

XX.—*On General Differentiation. Part III.* By the Rev. P. KELLAND, M.A., F.R.S.S.L. & E., F.C.P.S., late Fellow of Queen's College, Cambridge; Professor of Mathematics, &c., in the University of Edinburgh.

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Nearly six years ago, I presented to the Society two Memoirs on the subject of Differentiation, with fractional indices. The method which I adopted to extend the signification of a differential coefficient consisted in assuming that the function $\sqrt[n]{x}$, which enters into the value of the coefficient deduced from a particular hypothesis, is limited only by the definition $\sqrt[n+1]{x} = n\sqrt[n]{x}$. This generalization appears to be perfectly satisfactory, and promises to offer, if not the only, certainly the best extension of the Differential Calculus. Considering the length of the interval which has elapsed since the publication of my former Memoirs, it is remarkable that so little addition has been made to our knowledge of this branch of analysis. With the exception of one or two papers in LIOUVILLE'S *Journal*, and a few remarks by Professor DE MORGAN, in his Treatise on the Differential Calculus (pp. 598–600), I am not aware that anything has been written on this subject since that time. Seeing, therefore, that others are not willing to enter on this very promising field, I consider it not improper that I should make known a number of extensions of this science to which I have been subsequently led, many of which have been in my possession a considerable time.

I must premise, that the object of this generalization of the differential calculus is, not only to extend the bounds of research beyond the limits of that science, but also to group and classify the results of the science itself. It is, perhaps, as important in the latter aspect as in the former; for its very first consequence is the union of the elementary forms of the two separate branches of that science—the differential and the integral calculus—into one, so that the integral becomes simply the negative differential. Now it is evident that this can only be done by extending to some form, which is general for the existing calculus, a universal and unrestrained interpretation. Such a form, properly selected, becomes, in the new science, a defining property, precisely in the same way that the common differential coefficient is the defining property of the differential calculus. There are several forms which might appear appropriate to this purpose: that which I have adopted is the differential coefficient of x^n . The assumption, therefore, on which the science is based, is the following: that

$\frac{d^\mu x^n}{dx^\mu} = (-1)^\mu \frac{\sqrt{-n+\mu}}{\sqrt{-n}} x^{n-\mu}$, whatever be n and μ . This form can be proved to be the correct one in every interpretable case, and can be deduced from the generalization of $\frac{d^\mu e^{cx}}{dx^\mu}$ when n is negative.* We shall at present assume it as the

defining property or definition of $\frac{d^\mu x^n}{dx^\mu}$.

When, from this definition, we can deduce the differential coefficients of e^{cx} and of $\log x$, that is, of the ascending and descending index-function, we are in possession of the three fundamental forms from which all others may be derived. The following mode of arriving at those differential coefficients is different from that which has hitherto been given, and appears to leave nothing to be desired.

1. To find $\frac{d^\mu e^{cx}}{dx^\mu}$.

$$e^{cx} = 1 + cx + \frac{c^2 x^2}{1 \cdot 2} + \frac{c^3 x^3}{1 \cdot 2 \cdot 3} + \&c. = \sum \frac{c^r x^r}{r!}.$$

$$\therefore \frac{d^\mu e^{cx}}{dx^\mu} = (-1)^\mu \sum \frac{\sqrt{-r+\mu}}{\sqrt{-r}} \frac{c^r x^{r-\mu}}{r!}$$

$$= (-c)^\mu \frac{\sqrt{\mu}}{\sqrt{0}} \left\{ (cx)^{-\mu} + \frac{(cx)^{1-\mu}}{1-\mu} + \frac{(cx)^{2-\mu}}{(1-\mu)(2-\mu)} + \&c. \right\}$$

$$= (-c)^\mu \frac{\sqrt{\mu}}{\sqrt{0}} \left\{ z^{-\mu} + \frac{d^{-1}}{dz^{-1}} z^{-\mu} + \frac{d^{-2}}{dz^{-2}} z^{-\mu} + \&c. \right\}, \text{ where } z = cx;$$

* See Part I., and the excellent Memoir of M. LIOUVILLE, referred to in that Treatise.

Another formula has been proposed, viz.

$$\frac{d^\mu x^n}{dx^\mu} = \frac{\sqrt{1+n}}{\sqrt{1+n-\mu}} x^{n-\mu}.$$

I have lately received from Mr W. CENTER, of Langside, some judicious remarks on these formulæ, contrasting the results arrived at by them respectively. He shews that (without continual introduction of an infinite arbitrary constant) the latter formula is inapplicable in many of the most simple cases: for example, in d^μ of $\frac{1}{1+x}$ expanded positively, it gives, when applied, infinity on one side and not on

the other, and when expanded negatively, infinity on both sides; and again, it gives for $\frac{d^\mu a}{dx^\mu}$ or

$\frac{d^\mu a x^0}{dx^\mu}$ the value $\frac{1}{\sqrt{1-n}} x^{-\mu}$, which is a function of x when μ is a positive proper fraction.

$$=(-c)^{\mu} \frac{\sqrt{\mu}}{\sqrt{0}} \left\{ 1 - \left(\frac{d}{dz} \right)^{-1} \right\}^{-1} z^{-\mu} = c^{\mu} \left(\frac{d}{dz} - 1 \right)^{-1} \frac{d^{\mu+1}}{dz^{\mu+1}} \cdot 1$$

Let $y = \left(\frac{d}{dz} - 1 \right)^{-1} \frac{d^{\mu+1}}{dz^{\mu+1}} \cdot 1$; then

$$\frac{dy}{dz} - y = \frac{d^{\mu+1}}{dz^{\mu+1}} \cdot 1$$

$$y = e^z \left(C + \int dz e^{-z} \frac{d^{\mu+1}}{dz^{\mu+1}} \cdot 1 \right)$$

Now $\frac{d^{\mu+1}}{dz^{\mu+1}} 1 = 0$, except when μ is a negative whole number; in which case

$$\frac{d^{\mu+1}}{dz^{\mu+1}} 1 = \frac{z^{-\mu-1}}{-\mu-1}.$$

$\therefore y = C e^z$; except when μ is a negative whole number, in which case

$$y = C e^z - \frac{z^{-\mu-1}}{\mu-2} - \frac{z^{-\mu-1}}{\mu-2} - \&c.$$

Now, in all cases we omit the arbitrary functions in differentiation to any index; they being readily supplied when required. But $\frac{z^{-\mu-1}}{-\mu-1} + \&c.$, is evidently included in the arbitrary function, in the case in question; we may therefore omit it, and write generally,

$$y = C e^z, \text{ or}$$

$$\frac{d^{\mu} e^{c x}}{dx^{\mu}} = c^{\mu} C e^z = c^{\mu} C e^{c x} \quad . \quad . \quad . \quad (1)$$

This result has been deduced from the definition without any assumption whatever relative to the function $\sqrt{}$, except that it satisfies the condition $\sqrt{n+1} = n\sqrt{n}$. We may, consequently, obtain the value of the constant C, by admitting, that when n is positive, \sqrt{n} coincides with LEGENDRE'S function $\sqrt{}$. In this case,

$$\frac{\sqrt{n}}{x^n} = \int_0^{\infty} e^{-\alpha x} \alpha^{n-1} d\alpha.$$

Therefore, differentiating, to the index μ ,

$$\frac{\sqrt{n+\mu}}{x^{n+\mu}} = C \int \alpha^{\mu+n-1} e^{-\alpha x} d\alpha, \text{ by the definition and equation (1).}$$

But if $n+\mu$ be positive, $\sqrt{n+\mu}$ also coincides with LEGENDRE'S function, therefore,

$$\frac{\sqrt{n+\mu}}{x^{n+\mu}} = \int e^{-\alpha x} \alpha^{n+\mu-1} d\alpha, \text{ or } C=1.$$

Now C is altogether independent of n : if, therefore, we take n positive and greater than $(-\mu)$, which can always be done, we shall have proved generally, that

$$\frac{d^\mu e^{cx}}{dx^\mu} = c^\mu e^{cx} \dots (2).$$

It will be observed that the properties on which the truth of equation (2) is based, are these,—

1. $\frac{d^\mu x^n}{dx^\mu} = (-1)^\mu \frac{\sqrt{-n+\mu}}{\sqrt{-n}} x^{n-\mu}$
 2. $\sqrt{n+1} = n\sqrt{n}$
 3. $\frac{\sqrt{n}}{x^n} = \int_0^x e^{-ax} a^{n-1} da$, when n is positive.
- } whatever be n .

2. To find $\frac{d^\mu \log x}{dx^\mu}$.

In my previous Memoir, Art. 19, I obtained an expression for $\frac{d^\mu \log x}{dx^\mu}$, by assuming that $\int \frac{dx}{x} = \log x$; an assumption which owes its correctness to the admitted possibility of the introduction of an arbitrary constant of integration. Consequently, the conclusions at which I arrived can only be correct generally, by the aid of an arbitrary function of differentiation. Now, it is our object to avoid the use of such functions, and to obtain expressions for the general differential coefficient of all functions which shall be complete in themselves, so far as relates to the satisfaction of every law of combination to which they may be subjected. It becomes necessary, therefore, to reject the equation $\int \frac{dx}{x} = \log x$, and to substitute in its place some other function of x . The following process appears to be perfectly satisfactory.

The value of $\frac{x^p - x^q}{p}$ is $\frac{1 + p \log x + \&c. - 1 - q \log x - \&c.}{p}$

$$= \log x - \frac{q}{p} \log x + A p + \&c.$$

If, therefore, q be of a higher order than p , such as p^2 , it is manifest that $\frac{x^p - x^q}{p}$ will be a simple representation of $\log x$, provided $p=0$ and $\frac{q}{p}=0$.

By adopting this mode of representation we obtain,

$$\frac{d^\mu \log x}{dx^\mu} = (-1)^\mu \frac{\sqrt{\mu-p}}{p\sqrt{-p}} \frac{1}{x^{\mu-p}} - (-1)^\mu \frac{\sqrt{\mu-q}}{p\sqrt{-q}} \frac{1}{x^{\mu-q}}.$$

This expression comprehends every case, and appears to be the most simple

form under which the μ th differential coefficient of a logarithm can be represented.

We shall reduce it in the different cases :

1. When μ is a negative whole number $= -m$.

$$\begin{aligned}\sqrt{\mu-p} &= \sqrt{-(m+p)}; \text{ and } \sqrt{-p} = (-p-1)\sqrt{-p-1} \\ &= (-p-1)(-p-2)\dots(-p-m)\sqrt{-(m+p)} \\ &= (-1)^m \frac{\sqrt{1+m+p}}{\sqrt{1+p}} \sqrt{-(m+p)}\end{aligned}$$

$$\therefore \frac{\sqrt{\mu-p}}{\sqrt{-p}} = (-1)^\mu \frac{\sqrt{1+p}}{\sqrt{1-\mu+p}}$$

$$\text{and } \frac{\sqrt{\mu-q}}{\sqrt{-q}} = (-1)^\mu \frac{\sqrt{1+q}}{\sqrt{1-\mu+q}}$$

Hence
$$\frac{d^\mu \log x}{d x^\mu} = (-1)^{2\mu} \frac{\sqrt{1+p}}{\sqrt{1-\mu+p}} \frac{x^{-\mu+p}}{p} - (-1)^{2\mu} \frac{\sqrt{1+q}}{\sqrt{1-\mu+q}} \frac{x^{-\mu+q}}{p}$$

But
$$\frac{\sqrt{1+p}}{\sqrt{1-\mu+p}} = \frac{1}{\sqrt{1-\mu}} (1-pA + \&c.) \text{ where } A = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{-\mu}$$

and $x^p = 1 + p \log x + \&c.$

also
$$\frac{\sqrt{1+q}}{\sqrt{1-\mu+q}} = \frac{1}{\sqrt{1-\mu}} (1-qA + \&c.)$$

 $x^q = 1 + q \log x + \&c.$

$$\begin{aligned}\therefore \frac{d^\mu \log x}{d x^\mu} &= \frac{x^{-\mu}}{\sqrt{1-\mu}} \cdot \frac{(1-pA + \&c.) (1+p \log x + \&c.)}{p} \\ &\quad - \frac{x^{-\mu}}{\sqrt{1-\mu}} \cdot \frac{(1-qA + \&c.) (1+q \log x + \&c.)}{p} \\ &= \frac{x^{-\mu}}{\sqrt{1-\mu}} \left(\log x - A - \frac{q}{p} \log x + \frac{qA}{p} + \&c. \right) \\ &= \frac{x^{-\mu}}{\sqrt{1-\mu}} (\log x - A), \text{ since } p \text{ and } \frac{q}{p} \text{ are both equal to } 0.\end{aligned}$$

Hence
$$\frac{d^{-m} \log x}{d x^{-m}} = \frac{x^m}{\sqrt{m+1}} \left\{ \log x - \left(\frac{1}{1} + \frac{1}{2} + \&c. + \frac{1}{m} \right) \right\} \text{ which is a well known}$$

expression for $\int^{(m)} d x^m \log x$

2. If μ be not a negative whole number, $\sqrt{\mu}$ is finite ; and

$$\frac{\sqrt{\mu-p}}{p\sqrt{-p}} = -\frac{\sqrt{\mu-p}}{\sqrt{1-p}} = -\sqrt{\mu} (1 + Bp + \&c.)$$

by supposing this function (which is finite) expanded in terms of p ;

similarly

$$\frac{\sqrt{\mu-q}}{q\sqrt{-q}} = -\sqrt{\mu} (1 + Bq + \&c.);$$

and from the expression in Art 2.

$$\begin{aligned} \frac{d^\mu \log x}{d x^\mu} &= (-1)^{\mu+1} \frac{\sqrt{\mu}}{x^\mu} (1 + Bp + \&c.) (1 + p \log x + \&c.) \\ &\quad - (-1)^{\mu+1} \frac{\sqrt{\mu}}{x^\mu} (1 + Bq + \&c.) (1 + q \log x + \&c.) \frac{q}{p} \\ &= (-1)^{\mu+1} \frac{\sqrt{\mu}}{x^\mu}. \end{aligned}$$

3. The expression given above for the differential coefficient of a logarithm is, therefore, perfectly general, and is applicable to all cases. It is essentially analytical in its nature, and does not appear to be reducible to a more arithmetical form so as to retain its general character. The expression which I previously gave exhibits very simply the n th differential coefficient of a logarithm as well as its n th integral, when n is a whole number, and may be, consequently, regarded as the most comprehensive arithmetical form of this function which we can at present obtain.

It may not be considered out of place here to introduce the deduction of the value of $\frac{d^n \log x}{d x^n}$, when n is a positive or a negative whole number, from this form also.

The equation is

$$\begin{aligned} \frac{d^n \log x}{d x^n} &= \frac{\sqrt{n}(-1)^{n+1}}{\sqrt{-1} x^n} \left\{ \log x - (n+1)n \left(\frac{1}{(n+1)n} + \frac{1}{n(n-1)} + \right. \right. \\ &\quad \left. \left. \frac{1}{2} \frac{1}{(n-1)(n-2)} + \frac{1}{3} \frac{1}{(n-2)(n-3)} + \&c. \right) \right\} \end{aligned}$$

(Part I, Art. 21.)

(1.) If n be a positive whole number, the only terms in this expression which are not indefinitely small, are,

$$\begin{aligned} &\frac{\sqrt{n}(-1)^{n+2}}{\sqrt{-1} x^n} (n+1)n \left(\frac{1}{n(n-n+1)(n-n)} + \frac{1}{(n+1)(n-n)(n-n-1)} \right) \\ &= \frac{\sqrt{n}(-1)^{n+2}}{\sqrt{-1} x^n} (n+1)n \left(\frac{1}{n(n-n)} - \frac{1}{(n+1)(n-n)} \right) \\ &= \frac{\sqrt{n}(-1)^{n+2}}{\sqrt{-1} x^n (n-n)} = \frac{\sqrt{n}(-1)^{n+2} \sqrt{n-n}}{x^n \sqrt{n-n-1} \sqrt{n-n+1}} \\ &= \frac{\sqrt{n}(-1)^{n+2} (n-n-1)}{x^n} = \frac{\sqrt{n}(-1)^{n+3}}{x^n} = \frac{(-1)^{n+1} 1 \cdot 2 \dots (n-1)}{x^n} \end{aligned}$$

the well known form.

(2.) If n be a negative integer $= -m$;

$$\begin{aligned}\text{Let } y &= \frac{z^{m+1}}{n(n-1)} + \frac{1}{2} \frac{z^{m+2}}{(n-1)(n-2)} + \frac{1}{3} \frac{z^{m+3}}{(n-2)(n-3)} + \&c. \\ &= \frac{z^{m+1}}{m(m+1)} + \frac{1}{2} \frac{z^{m+2}}{(m+1)(m+2)} + \frac{1}{3} \frac{z^{m+3}}{(m+2)(m+3)} + \&c. \\ \therefore \quad \frac{d^2 y}{dz^2} &= z^{m-1} + \frac{1}{2} z^m + \&c. \\ &= -z^{m-2} \log(1-z)\end{aligned}$$

whence, by integration,

$$\begin{aligned}y &= -\frac{z^m}{m(m-1)} \log(1-z) + \frac{1}{m(m-1)} \left\{ \frac{z^m}{m} + \frac{z^{m-1}}{m-1} + \&c. + z \right\} \\ &+ \frac{1}{m(m-1)} \log(1-z) + \frac{1}{m-1} \left\{ \frac{z^m}{m(m-1)} + \frac{z^{m-1}}{(m-1)(m-2)} + \&c. + \frac{z^2}{2 \cdot 1} \right\} \\ &+ \frac{z}{m-1} \log(1-z) - \frac{z}{m-1} - \frac{1}{m-1} \log(1-z).\end{aligned}$$

Consequently, the value of y between the limits 0 and 1 is

$$\begin{aligned}y &= \frac{1}{m(m-1)} \left(\frac{1}{1} + \frac{1}{2} + \&c. \dots + \frac{1}{m} \right) \\ &+ \frac{1}{m-1} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \&c. + \frac{1}{(m-1)m} \right) - \frac{1}{m-1} \\ &= \frac{1}{m(m-1)} \left(\frac{1}{1} + \frac{1}{2} + \&c. + \frac{1}{m} \right) + \frac{1}{m} - \frac{1}{m-1} \\ &= \frac{1}{m(m-1)} \left\{ \frac{1}{2} + \frac{1}{3} + \&c. + \frac{1}{m} \right\} \\ \text{and } \frac{d^{-m} \log x}{dx^{-m}} &= \frac{(-m)(-1)^{-m+1}}{(-1)x^{-m}} \left\{ \log x - 1 - m(m-1)y \right\} \\ &= \frac{x^m(-1)^{-m+1}}{(-m)(-m+1) \dots (-2)} \left\{ \log x - \left(\frac{1}{1} + \frac{1}{2} + \&c. + \frac{1}{m} \right) \right\} \\ &= \frac{x^m}{m(m-1) \dots 2} \left\{ \log x - \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \&c. + \frac{1}{m} \right) \right\}\end{aligned}$$

which is the expression for $\int^{(m)} dx^m \log x$.

4. In my previous memoirs, I have obtained the general differential coefficients of several functions, and have applied the results to the solution of analytical and mechanical problems. It will be my object at present, to extend the science itself by exhibiting the solution of differential equations, and by investigating some of the properties of finite differences. In every instance I shall select the most simple problems which will serve to illustrate the process employed. Of the process itself, consisting entirely of the application of the *calculus of opera-*

tions, it is, perhaps, necessary to say a few words. The principle on which that calculus is founded is this :

If the laws which regulate the combinations of symbols of operation be the same as those which regulate the combinations of symbols of quantity, then all forms which would be equivalent relative to the latter, must also be equivalent relative to the former.

The laws to which symbols of quantity are subject, may be briefly classed under the seven following heads.

1. Their affections by numbers, or numerical quantities, are the same as if they themselves were numbers, or numerical quantities.

2. The law of signs.

3. The order of simple operations is indifferent.

4. The order of combined operations is indifferent.

5. Combined operations may be distributed.

6. and 7. The laws of indices.

