dots \dots \dots (12)$$

and when P is zero (equation (4))

$$\phi = \theta = 2.36502$$

for vibrations of the fundamental type.

* Cf. Rayleigh's 'Theory of Sound,' vol. i. 1894, p. 299.

Now, when Pl^2/EI is small, θ differs little from 2.36502, and we have, to the first order in this difference

$$\phi = \left[\phi \right]_{\theta=2.365} + \left[\frac{d\phi}{d\theta} \right]_{\theta=2.365} (\theta - 2.365),$$

that is

$$\alpha l = 10.5083 - 3.4432 \beta l. \quad . \quad . \quad . \quad (13)$$

Eliminating α from (12) and (13), we find, after an expansion by the Binomial as far as the second term

$$\beta^2 l^2 = 5.5933 - .2251 \frac{Pl^2}{EI},$$

by means of which the second of (10) gives

$$k^2 = 31.285 \frac{EI}{\rho \omega l^4} \left\{ 1 + .09831 \frac{Pl^2}{EI} - .00557 \left(\frac{Pl^2}{EI} \right)^2 \right\}. \quad (14)$$

§ 12. For harmonics we may in all cases take

$$\tanh \phi = 1;$$

thus, reasoning as in the last paragraph,

$$\alpha l = (1 - \eta) \beta l + \frac{1}{2} \eta^2,$$

where

$$\eta \equiv (2i - \frac{1}{2})\pi,$$

whence

$$\beta^2 l^2 = \frac{1}{4} \eta^2 - \frac{1}{\eta} \frac{Pl^2}{EI}$$

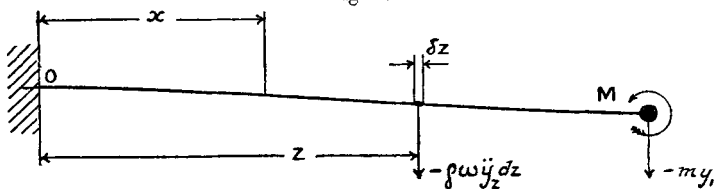
and

$$k^2 = \left(\frac{\eta}{2} \right)^4 \frac{EI}{\rho \omega l^4} + \frac{\eta}{2} \left(\frac{\eta}{2} - 1 \right) \frac{P}{\rho \omega l^2} - \frac{\eta - 1}{\eta^2} \frac{P^2}{\rho \omega EI}. \quad (15)$$

Section IV.—*Massive Bar, Load at Directed End.*

§ 13. When a massive clamped-directed bar carries a load concentrated at its directed end, the frequency and type of lateral vibration can be found to any required degree of accuracy by the method of continuous approximation*.

Fig. 4.



In figure 4, let the origin be at the clamped end and

* See Phil. Mag., July 1905, pp. 113-125, and March 1906, pp. 354-374.

assume for the first approximation to the vibration curve

$$y = y_1 \{3(x/l)^2 - 2(x/l)^3\}, \quad . \quad . \quad . \quad (16)$$

which, at the clamped end, satisfies $y = \frac{dy}{dx} = 0$, and, at the directed end, $y = y_1$, $\frac{dy}{dx} = 0$.

Let m = the mass concentrated at the end,

M = the couple required to maintain the direction at that end.

Then the ordinary approximate theory leads to

$$EI \frac{d^2 y}{dx^2} = -\rho \omega \frac{\ddot{y}_1}{y_1} \int_x^l y_z(z-x) dz - m \ddot{y}_1(l-x) + M. \quad (17)$$

Inserting the value of y_z from (16) and performing the integrations,

$$\begin{aligned} -EI y = \rho \omega \ddot{y}_1 \left(\frac{13}{120} l^2 x^2 - \frac{1}{12} l x^3 + \frac{1}{120} \frac{x^6}{l^2} - \frac{1}{420} \frac{x^7}{l^3} \right) \\ + m \ddot{y}_1 \left(\frac{l x^2}{4} - \frac{x^3}{6} \right), \quad . \quad (18) \end{aligned}$$

the constants of integration and the value of M having been determined by the end conditions. This expression is the second approximation to the vibration type, and is often sufficiently accurate.