Hence, if d , ϕ , ψ are any symbols of operation, subject to these laws (a and b being numerical quantities) :

$$1. (a \pm b) \phi = a \phi \pm b \phi = a \phi \pm \phi b; \text{ \&c.}$$

$$2. (a \pm \phi) (b \mp \psi) = a b \mp a \psi \pm b \phi - \phi \psi; \text{ \&c.}$$

$$3. \phi + \psi = \psi + \phi$$

$$4. \phi \psi = \psi \phi$$

$$5. d(\phi + \psi) = d\phi + d\psi$$

$$6. d^a d^b = d^{a+b}$$

$$7. (d^a)^b = d^{a \cdot b}$$

results which would be equivalent were d , ϕ , ψ numerical quantities, are equivalent when they are operations. For example,

$$(d + \phi)^n = d^n + n d^{n-1} \phi + \frac{n(n-1)}{1 \cdot 2} d^{n-2} \phi^2 + \text{\&c.}$$

The symbols of differentiation $\frac{d}{dx}$, $\frac{d}{dy}$ and of difference Δ_x , Δ_y satisfy these conditions.

It must be observed, in applying the principle which I have laid down, that it is inapplicable, unless it hold with respect to *every symbol* which enters into the operation. It will evidently apply to the ordinary symbols d and Δ as combined with each other, and to the symbols x , y as combined with each other; but it will not apply to the symbols d and x as combined with each other, because the fourth law is violated by their combination: For example,

$$d \Delta x^2 = 2, \quad \Delta d x^2 = 2$$

$$\therefore d \Delta x^2 \neq \Delta d x^2 :$$

$$\begin{aligned} \text{But} \quad & x d x^2 = 2 x^2, \quad d x x^2 = 3 x^2 \\ \therefore \quad & x d x^2 \text{ is not equal to } d x x^2. \end{aligned}$$

In proof of the sufficiency of the principle here laid down, it may be remarked, that both symbols of operation and symbols of quantity are defined or characterized by the above laws. The symbols of combination are indeed originally framed from arithmetic, but are subsequently generalized, and the basis of generalization is *obedience to these laws*. Thus the symbols + and – are generalized by *collective* symbols the reverse of each other, expressed by the equation $+a - a = +0 = -0$; where +0 is arithmetical, or signifies (as an operation *strictly*) *increased* by 0: \times and \div are ‘*cumulative* symbols the reverse of each other,’ expressed by the equation $\times a \div a = \times 1 = \div 1$; where $\times 1$ signifies *strictly multiplied* by 1. These definitions are in perfect conformity with the above laws. And a similar remarks applies to the general definition of an index.

Now certain symbols of operation, although not, like symbols of quantity, framed with direct reference to the above laws, do, notwithstanding, satisfy them. Consequently, *algebraic formulæ which are results of these laws and of nothing else, must be correct forms also when the algebraic symbols are replaced by such symbols of operation.*

SECTION I. LINEAR DIFFERENTIAL EQUATIONS.

Preliminary Theorems.

5. Since $\left(\frac{d}{dx}\right)^\mu e^{cx} = c^\mu e^{cx}$, it is evident that if $f\left(\frac{d}{dx}\right)$ be any function whatever of $\frac{d}{dx}$, we shall have $f\left(\frac{d}{dx}\right)e^{cx} = f(c)e^{cx}$. (A).

Let u be a function of x , and suppose it expanded in the form $u = \sum a_m e^{mx}$; then

$$\begin{aligned} e^{rx} u &= \sum a_m e^{(m+r)x}; \text{ and hence} \\ \left(\frac{d}{dx}\right)^\mu \cdot e^{rx} u &= \sum a_m (m+r)^\mu e^{(m+r)x}, \text{ by (A)} \\ &= e^{rx} \sum a_m (m+r)^\mu e^{mx} \\ &= e^{rx} \sum a_m \left(\frac{d}{dx} + r\right)^\mu e^{mx} \text{ by (A)} \\ &= e^{rx} \left(\frac{d}{dx} + r\right)^\mu \sum a_m e^{mx} \\ &= e^{rx} \left(\frac{d}{dx} + r\right)^\mu \cdot u \end{aligned}$$

$$\therefore f\left(\frac{d}{dx}\right) \cdot e^{rx} u = e^{rx} f\left(\frac{d}{dx} + r\right) \cdot u \quad (\text{B})$$

Let $x = e^\theta$, and suppose u expanded in the form $u = \sum a_n x^{-n}$: also write D for $\frac{d}{d\theta}$: then

$$\begin{aligned} x^\mu \left(\frac{d}{dx}\right)^\mu \cdot \frac{1}{x^n} &= (-1)^\mu \frac{\sqrt{n+\mu}}{\sqrt{n}} x^{-n} \\ \therefore x^\mu \left(\frac{d}{dx}\right)^\mu \cdot u &= (-1)^\mu \sum a_n \frac{\sqrt{n+\mu}}{\sqrt{n}} x^{-n} \\ &= (-1)^\mu \sum a_n \frac{\sqrt{n+\mu}}{\sqrt{n}} e^{-n\theta} \\ &= (-1)^\mu \sum a_n \frac{\sqrt{-D+\mu}}{\sqrt{-D}} e^{-n\theta} \quad \text{by (A)} \\ &= (-1)^\mu \frac{\sqrt{-D+\mu}}{\sqrt{-D}} \sum a_n x^{-n} \\ &= (-1)^\mu \frac{\sqrt{-D+\mu}}{\sqrt{-D}} \cdot u \quad (\text{C}) \end{aligned}$$

As a particular case of formula (B) we have

$$e^{r\theta} \frac{\sqrt{-D+\mu}}{\sqrt{-D}} \cdot u = \frac{\sqrt{-D+\mu+r}}{\sqrt{-D+r}} \cdot e^{r\theta} u \quad (\text{D})$$

These four theorems will be found of the utmost importance in reducing differential equations. Formulæ somewhat analogous have been applied to the solution of common differential equations by M. CAUCHY, *Exercices*, vol. i., p. 163, and *Exercices d'Analyse*, ii., 343; by Mr GREGORY, *Cambridge Mathematical Journal*, i., 22, &c.: and by Mr BOOLE, *Philosophical Transactions*, 1844, 225. Under the different heads in which we shall arrange differential equations, we shall solve only the most simple examples, our object being to illustrate the method of proceeding rather than to exhibit its power.

CLASS I. *Equations which are capable of solution without transformation.*

6. EX. I. $\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} - a^{\frac{1}{2}} y = 0.$

By writing d for $\frac{d}{dx}$, this equation becomes

$$(d^{\frac{1}{2}} - a^{\frac{1}{2}}) y = 0 \quad \text{or} \quad y = (d^{\frac{1}{2}} - a^{\frac{1}{2}})^{-1} \cdot 0.$$

Suppose $y = \sum b_m e^{m x}$; then by (A)

$$\sum b_m (m^{\frac{1}{2}} - a^{\frac{1}{2}}) e^{m x} = 0; \text{ which can only be satisfied when } m = a.$$

$\therefore y = A e^{a x}$ is the solution of the equation.

We might have proceeded in a somewhat different manner, as follows:

Put $0 e^{m x}$ for 0, then

$$y = (a^{\frac{1}{2}} - a^{\frac{1}{2}})^{-1} \cdot 0 e^{m x} = \frac{0 e^{m x}}{m^{\frac{1}{2}} - a^{\frac{1}{2}}} \text{ by (A).}$$

But $\frac{0}{m^{\frac{1}{2}} - a^{\frac{1}{2}}}$ is finite only when $m = a$; and then it is constant; $\therefore y = A e^{a x}$, as before.

Ex. 2. $\frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} - a^{\frac{1}{2}} y = X$; X being any function of x .

We have $y = (a^{\frac{1}{2}} - a^{\frac{1}{2}})^{-1} \cdot X + (a^{\frac{1}{2}} - a^{\frac{1}{2}})^{-1} \cdot 0$.

If $X = \sum b_r e^{r x}$

$$y = A e^{a x} + \sum \frac{b_r}{r^{\frac{1}{2}} - a^{\frac{1}{2}}} e^{r x} \quad (\text{Ex. 1.})$$

Cor. 1. If $r = a$, $\frac{b_r}{r^{\frac{1}{2}} - a^{\frac{1}{2}}} e^{r x}$ becomes infinite. In this case put $a + \alpha$ in place of r ;

then $\frac{b_r}{r^{\frac{1}{2}} - a^{\frac{1}{2}}} e^{r x}$ becomes $b_r e^{a x} \frac{1 + \alpha x + \&c.}{\alpha}$

$$= \frac{2 a^{\frac{1}{2}} b_r}{\alpha} e^{a x} + 2 b_r a^{\frac{1}{2}} x e^{a x}, \text{ when } \alpha = 0;$$

of which the first term may be incorporated with $A e^{a x}$; and the complete solution is

$$y = A e^{a x} + 2 b_r a^{\frac{1}{2}} x e^{a x} + \sum \frac{b_s e^{s x}}{s^{\frac{1}{2}} - a^{\frac{1}{2}}}$$

Cor. 2. If $X = b x^{-n}$, we have, by the well-known formula

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^\infty e^{-\alpha x} \alpha^{n-1} d\alpha,$$

$$\therefore (a^{\frac{1}{2}} - a^{\frac{1}{2}})^{-1} \cdot \frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^\infty \frac{e^{-a x} \alpha^{n-1} d\alpha}{(-\alpha)^{\frac{1}{2}} - a^{\frac{1}{2}}} \text{ by (A.)}$$

$$= -\frac{1}{\Gamma(n)} \int_0^\infty \left(\frac{1}{a^{\frac{1}{2}}} + \frac{(-\alpha)^{\frac{1}{2}}}{a} + \frac{(-\alpha)}{a^{\frac{3}{2}}} + \&c. \right) e^{-\alpha x} \alpha^{n-1} d\alpha$$

$$\begin{aligned}
&= -\frac{1}{\sqrt{n}} \left(\frac{\sqrt{n}}{a^{\frac{1}{2}} x^n} + \frac{(-1)^{\frac{1}{2}} \sqrt{n+\frac{1}{2}}}{a x^{n+\frac{1}{2}}} - \frac{\sqrt{n+1}}{a^{\frac{3}{2}} x^{n+1}} - \&c. \right) \\
&= -\frac{1}{a^{\frac{1}{2}}} \left(\frac{1}{x^n} - \frac{n}{a x^{n+1}} + \frac{n(n+1)}{a^2 x^{n+2}} - \&c. \right) \\
&\quad - \sqrt{-1} \frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n}} \left(\frac{1}{a x^{n+\frac{1}{2}}} - \frac{n+\frac{1}{2}}{a^2 x^{n+\frac{3}{2}}} + \frac{(n+\frac{1}{2})(n+\frac{3}{2})}{a^3 x^{n+\frac{5}{2}}} + \&c. \right) \\
&= a^{\frac{1}{2}} e^{ax} \int \frac{e^{-ax}}{x^n} dx + \sqrt{-1} \frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n}} e^{ax} \int \frac{e^{-ax}}{x^{n+\frac{1}{2}}} dx \\
\therefore y &= A e^{ax} + B e^{ax} \left\{ a^{\frac{1}{2}} \int \frac{e^{-ax}}{x^n} dx + \sqrt{-1} \frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n}} \int \frac{e^{-ax}}{x^{n+\frac{1}{2}}} dx \right\}
\end{aligned}$$

7. The solution of the foregoing examples might have been obtained very differently, thus :

$$\text{If} \quad d^{\frac{1}{2}} y - a^{\frac{1}{2}} y = X; \quad y = \frac{x}{d^{\frac{1}{2}} - a^{\frac{1}{2}}} = \frac{d^{\frac{1}{2}} + a^{\frac{1}{2}}}{d - a} \cdot X$$

Now $\frac{1}{d-a} X$ is the solution of the ordinary differential equation $\frac{dv}{dx} - av = X$; its value is, consequently, $e^{ax} \left(\int e^{-ax} X dx + C \right)$. Hence

$$y = \frac{d^{\frac{1}{2}}}{d x^{\frac{1}{2}}} \cdot e^{ax} \left(\int e^{-ax} X dx + C \right) + a^{\frac{1}{2}} e^{ax} \int (e^{-ax} X dx + C)$$

For instance, if $X=0$, the solution of the equation is

$$y = 2 a^{\frac{1}{2}} C e^{ax};$$

which is the same as that given above.

$$8. \text{ Ex. 3. } \quad \frac{dy}{dx} + \frac{a d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} + b y = 0$$

This may be written $(d + a d^{\frac{1}{2}} + b) \cdot y = 0$; or $(d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(d^{\frac{1}{2}} - \beta^{\frac{1}{2}}) \cdot y = 0$; where $\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} = -a$, and $(\alpha \beta)^{\frac{1}{2}} = b$, or $\alpha^{\frac{1}{2}}, \beta^{\frac{1}{2}}$ are the roots of the equation $z^2 + az + b = 0$.

$$\begin{aligned}
\therefore y &= A(d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})^{-1} \cdot 0 + B(d^{\frac{1}{2}} - \beta^{\frac{1}{2}})^{-1} \cdot 0 \\
&= A e^{\alpha x} + B e^{\beta x} \quad (\text{Ex. 1.})
\end{aligned}$$

COR. 1. If $\alpha = \beta$, we must write $\alpha + e$ instead of β , and proceed as in similar cases.

$$\text{The result is} \quad y = A e^{\alpha x} + B x e^{\alpha x}$$

COR. 2. In precisely the same way we may find the solution of the equation

$$\frac{d^{\frac{3}{2}} y}{d x^{\frac{3}{2}}} + a \frac{dy}{dx} + b \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} + c y = 0.$$

If $\alpha^{\frac{1}{2}}, \beta^{\frac{1}{2}}, \gamma^{\frac{1}{2}}$ be the roots of the equation $z^3 + a z^2 + b z + c = 0$, the solution is

$$y = A e^{\alpha x} + B e^{\beta x} + C e^{\gamma x}$$

And a similar process applies to equations of all orders, with constant coefficients.

9. It will be seen that in solving these equations, we treat symbols of operation in exactly the same way as if they were symbols of quantity. Our justification for so doing is an appeal to the fact, that the laws which regulate the combination of the former symbols are precisely the same as those which regulate the combination of the latter. Were it otherwise,—were one of the symbols, for instance, to be subject to a different law relative to its combination with one class of symbols from that which regulates its combination with another, we should not be at liberty to separate the operations of such symbols, nor even to combine them otherwise than in the form in which they are actually presented to us. An example will illustrate this remark. The combination $(d^m d^n) \times (d^m d^n) \cdot u$ may be written $(d^m \times d^n)^2 \cdot u$, in which form it is equivalent to $d^{2m} d^{2n} \cdot u$: but the combination $(d^m x^n) \times (d^m x^n) \cdot u$, when written (as we shall write it) $(d^m x^n)^2 \cdot u$, is not equivalent to $d^{2m} x^{2n} \cdot u$. The commutative law, or the law according to which operations may be taken in *any order*, is not true of the symbols d^m, x^n , in their combination with one another.

We may remark, in addition, that when an operation on y has been changed into the reciprocal operation on 0 or on X , giving the solution

$$y = \frac{1}{(D^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(D^{\frac{1}{2}} - \beta^{\frac{1}{2}})} 0, \text{ for instance; the operation } \frac{1}{(D^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(D^{\frac{1}{2}} - \beta^{\frac{1}{2}})} \text{ is resolved}$$

$$\text{into the two operations } \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{1}{D^{\frac{1}{2}} - \alpha^{\frac{1}{2}}} - \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{1}{D^{\frac{1}{2}} - \beta^{\frac{1}{2}}}, \text{ in the same manner as a}$$

fraction is resolved into its equivalent partial fractions. On this subject the reader may consult an excellent paper by Mr BOOLE, in the Cambridge Mathematical Journal, vol. ii., p. 114, where this method is first employed.

10. Ex. 4.
$$\frac{dy}{dx} + a \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + b y = X.$$

This gives
$$y = (d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})^{-1} (d^{\frac{1}{2}} - \beta^{\frac{1}{2}})^{-1} \cdot (X + 0)$$

Now
$$\frac{X}{(d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(d^{\frac{1}{2}} - \beta^{\frac{1}{2}})} = \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{X}{d^{\frac{1}{2}} - \alpha^{\frac{1}{2}}} - \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{X}{d^{\frac{1}{2}} - \beta^{\frac{1}{2}}}$$

$\therefore y = A e^{\alpha x} + B e^{\beta x} + \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \left\{ (d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})^{-1} X - (d^{\frac{1}{2}} - \beta^{\frac{1}{2}})^{-1} X \right\} \quad (\text{Ex. 3.})$

Cor. 1. If
$$X = \sum b_r e^{r x};$$

$$y = A e^{\alpha x} + B e^{\beta x} + \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \sum b_r e^{r x} \left(\frac{1}{r^{\frac{1}{2}} - \alpha^{\frac{1}{2}}} - \frac{1}{r^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \right)$$

$$= A e^{\alpha x} + B e^{\beta x} + \Sigma \frac{b_r e^{r x}}{r + a r^{\frac{1}{2}} + b}$$

Cor. 2. If $X = b_r$, a constant, $\therefore r = 0$ and

$$y = A e^{\alpha x} + B e^{\beta x} + \frac{b_r}{b}$$

Cor. 3. If $r + a r^{\frac{1}{2}} + b = 0$, r must be equal either to a or to β . Suppose $r = a$;

then $b_r e^{r x} \frac{1}{r + a r^{\frac{1}{2}} + b}$ becomes, by writing $a^{\frac{1}{2}} + c$ in place of $r^{\frac{1}{2}}$,

$$b_r e^{\alpha x} \frac{(1 + 2 a^{\frac{1}{2}} c x + \&c.)}{2 a^{\frac{1}{2}} c + a c} = C e^{\alpha x} + \frac{b_r x e^{\alpha x} 2 a^{\frac{1}{2}}}{2 a^{\frac{1}{2}} + a}$$

$$\text{and } y = A e^{\alpha x} + B e^{\beta x} + b_r \frac{2 a^{\frac{1}{2}} x e^{\alpha x}}{2 a^{\frac{1}{2}} + a} + \Sigma b_s \frac{e^{s x}}{s + a s^{\frac{1}{2}} + b}$$

Ex. 5.
$$\frac{d^{-1} y}{d x^{-1}} + a \frac{d^{-\frac{1}{2}} y}{d x^{-\frac{1}{2}}} + b y = X.$$

This gives $(d^{-1} + a d^{-\frac{1}{2}} + b) \cdot y = X.$

or $(d^{-\frac{1}{2}} - \alpha^{-\frac{1}{2}})(d^{-\frac{1}{2}} - \beta^{-\frac{1}{2}}) \cdot y = X$; where $\alpha^{-\frac{1}{2}}, \beta^{-\frac{1}{2}}$ are the roots of the equation $z^2 + a z + b = 0$;

$$\begin{aligned} \therefore y &= A e^{\alpha x} + B e^{\beta x} + \frac{(d^{-\frac{1}{2}} - \alpha^{-\frac{1}{2}})^{-1} \cdot X}{\alpha^{-\frac{1}{2}} - \beta^{-\frac{1}{2}}} - \frac{(d^{-\frac{1}{2}} - \beta^{-\frac{1}{2}})^{-1} \cdot X}{\alpha^{-\frac{1}{2}} - \beta^{-\frac{1}{2}}} \\ &= A e^{\alpha x} + B e^{\beta x} + \frac{\alpha^{\frac{1}{2}} \beta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \left\{ (d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})^{-1} \cdot \alpha^{\frac{1}{2}} \frac{d^{\frac{1}{2}} X}{d x^{\frac{1}{2}}} - (d^{\frac{1}{2}} - \beta^{\frac{1}{2}})^{-1} \beta^{\frac{1}{2}} \frac{d^{\frac{1}{2}} X}{d x^{\frac{1}{2}}} \right\} \end{aligned}$$

which is reduced to Ex. 2.

In precisely the same manner we may solve the more general equation $\frac{d^n y}{d x^n}$

$$+ a \frac{d^{n-\alpha} y}{d x^{n-\alpha}} + b \frac{d^{n-2\alpha} y}{d x^{n-2\alpha}} + \&c. + y = X, n \text{ being a multiple of } \alpha.$$

CLASS II. *Elementary Equations.*

11. The form to which more complicated equations can generally be reduced is $y - m x^n \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = X$; and it is with equations of this form that we are now to be occupied. The simplest case, when $n=0$, we have already solved.