Putting $x = l$ we have

$$k^2 = \frac{EI}{\frac{13}{420} \rho \omega l^4 + \frac{1}{12} m l^3}. \quad . \quad . \quad . \quad (19)$$

§ 14. Proceeding to a still closer approximation we can insert in (17) the value of y_z given by (18) and again integrate. In the resulting expression for y , we can again put $x = l$ and get

$$k^2 = \frac{EI}{\left(\frac{.3832 \rho \omega l + m}{.3714 \rho \omega l + m} \right) \frac{13}{420} \rho \omega l^4 + \frac{m l^3}{12}}. \quad . \quad . \quad (20)$$

The portion shown in brackets is the correction which this result gives to that first obtained. When the mass of the load is equal to that of the bar, (20) reduces to

$$k^2 = \frac{8.730 EI}{\rho \omega l^4}.$$

Section V.—Loaded Bars; Mass of Bar Neglected.

§ 15. When the mass of the bar is neglected and vibrations occur under the action of a concentrated load only, the exact solutions can be determined directly.

If there be no longitudinal force and the mass m be at the directed end, the equation of equilibrium is

$$EI \frac{d^2 y}{dx^2} = -m\ddot{y}_1(l-x) - M.$$

$$\therefore EI y = -m\ddot{y}_1 \left(\frac{1}{4}lx^2 - \frac{1}{6}x^3 \right),$$

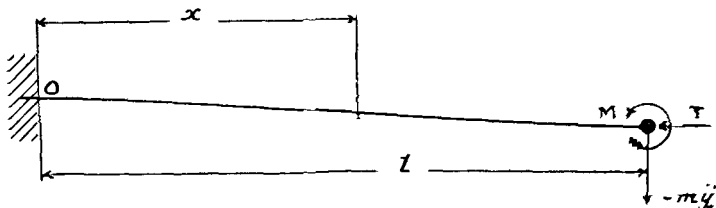
and

$$k^2 = \frac{12 EI}{ml^3} \dots \dots \dots (21)$$

§ 16. *Longitudinal Thrust*.—If in addition to the load m there be a longitudinal thrust T acting on the massless bar as represented in figure 5,

$$EI \frac{d^2 y}{dx^2} = -my_1(l-x) + T(y_1 - y) - M.$$

Fig. 5.



By virtue of the conditions that at $x = 0$, $y = \frac{dy}{dx} = 0$, and, at $x = l$, $\frac{dy}{dx} = 0$, the solution may be written

$$y = \frac{m\ddot{y}_1}{nT} \left\{ \frac{\cos nl - 1}{\sin nl} (1 - \cos nx) + nx - \sin nx \right\}$$

in which $n^2 = T/EI$; and putting $x = l$,

$$k^2 = \frac{T}{ml} \frac{nl \sin nl}{2 - 2 \cos nl - nl \sin nl} \dots \dots \dots (22)$$

This result might have been deduced from that for a bar clamped at each end with a load at the centre*.

* Phil. Mag. September 1906, p. 243.

Equation (22) shows that $k^2 = 0$ and vibration ceases when $nl = \pi$, that is, when

$$T = EI \pi^2 / l^2.$$

After expansion, (22) becomes

$$k^2 = \frac{12EI}{ml^3} \left(1 - \frac{4}{15} n^2 l^2 + \dots \right),$$

agreeing with (21) when T , and therefore n , is zero.

When $nl = 2i\pi$ (i being an integer) the right-hand side of (22) is indeterminate but has the limit $-T/ml$, vibration being impossible if T is positive.

§ 17. The result may also be written

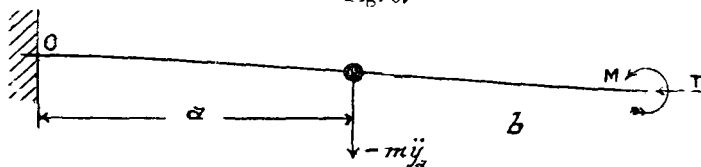
$$k^2 = \frac{T}{ml} \frac{\frac{1}{2} nl}{\tan \frac{1}{2} nl - \frac{1}{2} nl},$$

from which (nl being positive) the expression for the frequency is real when $\tan \frac{1}{2} nl > \frac{1}{2} nl$, that is, for values of nl between 0 and π , and for decreasing intervals in the neighbourhood of 3π , 5π , &c.; vibration always ceasing when $nl = (2i-1)\pi$.