Ex. 1.
$$y - m \sqrt{x} \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0.$$

By (C) this is reduced to $y - m \sqrt{-1} \frac{-D + \frac{1}{2}}{-D} \cdot y = 0,$

or
$$y = \left(1 - m \sqrt{-1} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}}\right)^{-1} \cdot 0.$$

Suppose $y = \sum a_n e^{-n\theta}$; then

$$\sum a_n \left(e^{n-\theta} - m \sqrt{-1} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}} e^{-n\theta} \right) = 0 \text{ by (A);}$$

which can be satisfied only by making $1 - m \sqrt{-1} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}} = 0$; giving, consequently, only one value of n :

Hence $y = \frac{A}{x^n}$ is the complete solution.

COR. If
$$m = \frac{2}{\sqrt{-1} \sqrt{\pi}} = \frac{2}{\sqrt{-1} \frac{1}{2}}$$

$$\frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}} = \frac{1}{2} \frac{1}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \frac{\sqrt{1 + \frac{1}{2}}}{\sqrt{1}};$$

$$\therefore n = 1 \text{ and } y = \frac{A}{x}.$$

Ex. 2.
$$y - m \sqrt{x} \frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = X.$$

The equation in θ is $y - m \sqrt{-1} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}} y = F e^{-\theta}$

$$\begin{aligned} \therefore y &= \left(1 - m \sqrt{-1} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}}\right)^{-1} \cdot 0 + \left(1 - m \sqrt{-1} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}}\right)^{-1} \cdot F e^{-\theta} \\ &= \frac{A}{x^n} + \sum b_r \left(1 - m \sqrt{-1} \frac{\sqrt{r + \frac{1}{2}}}{\sqrt{r}}\right)^{-1} e^{-r\theta} \text{ (Ex. 1 and A)} \end{aligned}$$

COR. If $r = n$; this expression becomes infinite. We must, in this case, write $n + c$ in place of r , expand in terms of c , and finally put $c = 0$.

$$\begin{aligned} \text{We have, thus, } \frac{e^{-r\theta}}{1 - m \sqrt{-1} \frac{\sqrt{r + \frac{1}{2}}}{\sqrt{r}}} &= \frac{e^{-n\theta} (1 - c\theta + \&c.)}{1 - m \sqrt{-1} \left(\frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}} + \frac{d}{dn} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}} \cdot c + \&c. \right)} \\ &= \frac{e^{-n\theta}}{1 - m \sqrt{-1} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}}} + \frac{c e^{-n\theta} \theta}{c m \sqrt{-1} \frac{d}{dn} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}}} + \&c. \\ &= \frac{e^{-n\theta}}{1 - m \sqrt{-1} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}}} + \frac{e^{-n\theta} \theta}{m \sqrt{-1} \frac{d}{dn} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}}} \end{aligned}$$

$$\therefore y = \frac{A}{x^n} + \frac{b_r}{m\sqrt{-1}} \frac{\log x}{x^n} \frac{1}{\frac{d}{dn} \frac{n+\frac{1}{2}}{n}} \\ + \Sigma \frac{b_s}{x^s} \frac{1}{1-m\sqrt{-1} \frac{s+\frac{1}{2}}{s}}$$

Ex. 3. $y - a\sqrt{-1} x \frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = 0.$

Suppose $y = \Sigma a_r x^{-r}$; then

$$\Sigma (a_r x^{-r} + a \frac{\sqrt{r+1}}{\sqrt{r+\frac{1}{2}}} a_{r+\frac{1}{2}} x^{-r}) = 0$$

or $a_{r+\frac{1}{2}} = -\frac{1}{a} \frac{\sqrt{r+1}}{\sqrt{r+\frac{1}{2}}} a_r \quad (1.)$

Hence the lowest value of r is 0, and the values succeed at intervals of $\frac{1}{2}$.

$\therefore y = A + \frac{A_1}{\sqrt{x}} + \frac{A_2}{x} + \&c.$, with the relation expressed by (1). By substitution

$$A_1 = -\frac{1}{a} \frac{\sqrt{\frac{1}{2}}}{\sqrt{1}} A, A_2 = -\frac{1}{a} \frac{\sqrt{1}}{\sqrt{\frac{3}{2}}} A_1, A_3 = -\frac{1}{a} \frac{\sqrt{\frac{3}{2}}}{\sqrt{2}} \&c.$$

$$A_1 = -\frac{1}{a} \sqrt{\pi} A; A_2 = \frac{1}{a^2} \frac{\sqrt{1} \sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}} \sqrt{1}} A = \frac{2}{a^2} A$$

$$A_3 = -\frac{1}{a^3} \frac{\sqrt{\frac{1}{2}} \sqrt{1} \sqrt{\frac{3}{2}}}{\sqrt{1} \sqrt{\frac{3}{2}} \sqrt{2}} A = -\frac{\sqrt{\pi}}{a^3} A, A_4 = \frac{1}{a^4} \frac{\sqrt{\frac{1}{2}} \sqrt{1} \sqrt{\frac{3}{2}} \sqrt{2}}{\sqrt{1} \sqrt{\frac{3}{2}} \sqrt{2} \sqrt{\frac{5}{2}}} = \frac{1}{a^4} \frac{2^2}{1 \cdot 3} A$$

$$A_5 = -\frac{\sqrt{\pi}}{a^5} \cdot \frac{\sqrt{2} \sqrt{\frac{5}{2}}}{\sqrt{\frac{5}{2}} \sqrt{3}} A = -\frac{\sqrt{\pi}}{a^5} \frac{1}{1 \cdot 2} A, A_6 = \frac{1}{a^6} \frac{2^2}{1 \cdot 3} A \frac{\sqrt{\frac{5}{2}} \sqrt{3}}{\sqrt{3} \sqrt{\frac{7}{2}}} = \frac{1}{a^6} \frac{2^3}{1 \cdot 3 \cdot 5} A$$

&c., &c., so that

$$y = A \left\{ 1 + \frac{2}{a^2 x} + \frac{2^2}{1 \cdot 3 a^4 x^2} + \frac{2^3}{1 \cdot 3 \cdot 5 a^6 x^3} + \&c. \right.$$

$$\left. - \sqrt{\pi} \left(\frac{1}{a \sqrt{x}} + \frac{1}{1 \cdot a^3 x^{\frac{3}{2}}} + \frac{1}{1 \cdot 2 a^5 x^{\frac{5}{2}}} + \&c. \right) \right\}$$

Let

$$y_1 = 1 + \frac{2}{a^2 x} + \frac{2^2}{1 \cdot 3 a^4 x^2} + \&c.$$

then

$$\frac{d}{dx} (\sqrt{x} \cdot y_1) = \frac{1}{2 \sqrt{x}} - \frac{1}{a^2 x^{\frac{3}{2}}} - \&c.$$

$$= \frac{1}{2 \sqrt{x}} - \frac{1}{a^2 x^{\frac{3}{2}}} y_1$$

or

$$\frac{dy_1}{dx} + \left(\frac{1}{2x} + \frac{1}{a^2 x^2} \right) y_1 = \frac{1}{2x}.$$

Again, let
$$y_2 = \frac{1}{\sqrt{x}} + \frac{1}{1 \cdot \alpha^3 x^{\frac{3}{2}}} + \&c.$$

then
$$\frac{d}{dx}(\sqrt{x} y_2) = -\frac{1}{1 \cdot \alpha^3 x^2} - \&c.$$
$$= -\frac{1}{\alpha^2 x^{\frac{3}{2}}} y_2$$

or
$$\frac{dy_2}{dx} + \left(\frac{1}{2x} + \frac{1}{\alpha^2 x^2}\right) y_2 = 0$$

and
$$y = A \left(y_1 - \frac{\sqrt{\pi}}{\alpha} y_2\right).$$

By solving the equations for y_1 and y_2 we obtain finally

$$y = \frac{A e^{\frac{1}{\alpha^2 x}}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\alpha} \right\}$$

The equations from which y_1 and y_2 are determined differ only in the term which does not contain y ; and it will be seen hereafter that similar equations serve to give the solution of the other differential equations of this class, when n is an integer. If $\alpha\sqrt{-1}=m$, these equations are

$$\frac{dy_1}{dx} + \left(\frac{1}{2x} - \frac{1}{m^2 x^2}\right) y_1 = \frac{1}{2x}$$

$$\frac{dy_2}{dx} + \left(\frac{1}{2x} - \frac{1}{m^2 x^2}\right) y_2 = 0.$$

12. OTHERWISE. The following method of solving this equation has the advantage of not appearing to take for granted the form in which y is expressed in terms of x .

$$y - \alpha\sqrt{-1} x \frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = 0 \text{ gives } y = \frac{0}{1 - \alpha\sqrt{-1} x d^{\frac{1}{2}}}$$

$$= \frac{1 + \alpha\sqrt{-1} x d^{\frac{1}{2}}}{1 + \alpha^2 x d^{\frac{1}{2}} x d^{\frac{1}{2}}} \cdot 0$$

Now $\frac{0}{1 + \alpha^2 x d^{\frac{1}{2}} x d^{\frac{1}{2}}}$ is the solution of the equation

$$v + \alpha^2 x d^{\frac{1}{2}} x d^{\frac{1}{2}} v = 0, \text{ or of}$$

$$\frac{v}{\alpha^2} + x^2 \frac{dv}{dx} + \frac{1}{2} x v = 0,$$

or of

$$\frac{dv}{dx} + \left(\frac{1}{2x} + \frac{1}{\alpha^2 x^2}\right) v = 0 :$$

which is the equation for determining y_2 given above.

$$\therefore v = \frac{A}{\sqrt{x}} e^{\frac{1}{\alpha^2 x}};$$

and

$$\begin{aligned} y &= (1 + \alpha \sqrt{-1} x d^{\frac{1}{2}}) v \\ &= \frac{A}{\sqrt{x}} e^{\frac{1}{\alpha^2 x}} + \alpha \sqrt{-1} x \frac{d^{\frac{1}{2}} v}{d x^{\frac{1}{2}}} \end{aligned}$$

which will be seen to coincide with the solution already given.

This second method of solving the equation is by far the most simple and satisfactory, when once the principles of the calculus of operations are thoroughly mastered. For the purpose, however, of exhibiting the analogy amongst the differential equations which determine the values of the different series which make up a function satisfying the conditions $y - m x^2 \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0$, I shall employ the first method in the three following examples.

13. Ex. 4. $y - m x^{\frac{3}{2}} \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0.$

let

$$y = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \&c.$$

then

$$\frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = (-1)^{\frac{1}{2}} \left\{ \frac{\sqrt{\frac{3}{2}}}{\sqrt{1}} \frac{A_1}{x^{\frac{3}{2}}} + \frac{\sqrt{2}}{\sqrt{\frac{3}{2}}} \frac{A_2}{x^{\frac{5}{2}}} - \&c. \right\}$$

and

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \&c. = m(-1)^{\frac{1}{2}} \left\{ \frac{\sqrt{\frac{3}{2}}}{\sqrt{1}} A_1 + \frac{\sqrt{2}}{\sqrt{\frac{3}{2}}} \frac{A_2}{x} + \&c. \right\}$$

$$A_1 = \frac{\sqrt{1}}{\sqrt{\frac{3}{2}}} \frac{A_0}{m\sqrt{-1}}; \quad A_2 = \frac{\sqrt{2}}{\sqrt{\frac{5}{2}}} \frac{A_1}{m\sqrt{-1}}; \quad A_3 = \frac{\sqrt{3}}{\sqrt{\frac{7}{2}}} \frac{A_2}{m\sqrt{-1}} \&c.$$

or $A_1 = \frac{2}{1} \cdot \frac{A_0}{m\sqrt{-1}}; \quad A_2 = -\frac{2^2}{3 \cdot 1} \cdot \frac{2}{1} \frac{1 \cdot A_0}{m^2 \pi}$

$$A_3 = -\frac{2^3}{5 \cdot 3 \cdot 1} \cdot \frac{2^2}{3 \cdot 1} \cdot \frac{2}{1} \cdot \frac{1 \cdot 2 \cdot A_0}{m^3 \pi \sqrt{-1}}; \quad A_4 = \frac{2^4}{7 \cdot 5 \cdot 3 \cdot 1} \cdot \frac{2^3}{5 \cdot 3 \cdot 1} \cdot \frac{2^2}{3 \cdot 1} \cdot \frac{2}{1} \cdot \frac{1 \cdot 2 \cdot 3 \cdot A_0}{m^4 \pi^2}$$

&c. = &c.

and $y = A_0 \left\{ 1 + \frac{2}{1} \frac{1}{m x \sqrt{-1}} - \frac{2^2}{3 \cdot 1} \cdot \frac{2}{1} \cdot \frac{1}{m^2 x^2 \pi} - \frac{2^3}{5 \cdot 3 \cdot 1} \cdot \frac{2^2}{3 \cdot 1} \cdot \frac{2}{1} \frac{1 \cdot 2}{m^3 x^3 \sqrt{-1}} \&c. \right\}$

Ex. 5. $y - m x^2 \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0.$

It is easily seen that the form of the series into which y may be expanded is this

$$\begin{aligned} y &= A x + \frac{B}{\sqrt{x}} + \frac{C}{x^2} + \frac{D}{x^{\frac{7}{2}}} + \&c. \\ &+ \alpha + \frac{\beta}{x^{\frac{3}{2}}} + \frac{\gamma}{x^3} + \frac{\delta}{x^{\frac{9}{2}}} + \&c. \end{aligned}$$

and that the result of substitution is

$$\left. \begin{aligned} & A x + \frac{B}{\sqrt{x}} + \frac{C}{x^2} + \frac{D}{x^{\frac{3}{2}}} + \&c. \\ & + a + \frac{\beta}{x^{\frac{3}{2}}} + \frac{\gamma}{x^3} + \frac{\delta}{x^{\frac{9}{2}}} + \&c. \end{aligned} \right\} = \begin{aligned} & m\sqrt{-1} \left(\frac{\bar{1}}{\frac{1}{2}} B x + \frac{\bar{5}}{2} \frac{C}{\sqrt{x}} + \frac{\bar{4}}{\frac{1}{2}} \frac{D}{x^2} + \&c. \right) \\ & + m\sqrt{-1} \left(\frac{\bar{2}}{\frac{3}{2}} \beta + \frac{\bar{7}}{3} \frac{\gamma}{x^{\frac{3}{2}}} + \frac{\bar{5}}{\frac{9}{2}} \frac{\delta}{x^3} + \&c. \right) \end{aligned}$$

so that

$$B = \frac{\bar{1}}{1} \frac{A}{m\sqrt{-1}}; \quad C = \frac{\bar{2}}{\frac{5}{2}} \frac{B}{m\sqrt{-1}} = -\frac{\bar{1}}{1} \frac{\bar{2}}{\frac{5}{2}} \frac{1}{m^2}$$

$$D = \frac{\bar{1}}{4} \frac{C}{m\sqrt{-1}} = -\frac{\bar{1}}{1} \frac{\bar{2}}{\frac{5}{2}} \frac{\bar{7}}{4} \frac{1}{m^3\sqrt{-1}}$$

$$E = \frac{\bar{1}}{1} \frac{\bar{2}}{\frac{5}{2}} \frac{\bar{7}}{4} \frac{\bar{5}}{\frac{1}{2}} \frac{1}{m^4} \&c.$$

and also

$$\beta = \frac{\bar{3}}{2} \frac{a}{m\sqrt{-1}}, \quad \gamma = -\frac{\bar{2}}{2} \frac{\bar{3}}{\frac{7}{2}} \frac{a}{m^2}, \quad \delta = -\frac{\bar{3}}{2} \frac{\bar{3}}{\frac{1}{2}} \frac{\bar{5}}{\frac{9}{2}} \frac{a}{m^3\sqrt{-1}}$$

$$\epsilon = \frac{\bar{3}}{2} \frac{\bar{3}}{\frac{1}{2}} \frac{\bar{5}}{\frac{9}{2}} \frac{\bar{6}}{\frac{1}{2}} \frac{a}{m^4} \&c.$$

$$\begin{aligned} \therefore y = A \left\{ x - \frac{1}{\frac{1}{2} \cdot \frac{3}{2} m^2 x^2} + \frac{1 \cdot 4}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} m^4 x^5} - \frac{1 \cdot 4 \cdot 7}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} \cdot \frac{13}{2} \cdot \frac{15}{2} m^6 x^8} + \&c. \right. \\ \left. + \frac{\sqrt{\pi}}{m\sqrt{-1}} \left(\frac{1}{\sqrt{x}} - \frac{\bar{5}}{2 \cdot 3 m^2 x^{\frac{5}{2}}} + \frac{\bar{5} \cdot \bar{1}^1}{2 \cdot 3 \cdot 5 \cdot 6 m^4 x^{\frac{6}{2}}} - \frac{\bar{5} \cdot \bar{1}^1 \cdot \bar{1}^7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 m^6 x^{\frac{9}{2}}} + \&c. \right) \right\} \\ + a \left\{ 1 - \frac{2}{\frac{3}{2} \cdot \frac{5}{2} m^2 x^3} + \frac{2 \cdot 5}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} \cdot \frac{13}{2} m^4 x^6} - \frac{2 \cdot 5 \cdot 8}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} \cdot \frac{13}{2} \cdot \frac{15}{2} \cdot \frac{17}{2} m^6 x^9} + \&c. \right. \\ \left. + \frac{\sqrt{\pi}}{m\sqrt{-1}} \left(\frac{\bar{1}}{x^{\frac{3}{2}}} - \frac{\bar{1} \cdot \bar{7}}{3 \cdot 4 m^2 x^{\frac{3}{2}}} + \frac{\bar{1} \cdot \bar{7} \cdot \bar{1}^3}{3 \cdot 4 \cdot 6 \cdot 7 m^4 x^{\frac{15}{2}}} - \&c. \right) \right\} \end{aligned}$$

Each of these four series is the integral of a differential equation of the second order.

$$\text{Let} \quad \frac{dy_1}{dx} = x - \frac{1}{\frac{1}{2} \cdot \frac{3}{2} m^2 x^2} + \frac{1 \cdot 4}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} m^4 x^5} - \&c.$$

$$\text{then} \quad y_1 = \frac{x^2}{2} + \frac{1}{\frac{1}{2} \cdot \frac{3}{2} m^2 x} - \&c.$$

$$\begin{aligned} \text{and} \quad \frac{d^2 \sqrt{x} y_1}{dx^2} &= \frac{5}{2} \cdot \frac{3}{2} \frac{x^{\frac{1}{2}}}{2} + \frac{1}{m^2 x^{\frac{5}{2}}} - \frac{1}{\frac{1}{2} \cdot \frac{3}{2} m^4 x^{\frac{11}{2}}} + \&c. \\ &= \frac{15}{8} \sqrt{x} + \frac{1}{m^2 x^{\frac{5}{2}}} \frac{dy_1}{dx} \end{aligned}$$

$$\text{or} \quad \frac{d^2 y_1}{dx^2} + \left(\frac{1}{x} - \frac{1}{m^2 x^4} \right) \frac{dy_1}{dx} - \frac{y_1}{4 x^2} = \frac{15}{8}$$

Again, let

$$\frac{dy_2}{dx} = \frac{1}{\sqrt{x}} - \frac{\frac{5}{2}}{2 \cdot 3 m^2 x^{\frac{7}{2}}} + \frac{\frac{5}{2} \cdot \frac{11}{2}}{2 \cdot 3 \cdot 5 \cdot 6 m^4 x^{\frac{13}{2}}} - \&c.$$

then

$$y_2 = 2\sqrt{x} + \frac{1}{2 \cdot 3 m^2 x^{\frac{5}{2}}} - \frac{\frac{5}{2}}{2 \cdot 3 \cdot 5 \cdot 6 m^4 x^{\frac{11}{2}}} + \&c.$$

and

$$\frac{d^2 \sqrt{x} y_2}{dx^2} = \frac{1}{m^2 x^4} - \frac{\frac{5}{2}}{2 \cdot 3 m^4 x^7} + \&c.$$

$$= \frac{1}{m^2 x^{\frac{7}{2}}} \frac{dy_2}{dx}$$

$$\therefore \frac{d^2 y_2}{dx^2} + \left(\frac{1}{x} - \frac{1}{m^2 x^4} \right) \frac{dy_2}{dx} - \frac{y_2}{4x^2} = 0$$