The first of these intervals is from $nl = 8.9868 \dots$ to $nl = 3\pi$, and this corresponds to the first harmonic.

With the massless bar harmonics are impossible with low values of T . As T is increased the frequency falls, becoming zero when $nl = \pi$, \ddot{y} being then zero. Between π and 8.9868 vibration is impossible, the deflexion increasing with the time. If this region be safely passed that between 8.9868 and 3π is reached, during which vibration may again occur, T being now sufficient to enable the bar to assume the curve of the first harmonic type.

Fig. 6.



§ 18. If the mass be situated at any point in the length, let it divide the bar into segments a and b , as indicated in fig. 6. Then for $x < a$ the differential equation becomes

$$EI \frac{d^2 y}{dx^2} = -m\ddot{y}_a (a-x) + T(y_1 - y) - M,$$

of which the solution is

$$y = A_1 \sin nx + B_1 \cos nx + \frac{m\ddot{y}_a}{T} (x-a) + y_1 - \frac{M}{T}.$$

For $x > a$ the equation is

$$EI \frac{d^2 y'}{dx^2} = T(y_1 - y') - M,$$

and the solution

$$y' = A_2 \sin nx + B_2 \cos nx + y_1 - \frac{M}{T},$$

where the dashed symbols refer to values of y in the region b .

The conditions at $x=0$ and $x=l$ lead respectively to

$$\left. \begin{aligned} y &= -\frac{c}{n^3} \sin nx + \left(\frac{m\ddot{y}_a}{T} a + \frac{M}{T} - y_1 \right) (\cos nx - 1) + \frac{m\ddot{y}_a}{T} x \\ y' &= \frac{M}{T} (\sin nl \sin nx + \cos nl \cos nx) + y_1 - \frac{M}{T} \end{aligned} \right\} \quad (23)$$

The conditions of continuity of y and $\frac{dy}{dx}$ when $x=a$ give an expression for the couple at the directed end, namely

$$M = m\ddot{y}_a \frac{\cos na - 1}{n \sin nl}.$$

Putting $x=a$ in the expressions for y and y' we find

$$k^{-2} = \frac{m}{nT} \left\{ \sin na (2 - \cos na) - \cot nl (1 - \cos na)^2 - na \right\}. \quad (24)$$

When $a=l$, this reduces to the result of § 16.

It can be shown, from equations (23), that the amplitude is a maximum at the directed end.

Section VI.—*Deduction of the Results of some Statical Problems.*

§ 19. The calculations in Section V. are similar to those required for the corresponding statical problems. Thus if a clamped-directed bar be subjected to a force W at and perpendicular to the directed end, the centre-line assumes the form (see § 15)

$$y = \frac{W}{EI} \left(\frac{1}{4} lx^2 - \frac{1}{6} x^3 \right).$$

Similar transformations may be made in §§ 16 and 18, and important results obtained. In each case the maximum deflexion is at the directed end.

When the bar is subjected to a force w per unit length acting normally to the x -axis, its deflexion curve is

$$y = \frac{w}{EI} \left(\frac{1}{6} l^2 x^2 - \frac{1}{6} l x^3 + \frac{1}{24} x^4 \right).$$

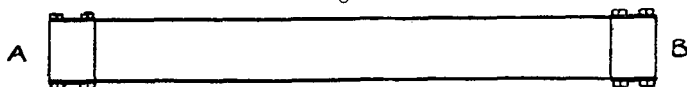
We notice that, for a force concentrated at the end, the maximum deflexion of a clamped-directed bar is one-quarter of that for a clamped-free bar; whilst when the force is uniformly distributed the ratio is one-third.

Section VII.—Some Experimental Results.

§ 20. The experiments described in this section were made merely to compare, in a few cases, the frequency calculated by the methods given in this paper with that obtained by direct observation. Care was taken that the accuracy should reach certain limits.

The composite beams used for the experiments consisted of two flat wrought-iron bars, each 3.00 cms. broad by 0.315 cm. thick, the latter dimension being in the plane of

Fig. 7.



vibration. The bars were rigidly connected together at their ends by being securely screwed to massive cast-iron blocks, as shown in fig. 7. The effective length was measured between the blocks, and this, as well as the distance between the bars, was varied.