Also let

$$\frac{dy_3}{dx} = 1 - \frac{2}{\frac{3}{2} \cdot \frac{5}{2} m^2 x^3} + \frac{2 \cdot 5}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} \cdot \frac{11}{2} m^4 x^5} - \&c.$$

then

$$y_3 = x + \frac{1}{\frac{3}{2} \cdot \frac{5}{2} m^2 x^2} - \frac{2}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{9}{2} \cdot \frac{11}{2} m^4 x^5} + \&c.$$

and

$$\frac{d^2 \sqrt{x} y_3}{dx^2} = \frac{3}{2} \frac{1}{2\sqrt{x}} + \frac{1}{m^2 x^{\frac{7}{2}}} - \frac{2}{\frac{3}{2} \cdot \frac{5}{2} m^4 x^{\frac{13}{2}}} + \&c.$$

$$= \frac{3}{4\sqrt{x}} + \frac{1}{m^2 x^{\frac{7}{2}}} \frac{dy_3}{dx}$$

or

$$\frac{d^2 y_3}{dx^2} + \left(\frac{1}{x} - \frac{1}{m^2 x^4} \right) \frac{dy_3}{dx} - \frac{y_3}{4x^2} = \frac{3}{4x}$$

Lastly, let

$$\frac{dy_4}{dx} = \frac{\frac{1}{2}}{x^{\frac{3}{2}}} - \frac{\frac{1}{2} \cdot \frac{7}{2}}{3 \cdot 4 m^2 x^{\frac{9}{2}}} + \frac{\frac{1}{2} \cdot \frac{7}{2} \cdot \frac{13}{2}}{3 \cdot 4 \cdot 6 \cdot 7 m^4 x^{\frac{15}{2}}} + \&c.$$

then

$$y_4 = -\frac{1}{x^{\frac{1}{2}}} + \frac{\frac{1}{2}}{3 \cdot 4 m^2 x^{\frac{7}{2}}} - \frac{\frac{1}{2} \cdot \frac{7}{2}}{3 \cdot 4 \cdot 6 \cdot 7 m^4 x^{\frac{11}{2}}} + \&c.$$

and

$$\frac{d^2 \sqrt{x} y_4}{dx^2} = \frac{\frac{1}{2}}{m^2 x^5} - \frac{\frac{1}{2} \cdot \frac{7}{2}}{3 \cdot 4 m^4 x^8} + \&c.$$

$$= \frac{1}{m^2 x^{\frac{7}{2}}} \frac{dy_4}{dx}$$

or

$$\frac{d^2 y_4}{dx^2} + \left(\frac{1}{x} - \frac{1}{m^2 x^4} \right) \frac{dy_4}{dx} - \frac{y_4}{4x^2} = 0$$

Having found y_1, y_2, y_3, y_4 from these equations, we obtain

$$y = A \left(\frac{dy_1}{dx} + \frac{\sqrt{\pi}}{m\sqrt{-1}} \frac{dy_2}{dx} \right) + a \left(\frac{dy_3}{dx} + \frac{\sqrt{\pi}}{m\sqrt{-1}} \frac{dy_4}{dx} \right)$$

The remarkable similarity between the equations which determine y_1, y_2, y_3, y_4 leads us to conclude that the form of this function is common to all similar equations. It may be seen that the equations for y_2 and y_4 are identical: the arbitrary constants must, however, be determined differently in the two: the one function vanishes when $x = \infty$, the other does not. By solving the equations in a more general form, and by a more purely symbolical method, we shall be able to

see the reason of this analogy. We shall, in Example 7, exhibit a complete and general solution of all equations of this form.

Ex. 6. $y - m x^3 \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0$

Let $y = A x^2 + B x + C + \frac{A_1}{x^{\frac{1}{2}}} + \frac{B_1}{x^{\frac{3}{2}}} + \frac{C_1}{x^{\frac{5}{2}}} + \frac{A_2}{x^{\frac{3}{2}}} + \frac{B_2}{x^{\frac{5}{2}}} + \frac{C_2}{x^{\frac{7}{2}}} + \frac{A_3}{x^{\frac{5}{2}}} + \frac{B_3}{x^{\frac{7}{2}}} + \frac{C_3}{x^{\frac{9}{2}}} + \&c.$

then $A x^2 + B x + C + \frac{A_1}{x^{\frac{1}{2}}} + \&c. = m \sqrt{-1} \left(\frac{1}{\frac{1}{2}} x^2 A_1 + \&c. \right)$

$$A_1 = \frac{A}{m \sqrt{-1} \frac{1}{2}}, A_2 = \frac{A_1}{m \sqrt{-1} \frac{3}{2}}, A_3 = \frac{A_2}{m \sqrt{-1} \frac{5}{2}} \&c.$$

$$B_1 = \frac{B}{m \sqrt{-1} \frac{3}{2}}, B_2 = \frac{B_1}{m \sqrt{-1} \frac{5}{2}}, B_3 = \frac{B_2}{m \sqrt{-1} \frac{7}{2}} \&c.$$

$$C_1 = \frac{C}{m \sqrt{-1} \frac{5}{2}}, C_2 = \frac{C_1}{m \sqrt{-1} \frac{7}{2}}, C_3 = \frac{C_2}{m \sqrt{-1} \frac{9}{2}} \&c.$$

This gives us six separate series.

1°. $A x^2 + \frac{A_2}{x^{\frac{3}{2}}} + \frac{A_4}{x^{\frac{9}{2}}} + \&c. = A \left(x^2 - \frac{1 \cdot 2}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} m^2 x^3} + \frac{1 \cdot 2 \cdot 6 \cdot 7}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} m^4 x^5} + \&c. \right)$

Let $\frac{d^2 y_1}{d x^2} = x^2 - \frac{1 \cdot 2}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} m^2 x^3} + \&c.$

then $y_1 = \frac{x^4}{3 \cdot 4} - \frac{1}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} m^2 x} + \frac{1 \cdot 2}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} m^4 x^5} - \&c.$

$$\frac{d^3 \sqrt{x} \cdot y_1}{d x^3} = \frac{\frac{3}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} x^{\frac{3}{2}}}{3 \cdot 4} + \frac{1}{m^2 x^{\frac{7}{2}}} - \frac{1 \cdot 2}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} m^4 x^{\frac{11}{2}}} + \&c.$$

$$= \frac{105}{32} x^{\frac{3}{2}} + \frac{1}{m^2 x^{\frac{7}{2}}} \frac{d^2 y_1}{d x^2}$$

or $\frac{d^3 y_1}{d x^3} + \frac{3}{2 x} \frac{d^2 y_1}{d x^2} - \frac{3}{4 x^2} \frac{d y_1}{d x} + \frac{3}{8 x^3} y_1 = \frac{105}{32} x + \frac{1}{m^2 x^5} \frac{d^2 y_1}{d x^2}$

or $\frac{d^3 y_1}{d x^3} + \left(\frac{3}{2 x} - \frac{1}{m^2 x^6} \right) \frac{d^2 y_1}{d x^2} - \frac{3}{4 x^2} \frac{d y_1}{d x} + \frac{3}{8 x^3} y_1 = \frac{105}{32} x$

2°. $\frac{A_1}{x^{\frac{1}{2}}} + \frac{A_3}{x^{\frac{5}{2}}} + \&c.,$ gives $\frac{\sqrt{\pi}}{m \sqrt{-1}} A \left(\frac{1}{x^{\frac{1}{2}}} - \frac{\frac{7}{2} \cdot \frac{3}{2}}{3 \cdot 4 \cdot 5 m^2 x^{\frac{3}{2}}} + \frac{\frac{7}{2} \cdot \frac{3}{2} \cdot \frac{17}{2} \cdot \frac{13}{2}}{3 \cdot 4 \cdot 5 \cdot 8 \cdot 9 \cdot 10 m^2 x^{\frac{5}{2}}} + \&c. \right)$

Let $\frac{d^2 y_2}{d x^2} = \frac{1}{x^{\frac{1}{2}}} - \frac{\frac{7}{2} \cdot \frac{3}{2}}{3 \cdot 4 \cdot 5 m^2 x^{\frac{3}{2}}} + \&c.$

$$\text{then } y_2 = 2 \cdot \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3 \cdot 4 \cdot 5 m^2 x^{\frac{7}{2}}} + \frac{\frac{7}{2} \cdot \frac{3}{2}}{3 \cdot 4 \cdot 5 \cdot 8 \cdot 9 \cdot 10 m^4 x^{\frac{11}{2}}} - \&c.$$

$$\begin{aligned} \frac{d^3 y_2 \sqrt{x}}{d x^3} &= \frac{1}{m^2 x^6} - \frac{\frac{7}{2} \cdot \frac{3}{2}}{3 \cdot 4 \cdot 5 m^4 x^{11}} + \&c. \\ &= \frac{1}{m^2 x^{\frac{11}{2}}} \frac{d^2 y_2}{d x^2} \end{aligned}$$

$$\therefore \frac{d^3 y_2}{d x^3} + \left(\frac{3}{2x} - \frac{1}{m^2 x^6} \right) \frac{d^2 y_2}{d x^2} - \frac{3}{4x^2} \frac{d y_2}{d x} + \frac{3}{8x^3} y_2 = 0$$

$$\begin{aligned} 3^\circ. \quad Bx + \frac{B_2}{x^4} + \frac{B_3}{x^9} + \&c. &= B \left(x - \frac{2 \cdot 3}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} m^2 x^4} + \frac{2 \cdot 3 \cdot 7 \cdot 8}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} m^4 x^9} + \&c. \right) \\ &= B \frac{d^2 y_3}{d x^2} \text{ suppose} \end{aligned}$$

$$\text{then } y_3 = \frac{x^3}{2 \cdot 3} - \frac{1}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} m^2 x^9} + \frac{2 \cdot 3}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} m^4 x^{17}} - \&c.$$

$$\begin{aligned} \frac{d^3 \sqrt{x} y_3}{d x^3} &= \frac{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} x^{\frac{1}{2}}}{2 \cdot 3} + \frac{1}{m^2 x^{\frac{9}{2}}} - \frac{2 \cdot 3}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} m^4 x^{\frac{17}{2}}} + \&c. \\ &= \frac{35}{16} x^{\frac{1}{2}} + \frac{1}{m^2 x^{\frac{17}{2}}} \frac{d^2 y_3}{d x^2} \end{aligned}$$

$$\therefore \frac{d^3 y_3}{d x^3} + \left(\frac{3}{2x} - \frac{1}{m^2 x^6} \right) \frac{d^2 y_3}{d x^2} - \frac{3}{4x^2} \frac{d y_3}{d x} + \frac{3}{8x^3} y_3 = \frac{35}{16}$$

$$\begin{aligned} 4^\circ. \quad \frac{B_1}{x^{\frac{3}{2}}} + \frac{B_3}{x^{\frac{9}{2}}} + \frac{B_5}{x^{\frac{15}{2}}} + \&c. &= \frac{B \sqrt{\pi}}{m \sqrt{-1}} \left(\frac{\frac{1}{2}}{x^{\frac{3}{2}}} - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{4 \cdot 5 \cdot 6 m^2 x^{\frac{9}{2}}} + \&c. \right) \\ &= \frac{\sqrt{\pi}}{m \sqrt{-1}} B_1 \frac{d^2 y_4}{d x^2} \text{ suppose} \end{aligned}$$

$$\text{then } y_4 = -2 x^{\frac{1}{2}} - \frac{\frac{1}{2}}{4 \cdot 5 \cdot 6 m^2 x^{\frac{9}{2}}} + \&c.$$

$$\frac{d^3 \sqrt{x} y_4}{d x^3} = \frac{1}{m^2 x^7} - \&c. = \frac{1}{m^2 x^{\frac{17}{2}}} y_4 \text{ the same equation as for } y_2.$$

$$\begin{aligned} 5^\circ. \quad C + \frac{C_2}{x^6} + \frac{C_4}{x^{10}} + \&c. &= C \left(1 - \frac{3 \cdot 4}{\frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} m^2 x^6} + \&c. \right) \\ &= C \frac{d^2 y_5}{d x^2} \end{aligned}$$

$$\therefore y_5 = \frac{x^2}{1 \cdot 2} - \frac{1}{\frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} m^2 x^6} + \&c.$$

$$\begin{aligned} \frac{d^3 \sqrt{x} y_5}{d x^3} &= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2 \sqrt{x}} + \frac{1}{m^2 x^{\frac{17}{2}}} - \&c. \\ &= \frac{15}{8 \sqrt{x}} + \frac{1}{m^2 x^{\frac{17}{2}}} \frac{d^2 y_5}{d x^2} \end{aligned}$$

$$\frac{d^3 y_5}{d x^3} + \left(\frac{3}{2x} - \frac{1}{m^2 x^6} \right) \frac{d^2 y_5}{d x^2} - \frac{3}{4x^2} \frac{d y_5}{d x} + \frac{3}{8x^3} y_5 = \frac{15}{8x}$$

$$6^{\circ}. \quad \frac{C_1}{x^{\frac{3}{2}}} + \frac{C_3}{x^{\frac{13}{2}}} + \&c. = \frac{\sqrt{\pi} C}{m \sqrt{-1}} \left(\frac{1}{2 x^{\frac{5}{2}}} - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{11}{2} \cdot \frac{13}{2}}{2 \cdot 5 \cdot 6 \cdot 7} \frac{1}{m^2 x^{\frac{15}{2}}} + \&c. \right) \\ = \frac{\sqrt{\pi} C}{m \sqrt{-1}} \frac{d^2 y_6}{d x^2}$$

$$\therefore y_6 = \frac{1}{2 \sqrt{x}} - \frac{\frac{1}{2} \cdot \frac{3}{2}}{2 \cdot 5 \cdot 6 \cdot 7} \frac{1}{m^2 x^{\frac{11}{2}}} + \&c.$$

$$\frac{d^3 \sqrt{x} y_6}{d x^3} = \frac{\frac{1}{2} \cdot \frac{3}{2}}{2 m^2 x^8} - \&c. \\ = \frac{1}{m^2 x^{\frac{11}{2}}} \frac{d^2 y_6}{d x^2} \text{ the same as } y_2 ;$$

$$\text{and } y = A \left(\frac{d^2 y_1}{d x^2} + \frac{\sqrt{\pi}}{m \sqrt{-1}} \frac{d^2 y_2}{d x^2} \right) + B \left(\frac{d^2 y_3}{d x^2} + \frac{\sqrt{\pi}}{m \sqrt{-1}} \frac{d^2 y_4}{d x^2} \right) \\ + C \left(\frac{d^2 y_6}{d x^2} + \frac{\sqrt{\pi}}{m \sqrt{-1}} \frac{d^2 y_6}{d x^2} \right)$$

It is scarcely necessary to point out the analogy which exists between the differential equations which determine the value of the transcendentals in this and in the preceding examples.

14. We proceed now to exhibit a general solution of equations of this kind.

$$\text{Ex. 7. } y - m x^n \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0 ; n \text{ being any integer.}$$

The symbolical form of this equation is

$$y = \frac{1}{1 - m x^n d^{\frac{1}{2}}} \cdot 0 = \frac{1 + m x^n d^{\frac{1}{2}}}{1 - m^2 x^n d^{\frac{1}{2}} x^n d^{\frac{1}{2}}} 0 \\ = (1 + m x^n d^{\frac{1}{2}}) v = v + m x^n \frac{d^{\frac{1}{2}} v}{d x^{\frac{1}{2}}} \quad (1)$$

where v is determined by the equation

$$\frac{1}{1 - m^2 x^n d^{\frac{1}{2}} x^n d^{\frac{1}{2}}} 0 = v ; \text{ or } \\ v - m^2 x^n d^{\frac{1}{2}} x^n d^{\frac{1}{2}} v = 0 \text{ or } v - m^2 x^n \frac{d^{\frac{1}{2}}}{d x^{\frac{1}{2}}} \left(x^n \frac{d^{\frac{1}{2}} v}{d x^{\frac{1}{2}}} \right) = 0 \quad (2)$$

$$\text{Let } v = \frac{d^{n-1} z}{d x^{n-1}} ; \text{ then } \frac{d^{\frac{1}{2}} v}{d x^{\frac{1}{2}}} = \frac{d^{n-\frac{1}{2}} z}{d x^{n-\frac{1}{2}}}.$$

$$\therefore \frac{d^{\frac{1}{2}}}{d x^{\frac{1}{2}}} \left(x^n \frac{d^{\frac{1}{2}} v}{d x^{\frac{1}{2}}} \right) = x^n \frac{d^n z}{d x^n} + \frac{1}{2} n x^{n-1} \frac{d^{n-1} z}{d x^{n-1}} \\ - \frac{1 \cdot 1}{2 \cdot 4} n (n-1) x^{n-2} \frac{d^{n-2} z}{d x^{n-2}} + \&c. + (-1)^{n-1} \frac{1 \cdot 1 \cdot 3 \dots (2n-3)}{2 \cdot 4 \dots 2n} n (n-1) \dots 1 \cdot z$$

(Part I. Art. 11.)

By substituting this in equation (2) we obtain

$$\begin{aligned} & \frac{d^{n-1}z}{dx^{n-1}} - m^2 x^n \left(x^n \frac{d^n z}{dx^n} + \frac{1}{2} n x^{n-1} \frac{d^{n-1}z}{dx^{n-1}} + \&c. \right) = 0 \\ \text{or} \quad & \frac{d^n z}{dx^n} + \left(\frac{n}{2x} - \frac{1}{m^2 x^{2n}} \right) \frac{d^{n-1}z}{dx^{n-1}} - \frac{1 \cdot 1}{2 \cdot 4} \frac{n(n-1)}{x^2} \frac{d^{n-2}z}{dx^{n-2}} \\ & + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \frac{n(n-1)(n-2)}{x^3} \frac{d^{n-3}z}{dx^{n-3}} - \&c. + (-1)^{n-1} \frac{1 \cdot 1 \cdot 3 \dots (2n-3)}{2 \cdot 4 \cdot 6 \dots 2n} \\ & \cdot \frac{n(n-1) \dots 1}{x^n} z = 0 \end{aligned}$$

When z has been determined from this equation, we shall have the complete value of y by means of Equation (1.), viz.

$$y = \frac{d^{n-1}z}{dx^{n-1}} + m x^n \frac{d^{n-\frac{1}{2}}z}{dx^{n-\frac{1}{2}}}$$

COR. If $n=3$;

$$\frac{d^3 z}{dx^3} + \left(\frac{3}{2x} - \frac{1}{m^2 x^6} \right) \frac{d^2 z}{dx^2} - \frac{3}{4x^2} \frac{dz}{dx} + \frac{3}{8x^3} z = 0;$$

which is the same equation as that which we obtained by a totally different process for determining y_2 and y_4 in Ex. 6.

Ex. 8.
$$y - m x^n \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = X$$

The solution is

$$\begin{aligned} y &= \frac{1}{1 - m x^n d^{\frac{1}{2}}} (X + 0) = \frac{1 + m x^n d^{\frac{1}{2}}}{1 - m^2 x^n d^{\frac{1}{2}} x^n d^{\frac{1}{2}}} (X + 0) \\ &= (1 + m x^n d^{\frac{1}{2}}) (v + w) \\ &= v + m x^n \frac{d^{\frac{1}{2}}v}{dx^{\frac{1}{2}}} + w + m x^n \frac{d^{\frac{1}{2}}w}{dx^{\frac{1}{2}}} \end{aligned}$$

where v is the same as in the last Example, and w is determined from the equation

$$w - m^2 x^n \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left(\frac{x^n d^{\frac{1}{2}}w}{dx^{\frac{1}{2}}} \right) = X$$

or by writing $\frac{d^{n-1}u}{dx^{n-1}}$ for w , and proceeding as in the last Example,

$$\frac{d^{n-1}u}{dx^{n-1}} - m^2 x^n \left(x^n \frac{d^n u}{dx^n} + \frac{n x^{n-1}}{2} \frac{d^{n-1}u}{dx^{n-1}} + \&c. \right) = X, \text{ or}$$

$$\frac{d^n u}{d x^n} + \left(\frac{n}{2x} - \frac{1}{m^2 x^{2n}} \right) \frac{d^{n-1} u}{d x^{n-1}} - \frac{1 \cdot 1}{2 \cdot 4} \frac{n(n-1)}{x^2} \frac{d^{n-2} u}{d x^{n-2}} + \&c.$$

$$+ (-1)^{n-1} \frac{1 \cdot 1 \cdot 3 \cdot \dots \cdot (2n-3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \cdot \frac{n(n-1) \cdot \dots \cdot 1}{x^n} u = - \frac{X}{m^2 x^{2n}}$$

COR. 1. If $n=1$, the equation for determining u is

$$\frac{du}{dx} + \left(\frac{1}{2x} - \frac{1}{m^2 x^2} \right) u = - \frac{X}{m^2 x^2},$$

of which the solution is
$$u = - \frac{e^{-\frac{1}{m^2 x}}}{\sqrt{x}} \int \frac{e^{\frac{1}{m^2 x}} X}{m^2 x^{\frac{3}{2}}} dx = v$$

$$\therefore v = \frac{A e^{-\frac{1}{m^2 x}}}{\sqrt{x}}$$

and
$$y = \left(1 + m x \frac{d^{\frac{1}{2}}}{d x^{\frac{1}{2}}} \right) \left(\frac{A e^{-\frac{1}{m^2 x}}}{\sqrt{x}} - e^{-\frac{1}{m^2 x}} \int \frac{e^{\frac{1}{m^2 x}} X dx}{m^2 x^{\frac{3}{2}}} \right)$$

COR. 2. If $n=1$, $X = \frac{b}{\sqrt{x}}$, it is evident that

$$u = \frac{b}{\sqrt{x}} \quad \therefore \quad y = y_0 + \frac{b}{\sqrt{x}} + \frac{m b \sqrt{-1}}{\sqrt{\pi}}$$

where y_0 is the solution of the equation without X (Ex. 3.)

It appears, therefore, that the complete solution of equations of this form is reduced to the solution of ordinary linear equations, and the determination of the half differential coefficient of the results.

Ex. 9. $y - m x^n \frac{d^{r+\frac{1}{2}} y}{d x^{r+\frac{1}{2}}} = X$, where n and r are any whole numbers.

We have
$$y = \frac{X+0}{1-m x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}}} = \frac{1+m x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}}}{1-m^2 x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}} x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}}} \cdot (X+0)$$

$$= \left(1 + m x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}} \right) (v+w) \quad \dots \quad (1)$$

where $v+w$ is the solution of the equation

$$\left(1 - m^2 x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}} x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}} \right) (v+w) = X+0.$$

Now
$$\frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}} x^n \frac{d^{r+\frac{1}{2}}}{d x^{r+\frac{1}{2}}} v = x^n \frac{d^{2r+1}}{d x^{2r+1}} v + (r+\frac{1}{2}) n x^{n+1} \frac{d^{2r}}{d x^{2r}} v$$

$$+ \frac{(r+\frac{1}{2})(r-\frac{1}{2})}{1 \cdot 2} n(n-1) x^{n-2} \frac{d^{2r-1}}{d x^{2r-1}} v - \&c.$$

$$\begin{aligned}
& + \frac{(r+\frac{1}{2})(r-\frac{1}{2}) \dots n(n-1) \dots 1}{1 \cdot 2 \dots n} \cdot \frac{d^{2r-n+1} v}{d x^{2r-n+1}} \\
& = x^n \frac{d^n z}{d x^n} + (r+\frac{1}{2}) n x^{n-1} \frac{d^{n-1} z}{d x^{n-1}} + \dots \\
& + \frac{(r+\frac{1}{2})(r-\frac{1}{2}) \dots (r-2n+\frac{5}{2})}{1 \cdot 2 \dots n} \cdot n(n-1) \dots 1 \cdot z
\end{aligned}$$

where

$$v = \frac{d^{n-(2r+1)} z}{d x^{n-(2r+1)}}$$

\therefore the equation for determining v is

$$\begin{aligned}
& \frac{d^n z}{d x^n} + \frac{(r+\frac{1}{2}) n}{x} \frac{d^{n-1} z}{d x^{n-1}} + \frac{(r+\frac{1}{2})(r-\frac{1}{2})}{1 \cdot 2} \cdot \frac{n(n-1)}{x^2} \frac{d^{n-2} z}{d x^{n-2}} + \&c. \\
& + \frac{(r+\frac{1}{2})(r-\frac{1}{2}) \dots (r-2n+\frac{5}{2})}{1 \cdot 2 \dots n} \cdot n(n-1) \dots 1 \cdot z \\
& - \frac{1}{m^2 x^{2n}} \frac{d^{n-(2r+1)} v}{d x^{n-(2r+1)}} = - \frac{1}{m^2 x^{2n}} X \quad \dots \quad (2)
\end{aligned}$$

w is the particular value of v corresponding with $X=0$. Having thus obtained v and w , equation (1) gives the complete value of y . It must be observed, that the transformation from v to z is only to be made when n is greater than $2r+1$.

CLASS III. *Equations which are capable of solution by transformation, without division of operations.*

15. Ex. 1. $y - m x^{\frac{3}{2}} \frac{d^{\frac{3}{2}} y}{d x^{\frac{3}{2}}} = 0$

By (C) this equation is transformed into

$$y - m (-1)^{\frac{3}{2}} \frac{\sqrt{-D + \frac{3}{2}}}{\sqrt{-D}} y = 0, \text{ or}$$

$$y = \left(1 + m \sqrt{-1} \frac{\sqrt{-D + \frac{3}{2}}}{\sqrt{-D}} \right)^{-1} \cdot 0.$$

Hence, as in Ex. 1, Class 2, the value of y is $y = \frac{A}{x^n}$, where n is determined

by the equation $1 + m \sqrt{-1} \frac{\sqrt{n + \frac{3}{2}}}{\sqrt{n}} = 0$.

COR. If $m = -\frac{4}{3\sqrt{\pi}\sqrt{-1}}$, $\sqrt{n + \frac{3}{2}} = \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \sqrt{n} = \frac{\sqrt{3}}{2} \sqrt{n}$

$$\therefore n = 1; \quad \text{and} \quad y = \frac{A}{x}.$$

Ex. 2.
$$y - m x^{\frac{3}{2}} \frac{d^{\frac{3}{2}} y}{d x^{\frac{3}{2}}} = X.$$

This gives
$$y = \frac{A}{x^n} + \left(1 + m \sqrt{-1} \frac{\sqrt{-D + \frac{3}{2}}}{\sqrt{-D}}\right)^{-1} \cdot X$$

$$= \frac{A}{x^n} + \sum b_r \left(1 + m \sqrt{-1} \frac{\sqrt{r + \frac{3}{2}}}{\sqrt{r}}\right)^{-1} e^{-r \theta}$$

COR. If $r=n$, this expression must be reduced, as in Ex. 2, Class 2, to

$$y = \frac{A}{x^n} + \frac{b_r}{m \sqrt{-1}} \frac{\log x}{x^n} \frac{1}{\frac{d}{d n} \frac{1}{\sqrt{n + \frac{3}{2}}}} + \sum b_s \left(1 + m \sqrt{-1} \frac{\sqrt{s + \frac{3}{2}}}{\sqrt{s}}\right)^{-1} \cdot \frac{1}{x^s}$$

COR. 2. As a particular case, the solution of

$$\frac{d^{\frac{3}{2}} y}{d x^{\frac{3}{2}}} + \frac{3}{4} \sqrt{-1} \sqrt{\pi} \frac{y}{x^{\frac{3}{2}}} = \frac{b}{x^{\frac{3}{2}}} \text{ is}$$

$$y = \frac{A}{x} - \frac{8 b}{9 \sqrt{-1} \sqrt{\pi}} \cdot \frac{1}{x^2}$$

These equations might have been included in the preceding Class, to which, both in their form and in the mode of their solution, they are very analogous. They are, however, particular cases of Example 5, below, which does not belong to that Class.

Ex. 3.
$$y + a \sqrt{x} \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} + b x \frac{d y}{d x} = 0.$$

The equation in θ is (by C),

$$y + a \sqrt{-1} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}} y - b \frac{\sqrt{-D + 1}}{\sqrt{-D}} \cdot y = 0$$

or
$$\left\{1 + a \sqrt{-1} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}} - b \frac{\sqrt{-D + 1}}{\sqrt{-D}}\right\} \cdot y = 0$$

Suppose $y = \sum a_n e^{-n \theta}$; then

$$\sum a_n \left\{1 + a \sqrt{-1} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}} - b \frac{\sqrt{n + 1}}{\sqrt{n}}\right\} \cdot e^{-n \theta} = 0 \text{ by (A)}$$

Hence any value of n which will satisfy the equation

$$1 + a \sqrt{-1} \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n}} - b \frac{\sqrt{n + 1}}{\sqrt{n}} = 0$$

will give a term in the solution.

COR. 1. If $a \sqrt{-1} = -\frac{2 \sqrt{\pi}}{4 - \pi}$, $b = \frac{4 - 2 \pi}{4 - \pi}$ we have

$$4 - \pi - n(4 - 2\pi) - 2\sqrt{\pi} \frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n}} = 0$$

which is satisfied by $n = \frac{1}{2}$ and $n = 1$.

Hence
$$y = \frac{A}{\sqrt{x}} + \frac{B}{x}.$$

COR. 2. If n be a whole number r ; $\sqrt{n} = 1.2 \dots (r-1)$

$$\text{and } \sqrt{n+\frac{1}{2}} = \frac{1}{2} \cdot \frac{3}{2} \dots (r-\frac{1}{2}) \sqrt{\pi}$$

$$\therefore 1.2 \dots (r-1) + a\sqrt{-1} \sqrt{\pi} \frac{1.3 \dots (2r-1)}{2^r} - b.1.2 \dots r = 0;$$

will determine the integral values of n .

If $n = r + \frac{1}{2}$, $\sqrt{n} = \frac{1.3 \dots (2r-1)}{2^r} \sqrt{\pi}$, $\sqrt{n+\frac{1}{2}} = 1.2 \dots r$

and
$$\frac{1.3 \dots (2r-1)}{2^r} \sqrt{\pi} + a\sqrt{-1}.1.2 \dots r - b \frac{1.3 \dots (2r+1)}{2^{r+1}} \sqrt{\pi} = 0,$$

which determines the fractional values of n which have 2 as their denominator.

Now it is evident that these are the only forms which n can assume; therefore the determination of the values of n is reduced to the solution of these two equations.

Ex. 4.
$$y + a\sqrt{x} \frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} + b x \frac{dy}{dx} = X.$$

Let $X = \sum b_r e^{-r\theta}$, then

$$\begin{aligned} y &= \sum a_n e^{-n\theta} + \sum b_r \left(1 + a\sqrt{-1} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} - b \frac{\sqrt{-D+1}}{\sqrt{-D}} \right)^{-1} \cdot e^{-r\theta} \\ &= \sum a_n e^{-n\theta} + \sum \frac{b_r x^{-r}}{1 + a\sqrt{-1} \frac{\sqrt{r+\frac{1}{2}}}{\sqrt{r}} - b \frac{\sqrt{r+1}}{\sqrt{r}}} \text{ by (A)} \end{aligned}$$

the values of n being determined as in Example 3.

COR. If $r=p$, $n=p$, we obtain, as in other instances,

$$y = \sum \frac{a_n}{x^n} + C \frac{\log x}{x^p} + \sum \frac{b_s x^{-s}}{1 - a\sqrt{-1} \frac{\sqrt{s+\frac{1}{2}}}{\sqrt{s}} - b \frac{\sqrt{s+1}}{\sqrt{s}}}$$

where

$$C = - \frac{1}{a\sqrt{-1} \frac{d}{dp} \frac{\sqrt{p+\frac{1}{2}}}{\sqrt{p}} - b}$$

Ex. 5.
$$x^m \frac{d^m y}{dx^m} + a x^{m-\frac{1}{2}} \frac{d^{m-\frac{1}{2}} y}{dx^{m-\frac{1}{2}}} + \&c. = X.$$

The equation in θ is

$$\left\{ (-1)^m \frac{\sqrt{-D+m}}{\sqrt{-D}} + a (-1)^{m-\frac{1}{2}} \frac{\sqrt{-D+m-\frac{1}{2}}}{\sqrt{-D}} + \&c. \right\} y = X$$

which may be written $f(-D)y = X$;

and

$$y = \{f(-D)\}^{-1} \cdot 0 + \{f(-D)\}^{-1} \cdot X$$

$$= \sum a_n x^{-n} + \sum \frac{b_r x^{-r}}{f(r)}$$

the values of n being determined by the equation $f(n)=0$.

$$\text{Ex. 6.} \quad (ax + \beta)^m \frac{d^m y}{dx^m} + a(ax + \beta)^{m-\frac{1}{2}} \frac{d^{m-\frac{1}{2}} y}{dx^{m-\frac{1}{2}}} + \&c. = X.$$

$$\text{Let } x' = ax + \beta, \text{ then } \frac{d^m y}{dx^m} = a^m \frac{d^m y}{dx'^m} \quad (\text{Part 1, Art. 27.})$$

$$\&c. = \&c.$$

$$\therefore a^m x'^m \frac{d^m y}{dx'^m} + a a^{m-\frac{1}{2}} x'^{m-\frac{1}{2}} \frac{d^{m-\frac{1}{2}} y}{dx'^{m-\frac{1}{2}}} + \&c. = X'$$

which coincides with Example 5.

$$\text{Ex. 7.} \quad \frac{dy}{dx} - a \cdot \frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} - \frac{1}{2} \frac{y}{x} = 0$$

By multiplying by x and reducing to differentials in θ , we get

$$\frac{\sqrt{-D+1}}{\sqrt{-D}} y + a \sqrt{x} (-1)^{\frac{1}{2}} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} y + \frac{1}{2} y = 0$$

$$\left(\frac{\sqrt{-D+1}}{\sqrt{-D}} + \frac{1}{2} \right) y + a (-1)^{\frac{1}{2}} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}} y = 0$$

$$\text{or} \quad (-D + \frac{1}{2}) y + a (-1)^{\frac{1}{2}} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}} y = 0$$

$$\text{or} \quad y + a (-1)^{\frac{1}{2}} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}} y = 0$$

$$\text{or} \quad y + a (-1)^{\frac{1}{2}} e^{\frac{3\theta}{x}} \frac{\sqrt{-D-\frac{1}{2}}}{\sqrt{-D}} \frac{y}{e^{\frac{\theta}{x}}} = 0$$

$$\text{or} \quad \frac{y}{x} - a (-1)^{-\frac{1}{2}} x^{\frac{1}{2}} \frac{\sqrt{-D-\frac{1}{2}}}{\sqrt{-D}} \frac{y}{x} = 0$$

$$\text{or} \quad \frac{y}{x} - a \cdot \frac{d^{-\frac{1}{2}} y}{dx^{-\frac{1}{2}}} = 0.$$

If $\frac{y}{x} = v$; this gives

$$v - a \frac{d^{-\frac{1}{2}} v}{d x^{-\frac{1}{2}}} = 0,$$

whence

$$v = A e^{a^2 x}$$

and

$$y = v x = A x e^{a^2 x}.$$

This equation may be integrated in the following manner. The equation

$$y + a \sqrt{-1} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{3}{2}}} e^{\frac{\theta}{2}} y = 0,$$

may be made to depend on the equation

$$v + a \sqrt{-1} \frac{\sqrt{-D}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{2}} v = 0$$

by the relation

$$y = P_{\frac{1}{2}} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{3}{2}}} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} v, \text{ where}$$

$$P_{\frac{1}{2}} f(D) = f(D) f(D - \frac{1}{2}) f(D - 1) \&c. \dots\dots$$

$$\begin{aligned} \therefore y &= P_{\frac{1}{2}} \frac{D}{D - \frac{1}{2}} v \\ &= \frac{D(D - \frac{1}{2})(D - 1)\dots\dots}{(D - \frac{1}{2})(D - 1)\dots\dots} v \\ &= D v \\ &= x \frac{d v}{d x} \end{aligned}$$

Now $v + a \sqrt{-1} \frac{\sqrt{-D}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{2}} v = 0$ is equivalent, by (D), to

$$v + a \sqrt{-1} e^{\frac{\theta}{2}} \frac{\sqrt{-D-\frac{1}{2}}}{\sqrt{-D}} v = 0$$

or

$$v - a \frac{d^{-\frac{1}{2}} v}{d x^{-\frac{1}{2}}} = 0$$

whence

$$v = A_1 e^{a^2 x}$$

\therefore

$$y = A x e^{a^2 x} \text{ the same result as before.}$$

This process, which is due to Mr BOOLE, is of great importance in the solution of certain classes of ordinary linear equations, but I have not, as yet, found it very extensively applicable to equations with fractional indices.

Ex. 8. *More generally, to investigate the conditions of integrability of the equation*

$$x \frac{d y}{d x} - c y + a x^{n+\frac{1}{2}} \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0$$

The symbolical form is

$$\left(-\frac{\sqrt{-D+1}}{\sqrt{-D}}-c\right)y+a\sqrt{-1}e^{n\theta}\frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}}\cdot y=0,$$

or
$$-(-D+c)y+a\sqrt{-1}\frac{\sqrt{-D+n+\frac{1}{2}}}{\sqrt{-D+n}}\cdot e^{n\theta}y=0, \text{ by (D).}$$

This is reducible,

1. When $c=n-\frac{1}{2}$; and it becomes, by dividing, by $-D+n-\frac{1}{2}$,

$$y-a\sqrt{-1}\frac{\sqrt{-D+n-\frac{1}{2}}}{\sqrt{-D+n}}\cdot e^{n\theta}y=0$$

or
$$a\sqrt{-1}e^{n\theta}y-\frac{\sqrt{-D+n}}{\sqrt{-D+n-\frac{1}{2}}}\cdot y=0$$

or
$$a\sqrt{-1}y-e^{-\frac{\theta}{2}}\frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}}\cdot e^{-(n-\frac{1}{2})\theta}y=0$$

or
$$ay+\frac{d^{\frac{1}{2}}yx^{-(n-\frac{1}{2})}}{dx^{\frac{1}{2}}}=0$$

If $yx^{-(n-\frac{1}{2})}=v$, this equation becomes

$$avx^{n-\frac{1}{2}}+\frac{d^{\frac{1}{2}}v}{dx^{\frac{1}{2}}}=0, \text{ or}$$

$$av+x^{-n+\frac{1}{2}}\frac{dv}{dx}=0$$

which is integrable when $n=\frac{1}{2}, 0, -\frac{1}{2}, -1$, &c. (Class. 2.)