A striking feature of the apparatus is the certainty with which the end conditions can be relied upon. When this is the case, the distance between the bars is immaterial; some of the experiments may, in this respect, be taken as a verification of the absolute rigidity of the ends.

§ 21. The flexural rigidity of a piece of one of the bars was determined by supporting it on knife-edges at each end of a span of 58.42 cms. (23 inches), and applying loads at the centre. The observed deflexions are given in the following table.

| Load in lbs. | Deflexion. Inches. | Difference. Inches. |
|------------------------------|-----------------------|------------------------|
| 0 | ·627 | ·047 |
| 1 | ·580 | ·048 |
| 2 | ·532 | ·049 |
| 3 | ·483 | ·050 |
| 4 | ·433 | ·050 |
| 3 | ·483 | ·048 |
| 2 | ·531 | ·049 |
| 1 | ·580 | ·049 |
| 0 | ·629 | |
| Average difference per pound | | ·0488 |

The average deflexion at the centre was thus found to be 0·0488 inch per pound, corresponding to a flexural rigidity of $15\cdot20 \times 10^6$ grammes weight-cms.² The value of E here involved is the static modulus, but the difference between it and the kinetic modulus would be too small to affect the calculations.

§ 22. The results of the experiments on the compound bars are given in the following table. In each case vibrations took place in a horizontal plane, that is, in the plane of fig. 7.

The clamped end was very securely bolted to a large lathe-bed in such a manner as to leave no room for doubt as to the applicability of the terminal conditions at that end.

| No. of Experiment. | Effective Length. cms. | $\frac{1}{2}$ Weight at Directed end, gms. | Distance apart of bars. cms. | Observed Frequency. | Calculated Frequency. |
|-----------------------|------------------------------|--|---------------------------------------|------------------------|--------------------------|
| 1 | 114·3 | 1604 | 16·5 | 1·27 | 1·26 |
| 2 | 114·3 | 780 | 16·5 | 1·66 | 1·67 |
| 3 | 91·5 | 1604 | 16·5 | 1·78 | 1·78 |
| 4 | 91·5 | 780 | 16·5 | 2·40 | 2·40 |
| 5 | 114·3 | 1604 | 6·0 | 1·26 | 1·26 |
| 6 | 114·3 | 802 | 6·0 | 1·65 | 1·65 |

The last column is calculated from equation (20); the values are, however, practically identical with those given

by (19). It will be seen that the agreement between the observed and calculated frequencies is very satisfactory.

I should like to take this opportunity of making a slight correction in my paper "On the Lateral Vibration of Bars subjected to Forces in the Direction of their Axes." On page 236 of the *Philosophical Magazine* for September 1906, and on page 227 of the *Proceedings of the Physical Society*, vol. xx., instead of $\tanh \phi$ I have used its reciprocal. The error makes very little difference to the final equation, the correct expression being

$$-\frac{\ddot{y}_1}{y_1} = 500 \cdot 56 \frac{EI}{\rho \omega l^4} + 12 \cdot 91 \frac{P}{\rho \omega l^2} - 167 \frac{P^2}{\rho \omega EI}.$$

Armstrong College,
January 1908.

LIV. *On the Electric Discharge through the Gases* HCl, HBr, and HI. By L. VEGARD, *Universitets stipendiat of Christiania University* *.

[Plate XIV.]

§ 1. **I**N some recent experiments made by Matthies† on the discharge through the vapours of HgCl, HgBr, HgI, as well as through the elements Cl₂, Br₂, and I, some very striking results are found regarding the potential gradient in the positive column. When compared for equal currents and pressures the gradients are found to be greatest for Br and smallest in the case of I. This remarkable result led me, at the suggestion of Sir J. J. Thomson, to undertake an investigation of the distribution of potential in the gases HCl, HBr, and HI.

As the properties of discharge depend on the form and size of the tube, it is difficult to reduce the numbers to universal units. In order therefore to compare our results with those found for other gases, measurements have been made with the same tube for oxygen.

Apparatus and Mode of Procedure.

§ 2. Fig. 1 gives the general arrangement of the tube-system. The air could be pumped out with a mercury-pump and the vacuum tested with a McLeod gauge. The chemically active gases were removed by blowing a current

* Communicated by Sir J. J. Thomson.

† W. Matthies, *Ann. d. Phys.* xvii. p. 675 (1905); id., *ibid.* xviii. p. 473 (1905).