2. When $c=n$, the equation becomes

$$-y+a\sqrt{-1}\frac{\sqrt{-D+n+\frac{1}{2}}}{\sqrt{-D+n+1}}\cdot e^{n\theta}y=0$$

or
$$a\sqrt{-1}e^{n\theta}y-\frac{\sqrt{-D+n+1}}{\sqrt{-D+n+\frac{1}{2}}}\cdot y=0$$

or
$$a\sqrt{-1}y-e^{\frac{\theta}{2}}\frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}}\cdot e^{-(n+\frac{1}{2})\theta}y=0$$

or
$$ay+x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}x^{n+\frac{1}{2}}}=0.$$

If $\frac{y}{x^{n+\frac{1}{2}}}=v$, this equation becomes

$$av+x^{-n+\frac{1}{2}}\frac{dv}{dx}=0; \text{ the same as before.}$$

Ex. 9.

$$x\frac{dy}{dx}-ax^3\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}}-\frac{3}{2}y=0.$$

The symbolical form is

$$\frac{\sqrt{-D+1}}{\sqrt{-D}} y + a(-1)^{\frac{3}{2}} x^{\frac{3}{2}} \frac{\sqrt{-D+\frac{3}{2}}}{\sqrt{-D}} y + \frac{3}{2} y = 0,$$

or
$$(-D + \frac{3}{2}) y + a(-1)^{\frac{3}{2}} \frac{\sqrt{-D+3}}{\sqrt{-D+\frac{3}{2}}} e^{\frac{3\theta}{2}} y = 0,$$

or
$$y + a(-1)^{\frac{3}{2}} \frac{\sqrt{-D+3}}{\sqrt{-D+\frac{3}{2}}} e^{\frac{3\theta}{2}} y = 0,$$

or
$$y + a(-1)^{\frac{3}{2}} e^{\frac{5\theta}{2}} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} \frac{y}{e^{\theta}} = 0$$

or
$$y - a e^{2\theta} \frac{d^{\frac{1}{2}} \frac{y}{x}}{d x^{\frac{1}{2}}} = 0$$

or
$$\frac{y}{x} - a x \frac{d^{\frac{1}{2}} \frac{y}{x}}{d x^{\frac{1}{2}}} = 0$$

$$y = A \sqrt{x} e^{-\frac{1}{a^2 x}} \left\{ \frac{1}{2} \int \frac{e^{\frac{a^2 x}{2}}}{\sqrt{x}} dx - \frac{\sqrt{\pi} \sqrt{-1}}{a} \right\} \quad (\text{Ex. 3, Class 2.})$$

Ex. 10.
$$y + a x \frac{dy}{dx} + b x^n \left(\frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} + c x \frac{d^{\frac{3}{2}} y}{d x^{\frac{3}{2}}} \right) = 0.$$

The symbolical form of the equation is

$$y - a \frac{\sqrt{-D+1}}{\sqrt{-D}} y + b e^{(n-\frac{1}{2})\theta} \sqrt{-1} \left(\frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} - c \frac{\sqrt{-D+\frac{3}{2}}}{\sqrt{-D}} \right) \cdot y = 0$$

or
$$(1 + a D) y - b \sqrt{-1} e^{(n-\frac{1}{2})\theta} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} \left(1 + c D - \frac{c}{2} \right) \cdot y = 0 \quad (1).$$

This equation may be reduced in several instances :

A. If $c = \frac{2}{3}$ the equation becomes

$$(1 + a D) y - b \sqrt{-1} \frac{2}{3} e^{(n-\frac{1}{2})\theta} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} \cdot y = 0,$$

or
$$(1 + a D) y - \frac{2}{3} b \sqrt{-1} \frac{\sqrt{-D+n}}{\sqrt{-D+n-\frac{3}{2}}} e^{(n-\frac{1}{2})\theta} y = 0 \quad \text{by (D.)}$$

or
$$y + \frac{2b}{3a} \sqrt{-1} \frac{\sqrt{-D+n}}{(-D-\frac{1}{a}) \sqrt{-D+n-\frac{3}{2}}} \cdot e^{(n-\frac{1}{2})\theta} y = 0. \quad (2.)$$

1. If $\frac{1}{a} = 1 - n$, equation (2) is reduced to

$$y + \frac{2b}{3a} \sqrt{-1} \frac{\sqrt{-D+n-1}}{\sqrt{-D+n-\frac{3}{2}}} \cdot e^{(n-\frac{1}{2})\theta} y = 0;$$

which is equivalent to $y + \frac{2b}{3a} \sqrt{-1} e^{(n-\frac{3}{2})\theta} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} \cdot xy=0$

or $xy + \frac{2b}{3a} x^n \frac{d^{\frac{1}{2}} xy}{dx^{\frac{1}{2}}} = 0$, or, if $v=xy$,

$v + \frac{2b}{3a} x^n \frac{dv}{dx^{\frac{1}{2}}} = 0$, which is the form integrated in Class 1.

2. If $\frac{1}{a} = \frac{3}{2} - n$, equation (2) becomes

$$y + \frac{2b}{3a} \sqrt{-1} \frac{\sqrt{-D+n}}{\sqrt{-D+n-\frac{1}{2}}} \cdot e^{(n-\frac{1}{2})\theta} y = 0$$

which is equivalent to $y + \frac{2b}{3a} \sqrt{-1} e^{(n-\frac{1}{2})\theta} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} \cdot y = 0$

or $y + \frac{2b}{3a} x^n \frac{dy}{dx^{\frac{1}{2}}} = 0$

which is of the same form as in the last case.

• B. If c is not equal to $\frac{2}{3}$, we have from equation (1) by (D.)

$$(1+aD)y + b\sqrt{-1} \frac{\sqrt{-D+n}(1+cD-cn)}{\sqrt{-D+n-\frac{1}{2}}} \cdot e^{(n-\frac{1}{2})\theta} y = 0$$

$$y + b\sqrt{-1} \frac{1+cD-cn}{1+aD} \frac{\sqrt{-D+n}}{\sqrt{-D+n-\frac{1}{2}}} e^{(n-\frac{1}{2})\theta} y = 0$$

3. If $\frac{c}{1-cn} = a$ this gives

$$y + b\sqrt{-1} (1-cn) e^{(n-\frac{1}{2})\theta} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} \cdot y = 0$$

or $y + b(1-cn)x^n \frac{dy}{dx^{\frac{1}{2}}} = 0$, the same form as before.

16. It would be improper to dismiss this equation without remarking the fact that it would appear to have been solved by M. BESGE in LIOUVILLE'S *Journal* 1844, ix., 294. The solution is, however, given without any demonstration, and is, if I mistake not, rather a differential equation *formed* than a differential equation *solved*. The *whole* which appears is as follows :

“ Let m, n, p, q be functions of x , and $\frac{d^2 y}{dx^2} + m \frac{dy}{dx} + n \frac{dy}{dx^{\frac{1}{2}}} + p y = q$, the proposed equation.

“ If we have $\frac{dm}{dx} + mn - p = 0$, the given equation can be reduced to the following, $\frac{d^2 y}{dx^2} + m y = z$, where z is obtained from the equation $\frac{dz}{dx} + n z = q$.”

Now, on examination, it appears that the proposed equation is nothing more than the differential coefficient of the quantity $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + my - z = 0$ added to n times the quantity itself: Thus,

$$\frac{d}{dx} \left(\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + my - z \right) + n \left(\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + my - z \right) = 0$$

gives

$$\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}} + m \frac{dy}{dx} + y \frac{dm}{dx} - \frac{dz}{dx} + n \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + mny - nz = 0$$

or

$$\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}} + m \frac{dy}{dx} + n \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + py = \frac{dz}{dx} + nz - \left(\frac{dm}{dx} + mn - p \right) y = q$$

$$\text{provided } \frac{dz}{dx} + nz = q \text{ and } \frac{dm}{dx} + mn - p = 0.$$

Thus it appears that the equation is not *solved* but *formed*: and this is probably all M. BESGE intends. How he can justify his additional remark, that $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + my = z$ can be solved if m is a constant, or a linear function of x , I am unable to conjecture.

CLASS 4. *Equations which are capable of solution by the division of operations.*

17. We have already met with several equations in Class 1, where the total operation was found to be equivalent to the product of two or more partial operations; and in Art. 9 we have pointed out the manner in which the partial operations are applied, viz., by decomposing the total operation in exactly the same way as an ordinary fraction is decomposed into partial fractions.

Ex. 1. $y + axy + bx \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + 2ax^2 \frac{dy}{dx} = 0.$

This equation, when reduced to the symbolical form, is

$$y + ae^{\theta}y + be^{\frac{\theta}{2}}\sqrt{-1} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}}y - 2ae^{\theta} \frac{\sqrt{-D+1}}{\sqrt{-D}}y = 0$$

or $y + b\sqrt{-1} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} \cdot e^{\frac{\theta}{2}}y - 2a \left(\frac{\sqrt{-D+2}}{\sqrt{-D+1}} - \frac{1}{2} \right) \cdot e^{\theta}y = 0$ by (D).

Now $\frac{\sqrt{-D+2}}{\sqrt{-D+1}} - \frac{1}{2} = -D + \frac{1}{2} = \frac{\sqrt{-D+\frac{3}{2}}}{\sqrt{-D+\frac{1}{2}}} = \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} \cdot \frac{\sqrt{-D+\frac{3}{2}}}{\sqrt{-D+1}}$

$\therefore \left(\frac{\sqrt{-D+2}}{\sqrt{-D+1}} - \frac{1}{2} \right) e^{\theta}y = \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{2}} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} \cdot e^{\frac{\theta}{2}}y$ by (D)

and the equation is reduced to

$$y + b \sqrt{-1} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} \cdot e^{\frac{\theta}{x}} y - 2a \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}} \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}} \cdot y = 0.$$

Let us abbreviate the operation $\frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}}$ by ϕ , and the equation becomes

$$(1 + b \sqrt{-1} \phi - 2a \phi^2) \cdot y = 0.$$

If $1 + b \sqrt{-1} z - 2a z^2 = (1 + \alpha z)(1 + \beta z)$; this equation is equivalent to

$$(1 + \alpha \phi)(1 + \beta \phi)y = 0$$

or
$$y = \frac{1}{(1 + \alpha \phi)(1 + \beta \phi)} \cdot 0 = \frac{\alpha}{\alpha - \beta} \cdot \frac{1}{1 + \alpha \phi} 0 - \frac{\beta}{\alpha - \beta} \frac{1}{1 + \beta \phi} \cdot 0$$

Now
$$1 + \alpha \phi = 1 - a \frac{\sqrt{-D+1}}{\sqrt{-D+\frac{1}{2}}} e^{\frac{\theta}{x}} = 1 + a e^{\frac{\theta}{x}} \frac{\sqrt{-D+1}}{\sqrt{-D}}$$
$$= 1 - a \sqrt{-1} x \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$$

Hence the solution of the given equation is reduced to the solution of the two equations

$$y_1 + a e^{\frac{\theta}{x}} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} y_1 = 0, \quad y_2 + \beta e^{\frac{\theta}{x}} \frac{\sqrt{-D+\frac{1}{2}}}{\sqrt{-D}} y_2 = 0$$

or
$$y_1 - a \sqrt{-1} x \frac{d^{\frac{1}{2}} y_1}{dx^{\frac{1}{2}}} = 0, \quad y_2 - \beta \sqrt{-1} x \frac{d^{\frac{1}{2}} y_2}{dx^{\frac{1}{2}}} = 0$$

Now these equations have been solved in Class 2, Ex. 3, and they give

$$y_1 = \frac{A e^{\frac{1}{\alpha^2 x}}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\alpha} \right\}$$

$$y_2 = \frac{B e^{\frac{1}{\beta^2 x}}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\beta^2 x}}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\beta} \right\}$$

and

$$y = \frac{1}{(1 - \alpha \phi)(1 - \beta \phi)} \cdot 0 = \frac{\alpha}{\alpha - \beta} \frac{1}{1 - \alpha \phi} \cdot 0 - \frac{\beta}{\alpha - \beta} \frac{1}{1 - \beta \phi} \cdot 0$$
$$= \frac{A \alpha}{\alpha - \beta} \frac{e^{\frac{1}{\alpha^2 x}}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\alpha} \right\}$$
$$- \frac{B \beta}{\alpha - \beta} \frac{e^{\frac{1}{\beta^2 x}}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\beta^2 x}}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\beta} \right\}$$

It will be readily seen that B is not an arbitrary constant, independent of A.

(See Art. 18.) For when $b=0$, the equation becomes an ordinary linear equation

of the first degree, of which the solution is $y = C \frac{1}{e^{\frac{2}{\alpha} x} \sqrt{x}}$.

In this case $\alpha = \beta$ and $A = -B$:

we may therefore write $B = -A$ generally, and we obtain as the complete solution

$$y = \frac{A e^{\frac{1}{\alpha^2} x}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\alpha^2} x}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\alpha} \right\} - \frac{A e^{\frac{1}{\beta^2} x}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\beta^2} x}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\beta} \right\}$$

The above equation may be reduced differently, thus. The symbolical form

$$y + a e^{\theta} y + b e^{\frac{\theta}{2}} \sqrt{-1} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}} y - 2 a e^{\theta} \frac{\sqrt{-D + 1}}{\sqrt{-D}} y = 0,$$

may be written

$$2 a (-D - \frac{1}{2}) y - b \sqrt{-1} e^{-\frac{\theta}{2}} \frac{\sqrt{-D + \frac{1}{2}}}{\sqrt{-D}} y - \frac{1}{2 a} e^{-\theta} y = 0,$$

or

$$y - \frac{b \sqrt{-1}}{2 a} \frac{\sqrt{-D}}{\sqrt{-D + \frac{1}{2}}} e^{-\frac{\theta}{2}} y - \frac{1}{2 a} \frac{1}{-D - \frac{1}{2}} e^{-\theta} y = 0,$$

or

$$y - \frac{b \sqrt{-1}}{2 a} \frac{\sqrt{-D}}{\sqrt{-D + \frac{1}{2}}} e^{-\frac{\theta}{2}} y - \frac{1}{2 a} \frac{\sqrt{-D}}{\sqrt{-D + \frac{1}{2}}} \frac{\sqrt{-D - \frac{1}{2}}}{\sqrt{-D}} e^{-\theta} y = 0,$$

or

$$y - \frac{b \sqrt{-1}}{2 a} \frac{\sqrt{-D}}{\sqrt{-D + \frac{1}{2}}} e^{-\frac{\theta}{2}} y - \frac{1}{2 a} \frac{\sqrt{-D}}{\sqrt{-D + \frac{1}{2}}} e^{-\frac{\theta}{2}} \frac{\sqrt{-D}}{\sqrt{-D + \frac{1}{2}}} e^{-\theta} y = 0,$$

which is of the form $(1 - \frac{b \sqrt{-1}}{2 a} \phi_1 - \frac{1}{2 a} \phi_1^2) y = 0$;

of which the solutions are

$$(1 + \frac{1}{\alpha} \phi_1) y = 0, \text{ and } (1 + \frac{1}{\beta} \phi_1) y = 0, \text{ or}$$

$$y + \frac{1}{\alpha} e^{\frac{\theta}{2}} \frac{\sqrt{-D - \frac{1}{2}}}{\sqrt{-D}} e^{-\theta} y = 0, \text{ and } y + \frac{1}{\beta} e^{\frac{\theta}{2}} \frac{\sqrt{-D - \frac{1}{2}}}{\sqrt{-D}} e^{-\theta} y = 0,$$

or

$$y - \frac{1}{\alpha \sqrt{-1}} \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \frac{y}{x} = 0, \text{ and } y - \frac{1}{\beta \sqrt{-1}} \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \frac{y}{x} = 0;$$

which, on differentiation to the index $\frac{1}{2}$, give the same results as before.

Ex. 2. $y + a x y + b x \frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} + 2 a x^2 \frac{d y}{dx} = X.$

The solution is, as in Example 1,

$$y = \frac{\alpha}{\alpha - \beta} \frac{1}{1 + \alpha \phi} (X + 0) - \frac{\beta}{\alpha - \beta} \frac{1}{1 + \beta \phi} (X + 0)$$

Now $\frac{X+0}{1+\alpha\phi}$ is the solution of the equation $y_1 - \alpha\sqrt{-1}x \frac{d^{\frac{1}{2}}y_1}{dx^{\frac{1}{2}}} = X + 0$; which (Class II, Ex. 8, Cor. 1) is

$$\left(1 + \alpha\sqrt{-1}x \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\right) \left(\frac{\Lambda e^{\frac{1}{\alpha^2}x}}{\sqrt{x}} + \frac{1}{\sqrt{x}} \int \frac{e^{-\frac{1}{\alpha^2}x}}{\alpha^2 x^{\frac{3}{2}}} X dx\right)$$

and a similar equation results for β . Hence the solution of the given equation is known.

18. It must be remarked of this solution, that it is not in all cases complete without the introduction of the complementary (or arbitrary) function. This arises from the circumstance that when y contains positive integral powers of x , $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = 0$, whereas $x \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ is not equal to 0.

Hence $x^2 \frac{dy}{dx} + \frac{1}{2}xy$ can be replaced by the latter function only by the convention that $\frac{d^{\frac{1}{2}}x^n}{dx^{\frac{1}{2}}}$ is not to be written 0 when n is a positive integer.

On account of this convention, the solution of the equation $\frac{1}{1+\alpha\phi}y = X$ must contain, besides the expression given for it above, a series of positive integral powers of x ; and hence y , the solution of Equation (2), is incomplete without the addition of such a function. It is probable, however, that the determination of a relation between the arbitrary constants may give a solution possessing all the generality which the science is capable of. We have already given an example of the mode of avoiding arbitrary functions by introducing such a relation in Example 1. We shall offer another as a corollary.

COR. If $X = \frac{e}{\sqrt{x}}$, the solution is (Class 2, Ex. 8, Cor 2.)

$$\begin{aligned} y &= y_0 + \frac{\alpha}{\alpha - \beta} \left(\frac{e}{\sqrt{x}} - \frac{\alpha e}{\sqrt{x}} \right) - \frac{\beta}{\alpha - \beta} \left(\frac{e}{\sqrt{x}} - \frac{\beta e}{\sqrt{x}} \right) + \text{arbitrary function} \\ &= y_0 + \frac{e}{\sqrt{x}} - (\alpha + \beta) \frac{e}{\sqrt{\pi}} + \text{arbitrary function} \\ &= y_0 + \frac{e}{\sqrt{x}} - (\alpha + \beta) \frac{e}{\sqrt{\pi}} + p x + q x^2 + \&c. \end{aligned}$$

Now if we examine the equation which connects together p , q , &c., we shall find that it is the same as that which determines y_1 in Class 2, Ex. 3, having only 2α in place of α^2 . Hence it is contained in the solution of the given equation when b and X are omitted. It is, therefore, itself only a supplementary term in the solution of the given equation, and its place may be supplied, appa-

rently without any sacrifice of generality, by the introduction of a relation between A and B. The relation is, $A + B = \frac{be\sqrt{-1}}{\sqrt{\pi}}$.

Hence the complete solution of the equation

$$y + axy + bx \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + 2ax^2 \frac{dy}{dx} = \frac{e}{\sqrt{x}}, \text{ is}$$

$$y = \frac{Ae^{\frac{1}{\alpha^2 x}}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\alpha} \right\} - \left(A - \frac{be\sqrt{-1}}{\sqrt{\pi}} \right) \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{\beta^2 x}}}{\sqrt{x}} dx - \frac{\sqrt{\pi}}{\beta} \right\} + \frac{e}{\sqrt{x}} - \frac{be\sqrt{-1}}{\sqrt{\pi}}$$

19. Ex. 3. $xy + ax \frac{d^{-1}y}{dx^{-1}} - (a + bx) \frac{d^{-2}y}{dx^{-2}} + 2b \frac{d^{-3}y}{dx^{-3}} = 0.$

Multiply by x^{-3} , and the result will be

$$x^{-2}y + ax^{-2} \frac{d^{-1}y}{dx^{-1}} - (ax^{-3} + bx^{-2}) \frac{d^{-2}y}{dx^{-2}} + 2bx^{-3} \frac{d^{-3}y}{dx^{-3}} = 0$$

of which the symbolical form is

$$ye^{-2\theta} - ae^{-\theta} \frac{\sqrt{-D-1}}{\sqrt{-D}} y - (ae^{-\theta} + b) \frac{\sqrt{-D-2}}{\sqrt{-D}} y - 2b \frac{\sqrt{-D-3}}{\sqrt{-D}} y = 0$$

which is equivalent to

$$ye^{-2\theta} + ae^{-\theta} \left(\frac{1}{D+1} - \frac{1}{(D+1)(D+2)} \right) y - b \left(\frac{1}{(D+1)(D+2)} - \frac{2}{(D+1)(D+2)(D+3)} \right) y = 0$$

or $ye^{-2\theta} + ae^{-\theta} \frac{1}{D+2} y - b \frac{1}{(D+2)(D+3)} y = 0$

Hence, by multiplication,

$$y - \frac{a}{b}(D+2)(D+3)e^{-\theta} \frac{1}{D+2} y - \frac{1}{b}(D+2)(D+3)e^{-2\theta} y = 0$$

or $y - \frac{a}{b}(D+2)e^{-\theta} y - \frac{1}{b}(D+2)e^{-\theta}(D+2)e^{-\theta} y = 0$ by (B)

which is of the form $\left(1 - \frac{a}{b}\phi - \frac{1}{b}\phi^2 \right) y = 0;$

which, being put under the form

$$(1 + \alpha\phi)(1 + \beta\phi)y = 0 \text{ gives}$$

$$y = A(1 + \alpha\phi)^{-1} \cdot 0 + B(1 + \beta\phi)^{-1} \cdot 0$$

Now $(1 + \alpha\phi)^{-1} \cdot 0$ is the solution of the equation

$$y_1 + \alpha(D+2)y_1 e^{-\theta} = 0 \text{ or } y_1 + \frac{\alpha}{x} \frac{dx y_1}{dx} = 0$$

of which the result is $y_1 = \frac{A_1}{x} e^{-\frac{x}{\alpha}}$.

Hence $y = \frac{A}{x} e^{-\frac{x}{\alpha}} + \frac{B}{x} e^{-\frac{x}{\beta}}$ is the complete solution of the given equation.

In my second Memoir on this subject, I exemplified the use of a theorem in general differentiation, by solving the problem of determining the law of force by which the particles of a sphere must act on a point, so that the whole attraction may be the same as if the sphere were collected at its centre of gravity. The solution of this problem led to a differential equation which was shewn, by an indirect process, to be satisfied by the law of force varying as the distance, or inversely as its square. I propose, at present, to solve this differential equation.

Ex. 4. The equation is (vol. xiv., p. 608).

$$\frac{\pi}{4a^2} \left\{ 8aR(a+R) \frac{d^{-2}y}{dz^{-2}} - 8(a^2 + 3aR + R^2) \frac{d^{-3}y}{dz^{-3}} \right. \\ \left. + 24(a+R) \frac{d^{-4}y}{dz^{-4}} - 24 \frac{d^{-5}y}{dz^{-5}} \right\} = \frac{4\pi}{3} R^3 f(a)$$

where $y = f(z+a)$, $z = 2R$, $a = a - R$.

This becomes, by substitution,

$$4az \left(a + \frac{z}{2} \right) \frac{d^{-2}y}{dz^{-2}} - 8 \left(a^2 + \frac{3az}{2} + \frac{z^2}{4} \right) \frac{d^{-3}y}{dz^{-3}} \\ + 24 \left(a + \frac{z}{2} \right) \frac{d^{-4}y}{dz^{-4}} - 24 \frac{d^{-5}y}{dz^{-5}} = \frac{2}{3} a^2 f a z^3.$$

Dividing by z^5 , we get

$$\frac{4a^2}{z^4} \frac{d^{-2}y}{dz^{-2}} + \frac{2a}{z^3} \frac{d^{-2}y}{dz^{-2}} - \frac{8a^2}{z^5} \frac{d^{-3}y}{dz^{-3}} - \frac{12a}{z^4} \frac{d^{-3}y}{dz^{-3}} - \frac{2}{z^3} \frac{d^{-3}y}{dz^{-3}} \\ + \frac{24a}{z^5} \frac{d^{-4}y}{dz^{-4}} + \frac{12}{z^4} \frac{d^{-4}y}{dz^{-4}} - \frac{24}{z^5} \frac{d^{-5}y}{dz^{-5}} = \frac{2a^2 f(a)}{3z^2}$$

Writing e^θ for z , and $(-1)^{-\mu} \frac{\sqrt{-D-\mu}}{\sqrt{-D}} y$ for $\frac{1}{z^\mu} \frac{d^{-\mu}y}{dz^{-\mu}}$, there results the symbolical form

$$4a^2 e^{-2\theta} \frac{\sqrt{-D-2}}{\sqrt{-D}} y + 2a e^{-\theta} \frac{\sqrt{-D-2}}{\sqrt{-D}} y + 8a^2 e^{-2\theta} \frac{\sqrt{-D-3}}{\sqrt{-D}} y \\ + 12a e^{-\theta} \frac{\sqrt{-D-3}}{\sqrt{-D}} y + 2 \frac{\sqrt{-D-3}}{\sqrt{-D}} y + 24a e^{-\theta} \frac{\sqrt{-D-4}}{\sqrt{-D}} y \\ + 12 \frac{\sqrt{-D-4}}{\sqrt{-D}} y + 24 \frac{\sqrt{-D-5}}{\sqrt{-D}} y = \frac{2}{3} a^2 f(a) e^{-2\theta};$$

or, collecting the terms,

$$\left\{ 24 \frac{\sqrt{-D-5}}{\sqrt{-D}} + 12 \frac{\sqrt{-D-4}}{\sqrt{-D}} + \frac{\sqrt{-D-3}}{\sqrt{-D}} \right\} y +$$

$$+ a e^{-\theta} \left\{ 24 \frac{\sqrt{-D-4}}{\sqrt{-D}} + 12 \frac{\sqrt{-D-3}}{\sqrt{-D}} + 2 \frac{\sqrt{-D-2}}{\sqrt{-D}} \right\} y$$

$$+ a^2 e^{-2\theta} \left\{ 8 \frac{\sqrt{-D-3}}{\sqrt{-D}} + 4 \frac{\sqrt{-D-2}}{\sqrt{-D}} \right\} y = \frac{2}{3} a^2 f(a) e^{-2\theta}$$

which being reduced gives

$$2(D+1)(D+2) \frac{\sqrt{-D-5}}{\sqrt{-D}} y + 2a e^{-\theta} D(D+1) \frac{\sqrt{-D-4}}{\sqrt{-D}} y$$

$$- 4a^2 e^{-2\theta} (D+1) \frac{\sqrt{-D-3}}{\sqrt{-D}} y = \frac{2}{3} a^2 f(a) e^{-2\theta};$$

or

$$- \frac{1}{(D+3)(D+4)(D+5)} y + a e^{-\theta} \frac{D}{(D+2)(D+3)(D+4)} y$$

$$+ 2a^2 e^{-2\theta} \frac{1}{(D+2)(D+3)} y = \frac{1}{3} a^2 f(a) e^{-2\theta}$$

Now $y = f(\alpha + z)$, but since α is itself a function of z , we cannot proceed further with the reduction of this equation by division, but must proceed to obtain a relation between α and R or α and z .

To do this we shall expand $f(\alpha + z)$ by TAYLOR'S Theorem.

The result is

$$y = \Sigma \frac{d^n f(\alpha)}{d \alpha^n} \frac{z^n}{n+1} \text{ which being substituted in the reduced equation, gives by (A)}$$

$$\Sigma \left(\frac{d^n f(\alpha)}{d \alpha^n} \frac{e^{n\theta}}{n+1} \right) \left\{ \frac{-1}{(n+3)(n+4)(n+5)} \right.$$

$$\left. + a \frac{n}{(n+2)(n+3)(n+4)} e^{-\theta} + \frac{2a^2 e^{-2\theta}}{(n+2)(n+3)} \right\} = \frac{a^2}{3} f a e^{-2\theta}$$

But $\alpha = a - R = a - \frac{z}{2}$: hence

$$\frac{d^n f(\alpha)}{d \alpha^n} = \frac{d^n f a}{d a^n} - \frac{d^{n+1} f a}{d a^{n+1}} \frac{z}{2} + \&c.$$

which being substituted for $\frac{d^n f(\alpha)}{d \alpha^n}$, the sum being taken for n and p , we get

$$\Sigma_{n,p} \frac{(-1)^p}{2^p} \frac{d^{n+p} f a}{d a^{n+p}} \frac{1}{n+1} \frac{1}{p+1} \left\{ \frac{-z^{n+p}}{(n+3)(n+4)(n+5)} \right.$$

$$\left. + \frac{a n z^{n+p-1}}{(n+2)(n+3)(n+4)} + \frac{2a^2 z^{n+p-2}}{(n+2)(n+3)} \right\} = \frac{a^2}{3} \frac{f a}{z^2}$$

every integer value of n and p being taken from 0 to ∞ .

When $n=0, p=0$, the left-hand side gives

$$\frac{-f a}{60} + \frac{a^2 f a}{3 z^2}$$

$$n=0, p=1, \dots \frac{z}{120} \frac{d f a}{d a} - \frac{a^2}{6z} \frac{d f a}{d a}$$

$$n=0, p=2, \dots \frac{-z^2}{480} \frac{d^2 f a}{d a^2} + \frac{a^2}{24} \frac{d^2 f a}{d a^2}$$

When $n=1, p=0, \dots \frac{-z}{120} \frac{d f a}{d a} + \frac{a}{60} \frac{d f a}{d a}$
 $+ \frac{a^2}{6z} \frac{d f a}{d a}$

$$n=1, p=1, \frac{z^2}{240} \frac{d^2 f a}{d a^2} - \frac{a z}{120} \frac{d^2 f a}{d a^2} - \frac{a^2}{12} \frac{d^2 f a}{d a^2}$$

$$n=2, p=0, \frac{-z^2}{420} \frac{d^2 f a}{d a^2} + \frac{a z}{120} \frac{d^2 f a}{d a^2} + \frac{a^2}{20} \frac{d^2 f a}{d a^2}, \&c., \&c.$$

Hence we obtain, by collecting the terms and equating their sum to $\frac{a^2 f a}{3 z^2}$,

$$\frac{a^2 f a}{3 z^2} - \frac{f a}{60} + \frac{a}{60} \frac{d f a}{d a} + \frac{a^2}{120} \frac{d^2 f a}{d a^2} + P z + Q z^2 + \&c.$$

$$= \frac{a^2 f a}{3 z^2}$$

Equating coefficients of like powers of z , we obtain

$$-\frac{f a}{60} + \frac{a}{60} \frac{d f a}{d a} + \frac{a^2}{120} \frac{d^2 f a}{d a^2} = 0.$$

This equation will determine $f(a)$, the only law of force by which a sphere can attract an external particle exactly as much as if it were all collected at its centre of gravity.

The symbolical form of the equation is

$$\{D(D-1) + 2D-2\} f a = 0$$

or $(D^2 + D - 2) f a = 0$, or $(D-1)(D+2) f a = 0$.

Hence $(D-1) f a = 0$, $(D+2) f a = 0$,

or $\frac{d f a}{d a} = f a$, and $\frac{d f a}{d a} = -2 f a$

that is $f a = A a$, and $f a = \frac{B}{a^2}$ are particular integrals, and the complete integral is $y = A a + \frac{B}{a^2}$; which is the law required.

SECTION II. SIMULTANEOUS EQUATIONS.

20. To effect the solution of simultaneous equations, we must eliminate one of the quantities differentiated. This is best effected by treating both the differ-

entiation and the multiplication by a constant in the same manner, regarding both the one and the other as an operation. A similar process has been employed for the solution of ordinary simultaneous equations by Mr GREGORY in the *Cambridge Mathematical Journal*, i. 173.

Ex. 1.
$$\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} + ay = 0, \quad \frac{d^{\frac{1}{2}}y}{dt^{\frac{1}{2}}} + bx = 0$$

By taking the $\frac{1}{2}$ differential of the first equation, we get $\frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}} + a \frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} = 0$; which, by virtue of the second, gives

$$\frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}} - abx = 0; \text{ or } x = A e^{abt} \quad \therefore y = -A \sqrt{\frac{b}{a}} e^{abt}.$$

Ex. 2.
$$\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} + ay + bx = 0, \quad \frac{d^{\frac{1}{2}}y}{dt^{\frac{1}{2}}} + a'y + b'x = 0.$$

These equations may be written

$$\left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} + b\right)x + ay = 0, \quad \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} + a'\right)y + b'x = 0;$$

whence, by eliminating y , we obtain

$$\left\{ \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} + a'\right) \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} + b\right) - ab' \right\} x = 0.$$

Let $\alpha^{\frac{1}{2}}, \beta^{\frac{1}{2}}$, be the roots of the equation

$$(z + a')(z + b) - ab' = 0$$

then
$$\left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} - \alpha^{\frac{1}{2}}\right) \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} - \beta^{\frac{1}{2}}\right) x = 0; \text{ which gives}$$

$$x = A e^{\alpha t} + B e^{\beta t}$$

and

$$y = -\frac{1}{a} \frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} - \frac{b}{a} x \\ = -\left(\frac{\alpha^{\frac{1}{2}}}{a} + \frac{b}{a}\right) A e^{\alpha t} - \left(\frac{\beta^{\frac{1}{2}}}{a} + \frac{b}{a}\right) B e^{\beta t}.$$

Ex. 3.
$$\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} + ay + bx = f(t)$$

$$\frac{d^{\frac{1}{2}}y}{dt^{\frac{1}{2}}} + a'y + b'x = \phi(t)$$

$$\left\{ \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} + a'\right) \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} + b\right) - ab' \right\} x = \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} + a'\right) f(t) - a\phi(t) \\ = \psi(t)$$

This coincides with Ex. 4, Class 1, and the solution is

$$x = A e^{\alpha t} + B e^{\beta t} + \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} - \alpha^{\frac{1}{2}}\right)^{-1} \psi(t) - \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \left(\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} - \beta^{\frac{1}{2}}\right)^{-1} \psi(t)$$

Ex. 4.
$$\frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + a x + b y + c z = 0$$

$$\frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + a' x + b' y + c' z = 0$$

$$\frac{d^{\frac{1}{2}} z}{d t^{\frac{1}{2}}} + a'' x + b'' y + c'' z = 0$$

These equations can be written in the form

$$A x + b y + c z = 0$$

$$a' x + B' y + c' z = 0$$

$$a'' x + b'' y + C'' z = 0$$

By eliminating y and z we obtain

$$\{A B' C'' - b'' c' A - a'' c B' - a' b C'' + a' b' c + a'' b c'\} x = 0$$

or
$$\left\{ \frac{d^{\frac{3}{2}}}{d t^{\frac{3}{2}}} + (a + b' + c) \frac{d}{d t} + (a b' + a c'' + b c'') \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a b' c'' - b'' c' \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a b'' c' - a'' c \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a'' b' c - a' b \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a' b c'' + a' b'' c + a'' b c' \right\} x = 0$$

or
$$\left\{ \frac{d^{\frac{3}{2}}}{d t^{\frac{3}{2}}} + (a + b + c) \frac{d}{d t} + (a b' + a c'' + b' c' - b c'' - a'' c' - a' b) \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a b' c'' + a b'' c' + a'' b' c + a' b c'' + a' b'' c + a'' b c' \right\} x = 0$$

which is of the same form as Cor. 2, Ex. 3, Class 1, and its integral is therefore known.

Knowing x , we have $b y + c z = f(t)$ by the first of the three equations, and $\frac{b d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + c \frac{d^{\frac{1}{2}} z}{d t^{\frac{1}{2}}} = \phi(t)$ by differentiation, whence, by substituting the values of $\frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}}$ and $\frac{d^{\frac{1}{2}} z}{d t^{\frac{1}{2}}}$ from the second and third equations, there results a second equation between y , z , and t . From these two equations y and z are determined in terms of t .

Ex. 5. Given
$$\frac{d x}{d t} + a \frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + b x = p \frac{d y}{d t} + q \frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + r y$$

$$\frac{d x}{d t} + a' \frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + b' x = p' \frac{d y}{d t} + q' \frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + r' y$$

These equations may be written

$$(d + a d^{\frac{1}{2}} + b) x = (p d + q d^{\frac{1}{2}} + r) y$$

$$(d + a' d^{\frac{1}{2}} + b') x = (p' d + q' d^{\frac{1}{2}} + r') y$$

$$\therefore (p'd + q'd^{\frac{1}{2}} + r')(d + a d^{\frac{1}{2}} + b)x = (p d + q d^{\frac{1}{2}} + r)(d + a' d^{\frac{1}{2}} + b')x$$

or

$$(p-p') \frac{d^2 x}{d t^2} + (p a' + q - p' a - q') \frac{d^{\frac{3}{2}} x}{d t^{\frac{3}{2}}} + (p b' + q a' + r - p' b - q' a - r') \frac{d x}{d t} + (q b' + r a' - q' b - r' a) \frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + r b' - r' b)x = 0$$

which coincides with the general form Cor. 2, Ex. 3, Class 1.

SECTION III. PARTIAL DIFFERENTIAL EQUATIONS.

21. In order to effect the solution of partial differential equations in which the operation with respect to x is totally independent of the operation with respect to y , we must distinguish the operation of differentiation in the two cases by different symbols. Let d stand for $\frac{d}{dx}$, δ for $\frac{d}{dy}$: then in solving the equation with respect to d , we may treat δ as a constant, and *vice versa*.

Ex. 1.
$$\frac{d^{\frac{1}{2}} z}{d x^{\frac{1}{2}}} - b \frac{d^{\frac{1}{2}} z}{d y^{\frac{1}{2}}} = 0.$$

Write this equation $(d^{\frac{1}{2}} - b \delta^{\frac{1}{2}})z = 0$: In this form it coincides with Ex. 1, Class 1, and its solution is $z = A e^{b^2 \delta x}$.

Now A is an arbitrary function of y ; call it $f(y)$: then $z = e^{b^2 x \delta} f(y)$, where $e^{b^2 x \delta}$ represents an operation on $f(y)$.

But since

$$\begin{aligned} f(y+k) &= f y + \frac{d f y}{d y} k + \\ &= (1 + k \delta + \&c.) f(y) \\ &= e^{k \delta} f(y), \\ \text{it is evident that} \quad z &= f(y + b^2 x). \end{aligned}$$

Ex. 2.
$$\frac{d^{\frac{1}{2}} z}{d x^{\frac{1}{2}}} = a \frac{d z}{d y}.$$

This equation may be written $(d^{\frac{1}{2}} - a \delta)z = 0$.

$$\therefore z = e^{a^2 x \delta^2} f(y).$$

Now
$$\int_{-\infty}^{\infty} d \omega e^{-(\omega-b)^2} = \sqrt{\pi} \quad (\text{GREGORY'S Examples, p. 499.})$$

$$\begin{aligned} \therefore z \sqrt{\pi} &= \int_{-\infty}^{\infty} d \omega e^{-\omega^2 + 2 b \omega - b^2} \cdot e^{a^2 x \delta^2} f(y) \\ &= \int_{-\infty}^{\infty} d \omega e^{-\omega^2} e^{2 \omega a \sqrt{x} \delta} f(y) \quad (\text{if } b = a \sqrt{x} \delta) \end{aligned}$$

$$= \int_{-\infty}^{\infty} d\omega e^{-\omega^2} f(y + 2\omega a \sqrt{x})$$

which is the solution of the equation in the form of a definite integral.

Ex. 3.
$$\frac{d^{\frac{1}{2}} z}{dx^{\frac{1}{2}}} = a \frac{dz}{dy} + cz$$

The first form of the solution is evidently

$$z = e^{(a\delta + c)^2 x} f(y)$$

which is reduced to $z = e^{c^2 x} e^{a^2 \delta^2 x} f(y + 2acx)$

$$= \frac{e^{c^2 x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\omega e^{-\omega^2} f(y + 2acx + 2\omega a \sqrt{x})$$

as in Example 2.

Ex. 4.
$$\frac{dz}{dx} - 2a \frac{d^{\frac{1}{2}} z}{dx^{\frac{1}{2}}} \cdot \frac{d}{dy} z + a^2 \frac{d^2 z}{dy^2} = c^2 z$$

This equation may be written $(d - 2a d^{\frac{1}{2}} \delta + a^2 \delta^2 - c^2) z = 0$, which is of the form of Ex. 3, Class 1, and the solution is

$$\begin{aligned} z &= e^{c^2 x} e^{a^2 x \delta^2} \{e^{2acx\delta} f(y) + e^{-2acx\delta} \phi(y)\} \\ &= e^{c^2 x} e^{a^2 x \delta^2} \{f(y + 2acx) + \phi(y - 2acx)\} \\ &= \frac{e^{c^2 x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\omega e^{-\omega^2} \{f(y + 2acx + 2\omega a \sqrt{x}) + \phi(y - 2acx + 2\omega a \sqrt{x})\} \end{aligned}$$

Ex. 5.
$$\frac{dz}{dx} - 2a \frac{d^{\frac{1}{2}} z}{dx^{\frac{1}{2}}} \cdot \frac{d^{\frac{1}{2}} z}{dy^{\frac{1}{2}}} + a^2 \frac{dz}{dy} = 0.$$

This equation gives $(d - 2a d^{\frac{1}{2}} \delta^{\frac{1}{2}} + a^2 \delta) z = 0$

or
$$\begin{aligned} z &= e^{a^2 x \delta} f(y) + x e^{a^2 x \delta} \phi(y) \quad (\text{Class 1, Ex. 3. Cor. 1.}) \\ &= f(y + a^2 x) + x \phi(y + a^2 x) \end{aligned}$$

Ex. 6.
$$\frac{dz}{dx} + a \frac{d^{\frac{1}{2}} z}{dx^{\frac{1}{2}}} \cdot \frac{d^{\frac{1}{2}} z}{dy^{\frac{1}{2}}} + b \frac{dz}{dy} = cz.$$

This equation may be written $(d + a \delta^{\frac{1}{2}} d^{\frac{1}{2}} + b \delta - c) z = 0$ which coincides with Ex. 3, Class 1; and the solution is

$$z = e^{\alpha x} f(y) + e^{\beta x} \phi(y)$$

where $\alpha^{\frac{1}{2}}, \beta^{\frac{1}{2}}$ are the roots of the equation in v

$$v + a \delta^{\frac{1}{2}} v^{\frac{1}{2}} + b \delta - c = 0:$$

or
$$\alpha^{\frac{1}{2}} = -\frac{a \delta^{\frac{1}{2}}}{2} + \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}$$

$$\beta^{\frac{1}{2}} = -\frac{a \delta^{\frac{1}{2}}}{2} - \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}$$

or

$$\alpha = \left(\frac{a^2}{2} - b\right) \delta + c - a \delta^{\frac{1}{2}} \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}$$

$$\beta = \left(\frac{a^2}{2} - b\right) \delta + c + a \delta^{\frac{1}{2}} \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}$$

$$\therefore z = e^{c x} e^{\left(\frac{a^2}{2} - b\right) \delta x} \left\{ e^{-a \delta^{\frac{1}{2}} x \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}} f(y) + e^{a \delta^{\frac{1}{2}} x \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}} \phi(y) \right\}$$

$$= e^{c x} \left\{ e^{-a \delta^{\frac{1}{2}} x \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}} f\left(y + \left(\frac{a^2}{2} - b\right)x\right) + e^{a \delta^{\frac{1}{2}} x \sqrt{\left(\frac{a^2}{4} - b\right) \delta + c}} \phi\left(y + \left(\frac{a^2}{2} - b\right)x\right) \right\}$$

These expressions do not appear to be susceptible of further reduction, except in particular cases.

COR. Let $b = \frac{a^2}{4}$; then

$$z = e^{c x} \left\{ e^{-a x \delta^{\frac{1}{2}} \sqrt{c}} f\left(y + \frac{a^2}{4} x\right) + e^{a x \delta^{\frac{1}{2}} \sqrt{c}} \phi\left(y + \frac{a^2}{4} x\right) \right\}$$

To reduce this expression, it may perhaps be sufficiently general to suppose the symbol $\delta^{\frac{1}{2}}$ to include both the positive and negative signs, in which case we may write only one of the functions $e^{-a \sqrt{c} x \delta^{\frac{1}{2}}} f\left(y + \frac{a^2}{4} x\right)$.

Now $\frac{\sqrt{\pi}}{2} e^{-2v} = \int_0^\infty d\omega e^{-\left(\omega^2 + \frac{v^2}{\omega^2}\right)}$. (See GREGORY'S *Examples*, p. 499.)

Let $v = \frac{a x \sqrt{c} \delta^{\frac{1}{2}}}{2}$; then

$$\frac{\sqrt{\pi}}{2} e^{-a \sqrt{c} x \delta^{\frac{1}{2}}} = \int_0^\infty d\omega e^{-\left(\omega^2 + \frac{a^2 c x^2 \delta}{4 \omega^2}\right)}$$

and

$$z = e^{c x} e^{-a \sqrt{c} x \delta^{\frac{1}{2}}} f\left(y + \frac{a^2}{4} x\right)$$

$$= e^{c x} \frac{2}{\sqrt{\pi}} \int_0^\infty d\omega e^{-\left(\omega^2 + \frac{a^2 c x^2 \delta}{4 \omega^2}\right)} f\left(y + \frac{a^2}{4} x\right)$$

$$= \frac{2}{\sqrt{\pi}} e^{c x} \int_0^\infty d\omega e^{-\omega^2} f\left(y + \frac{a^2}{4} x - \frac{a^2 c x^2}{4 \omega^2}\right).$$

SECTION IV. DIFFERENCES.

22. The definition of the difference of u_x , as it is commonly written by English authors is $u_{x+1} - u_x$. We shall retain this definition, and generalize it by writing $\frac{d}{d^x} u_x$ for u_{x+1} , and consequently $(e^{\frac{d}{d^x}} - 1) u_x$ for Δu_x .

The results which we shall produce from this definition, as applied to frac-

tional values of the index of difference will, in most cases, differ not at all from the results obtained in the ordinary calculus of differences. We offer them only for the purpose of exhibiting those formulæ which possess all the generality which can be desired, at a single glance.

Suppose, then, $\Delta u_x = f(u_x) = v_x$

$$\therefore \Delta^2 u_x = \Delta v_x = (e^{\frac{d}{dx}} - 1) v_x = (e^{\frac{d}{dx}} - 1)^2 u_x, \text{ \&c.}$$

and, according to the axiom of the calculus of operations that the repetitions of equivalent operations are equivalent, we shall have generally $\Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x$: whatever n may be. This, then, may be said to be the *definition* of $\Delta^n u_x$.

Also, since $u_{x+n} = e^{n\frac{d}{dx}} u_x$ by TAYLOR'S Theorem, and $\Delta u_x = (e^{\frac{d}{dx}} - 1) u_x$; it follows that $u_{x+n} = (1 + \Delta)^n u_x$.

We proceed now to apply it to the demonstration of the theorems which connect together $\Delta^r u_{x+s}$, and u_{x+p} , &c.

$$(1). \quad \Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x = \left(e^{n\frac{d}{dx}} - n e^{(n-1)\frac{d}{dx}} + \frac{n(n-1)}{1 \cdot 2} e^{(n-2)\frac{d}{dx}} - \&c. \right) u_x$$

$$= u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{1 \cdot 2} u_{x+n-2} - \&c.$$

COR. 1. If $n = -1$; $\Delta^{-1} u_x = u_{x-1} + u_{x-2} + u_{x-3} + \&c.$

OR $\Sigma u_x = u_{x-1} + u_{x-2} + u_{x-3} + \&c.$ together with an arbitrary constant;

OR $u_x = \Delta u_{x-1} + \Delta u_{x-2} + \Delta u_{x-3} + \&c.$

COR. 2. If $n = -2$; $\Sigma^2 u_x = u_{x-2} + 2 u_{x-3} + 3 u_{x-4} + \&c.$ together with $A + Bx$.

$$(2). \quad \Delta^n u_x = (-1)^n (1 - e^{\frac{d}{dx}})^n u_x = (-1)^n \left(1 - n e^{\frac{d}{dx}} + \frac{n(n-1)}{1 \cdot 2} e^{2\frac{d}{dx}} - \&c. \right) u_x$$

$$= (-1)^n (u_x - n u_{x+1} + \frac{n(n-1)}{1 \cdot 2} u_{x+2} - \&c.)$$

COR. 1. If $n = -1$, $\Delta^{-1} u_x = \Sigma u_x = -(u_x + u_{x+1} + u_{x+2} + \&c.)$; to which we may add an arbitrary constant.

COR. 2. If $n = -2$, $\Sigma^2 u_x = u_x + 2 u_{x+1} + 3 u_{x+2} + \&c.$, together with $A + Bx$.

$$(3). \quad \Delta^n u_x = e^{n\frac{d}{dx}} \left(\frac{e^{\frac{d}{dx}} - 1}{e^{\frac{d}{dx}}} \right)^n u_x = e^{n\frac{d}{dx}} \left(\frac{e^{\frac{d}{dx}}}{e^{\frac{d}{dx}} - 1} \right)^{-n} u_x$$

$$= e^{n\frac{d}{dx}} \left(1 + (e^{\frac{d}{dx}} - 1)^{-1} \right)^{-n} u_x$$

$$\begin{aligned}
&= e^{\frac{n}{d}x} \left\{ 1 - n(e^{\frac{d}{d}x} - 1)^{-1} + \frac{n(n+1)}{1 \cdot 2} (e^{\frac{d}{d}x} - 1)^{-2} - \&c. \right\} \\
&= e^{\frac{n}{d}x} \left\{ u_x - n \Delta^{-1} u_x + \frac{n(n+1)}{1 \cdot 2} \Delta^{-2} u_x - \&c. \right\} \\
&= u_{x+n} - n \Delta^{-1} u_{x+n} + \frac{n(n+1)}{1 \cdot 2} \Delta^{-2} u_{x+n} - \&c.
\end{aligned}$$

or

$$\begin{aligned}
&= u_{x+n} - n \Sigma u_{x+n} + \frac{n(n+1)}{1 \cdot 2} \Sigma^2 u_{x+n} - \&c. \\
(4). \quad \Delta^n u_x &= (-1)^{-n} \left(1 - \frac{e^{\frac{d}{d}x}}{e^{\frac{d}{d}x} - 1} \right)^{-n} u_x \\
&= (-1)^{-n} \left\{ 1 + n e^{\frac{d}{d}x} (e^{\frac{d}{d}x} - 1)^{-1} + \frac{n(n+1)}{1 \cdot 2} e^{\frac{2d}{d}x} (e^{\frac{d}{d}x} - 1)^{-2} + \&c. \right\} u_x \\
&= (-1)^{-n} \left\{ u_x + n \Sigma u_{x+1} + \frac{n(n+1)}{1 \cdot 2} \Sigma^2 u_{x+2} + \&c. \right\}
\end{aligned}$$

$$\begin{aligned}
(5). \quad \Delta^n u_x &= \left(\frac{e^{\frac{d}{d}x}}{e^{\frac{d}{d}x} - 1} - 1 \right)^{-n} u_x \\
&= \left\{ e^{-\frac{n}{d}x} (e^{\frac{d}{d}x} - 1)^n + n e^{-(n+1)\frac{d}{d}x} (e^{\frac{d}{d}x} - 1)^{n+1} + \&c. \right\} u_x \\
&= \Delta^n u_{x-n} + n \Delta^{n+1} u_{x-n-1} + \frac{n(n+1)}{1 \cdot 2} \Delta^{n+2} u_{x-n-2} + \&c.
\end{aligned}$$

These formulæ are all quite independent of the value of n , and serve to connect the n th difference of a function of x with differences of functions of $x+n$, &c., $x+1$, &c.

We shall now obtain the converse series of connections, those of u_{x+n} with u_x , &c.

$$\begin{aligned}
(6). \quad u_{x+n} &= (1 + \Delta)^n u_x = \left(1 + n \Delta + \frac{n(n-1)}{1 \cdot 2} \Delta^2 + \&c. \right) u_x \\
&= u_x + n \Delta u_x + \frac{n(n-1)}{1 \cdot 2} \Delta^2 u_x + \&c.
\end{aligned}$$

$$\begin{aligned}
(7). \quad u_{x+n} &= (\Delta + 1)^n u_x = \left(\Delta^n + n \Delta^{n-1} + \frac{n(n-1)}{1 \cdot 2} \Delta^{n-2} + \&c. \right) u_x \\
&= \Delta^n u_x + n \Delta^{n-1} u_x + \frac{n(n-1)}{1 \cdot 2} \Delta^{n-2} u_x + \&c.
\end{aligned}$$

If n were a positive integer, formula (1) would coincide with formula (2); and formula (6) with formula (7); but in our present calculus they are by no means the same thing.

$$\begin{aligned}
 (8.) \quad u_{x+n} &= \left(\frac{1}{1+\Delta}\right)^{-n} u_x = \left(1 - \frac{\Delta}{1+\Delta}\right)^{-n} u_x \\
 &= \left(1 + n \frac{\Delta}{1+\Delta} + \frac{n(n+1)}{1 \cdot 2} \frac{\Delta^2}{(1+\Delta)^2} + \&c.\right) u_x \\
 &= u_x + n \Delta u_{x-1} + \frac{n(n+1)}{1 \cdot 2} \Delta^2 u_{x-2} + \&c.
 \end{aligned}$$

COR. If $n=1$, $u_{x+1} = u_x + \Delta u_{x-1} + \Delta^2 u_{x-2} + \&c.$

$$\begin{aligned}
 (9.) \quad u_{x+n} &= (-1)^{-n} \left(\frac{\Delta}{1+\Delta} - 1\right)^{-n} u_x \\
 &= (-1)^{-n} (\Delta^{-n} (1+\Delta)^n + n \Delta^{-(n+1)} (1+\Delta)^{n+1} + \&c.) u_x \\
 &= (-1)^{-n} \left\{ \Sigma^n u_{x+n} + n \Sigma^{n+1} u_{x+n+1} + \frac{n(n+1)}{1 \cdot 2} \Sigma^{n+2} u_{x+n+2} + \&c. \right\}
 \end{aligned}$$

In strictness we ought to write Δ^{-n} for Σ^n , but the latter notation is more familiar to the eye.

$$\begin{aligned}
 (10.) \quad u_{x+n} &= \Delta^n \left(\frac{1+\Delta}{\Delta}\right)^n u_x = \Delta^n \left(1 - \frac{1}{1+\Delta}\right)^{-n} u_x \\
 &= \Delta^n \left\{ 1 + n (1+\Delta)^{-1} + \frac{n(n+1)}{1 \cdot 2} (1+\Delta)^{-2} + \&c. \right\} u_x \\
 &= \Delta^n u_x + n \Delta^n u_{x-1} + \frac{n(n+1)}{1 \cdot 2} \Delta^n u_{x-2} + \&c.
 \end{aligned}$$

Formula (10) is a particular form of formula (1), for by formula (1), $\Delta^{-m} u_{x+m} = u_x + m u_{x-1} + \&c.$, which is reduced to (10) by multiplying by Δ^m . In the same manner we may reproduce formula (2.)

The last class of relations which we shall produce are such as do not depend on the general expansion of the binomial.

$$\begin{aligned}
 (11.) \quad u_{x+n} &= e^{\frac{n}{d} x} u_x = e^{\frac{n}{d} x} \frac{e^{\frac{d}{d} x} - 1}{\frac{d}{d} x - 1} u_x \\
 &= (e^{\frac{d}{d} x} - 1) \left(e^{(n-1)\frac{d}{d} x} + e^{(n-2)\frac{d}{d} x} + e^{(n-3)\frac{d}{d} x} + \&c. \right) u_x \\
 &= (e^{\frac{d}{d} x} - 1) (u_{x+n-1} + u_{x+n-2} + u_{x+n-3} + \&c.) \\
 &= \Delta u_{x+n-1} + \Delta u_{x+n-2} + \Delta u_{x+n-3} + \&c.
 \end{aligned}$$

COR. 1. If $n=0$; $u_x = \Delta u_{x-1} + \Delta u_{x-2} + \Delta u_{x-3} + \&c.$, which coincides with Cor. 1., formula (1.)

COR. 2. If n be a positive integer

$$\begin{aligned}
 u_{x+n} &= \Delta u_{x+n-1} + \Delta u_{x+n-2} + \&c. + \Delta u_{x+1} + \Delta u_x + \Delta u_{x-1} + \&c. \\
 &= \Delta u_{x+n-1} + \Delta u_{x+n-2} + \&c. + \Delta u_{x+1} + \Delta u_x + u_x; \text{ by Cor. 1.}
 \end{aligned}$$

$$\begin{aligned}
 (12.) \quad u_{x+n} &= -e^{\frac{n}{d} \frac{d}{dx}} \frac{e^{\frac{d}{dx}} - 1}{\frac{d}{dx}} u_x \\
 &= -\left(e^{\frac{d}{dx}} - 1\right) \left(e^{\frac{n}{d} \frac{d}{dx}} + e^{\frac{(n+1)}{d} \frac{d}{dx}} + e^{\frac{(n+2)}{d} \frac{d}{dx}} + \&c.\right) u_x \\
 &= -(\Delta u_{x+n} + \Delta u_{x+n+1} + \Delta u_{x+n+2} + \&c.)
 \end{aligned}$$

COR. If $n=0$, $\Sigma u_x = -(u_x + u_{x+1} + u_{x+2} + \&c.)$,

which coincides with Cor. 1, formula (2.)

$$\begin{aligned}
 (13.) \quad u_{x+n} &= e^{\frac{(n-1)}{d} \frac{d}{dx}} \frac{d}{dx} e^{\frac{d}{dx}} u_x = e^{\frac{(n-1)}{d} \frac{d}{dx}} \frac{1}{1 - \frac{e^{\frac{d}{dx}} - 1}{\frac{d}{dx}}} u_x \\
 &= e^{\frac{(n-1)}{d} \frac{d}{dx}} \left(1 + \frac{\frac{d}{dx} - 1}{\frac{d}{dx}} + \left(\frac{\frac{d}{dx} - 1}{\frac{d}{dx}}\right)^2 + \&c.\right) u_x \\
 &= u_{x+n-1} + \Delta u_{x+n-2} + \Delta^2 u_{x+n-3} + \&c.
 \end{aligned}$$

which coincides with the Cor. to formula (8).

Thus formulæ (1), (2), and (8), include formulæ (11), (12), and (13).

It is evident that by the same process all the ordinary formulæ in finite differences, which are usually obtained by the aid of generating functions, may be easily obtained.

For example the following:

$$\begin{aligned}
 (14.) \quad u_{x+n} &= (n+1) \left\{ u_x + \frac{(n+1)^2 - 1^2}{1 \cdot 2 \cdot 3} \Delta^2 u_{x-1} + \frac{(n+1)^2 - 1^2 \cdot (n+1)^2 - 2^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \Delta^4 u_{x-2} + \&c. \right\} \\
 &\quad - n \left\{ u_{x-1} + \frac{n^2 - 1^2}{1 \cdot 2 \cdot 3} \Delta^2 u_{x-2} + \&c. \right\}
 \end{aligned}$$

We have

$$\frac{1}{1-a(1+\Delta)} = \frac{1 - \frac{a}{1+\Delta}}{\left\{1 - a(1+\Delta)\right\} \left\{1 - \frac{a}{1+\Delta}\right\}} =$$

$$\frac{1}{(1-a)^2 - \frac{a \Delta^2}{1+\Delta}} - \frac{\frac{a}{1+\Delta}}{(1-a)^2 - \frac{a \Delta^2}{1+\Delta}}$$

Now $\frac{1}{(1-a)^2 - a z}$ when expanded in terms of a , gives as the coefficient of a^n ,

$$(n+1) \left\{ 1 + \frac{(n+1)^2 - 1^2}{1 \cdot 2 \cdot 3} z + \&c. \right\}$$

Hence, if we equate the coefficients of α^n in the two equivalent expressions

$$\frac{1}{1-\alpha(1+\Delta)} u_x \text{ and } \left(\frac{1}{(1-\alpha)^2 - \frac{\alpha \Delta^2}{1+\Delta}} - \frac{\frac{\alpha}{1+\Delta}}{(1-\alpha)^2 - \frac{\alpha \Delta^2}{1+\Delta}} \right) u_x$$

the result will be

$$\begin{aligned} (1+\Delta)^n u_x &= (n+1) \left\{ 1 + \frac{(n+1)^2 - 1^2}{1 \cdot 2 \cdot 3} \frac{\Delta^2}{1+\Delta} + \&c. \right\} u_x \\ &\quad - n \left\{ 1 + \frac{n^2 - 1^2}{1 \cdot 2 \cdot 3} \frac{\Delta^2}{1+\Delta} + \&c. \right\} \frac{1}{1+\Delta} u_x \\ \text{or} \quad u_{x+n} &= (n+1) \left\{ u_x + \frac{(n+1)^2 - 1^2}{1 \cdot 2 \cdot 3} \Delta^2 u_{x-1} + \&c. \right\} \\ &\quad - n \left\{ u_{x-1} + \frac{n^2 - 1^2}{1 \cdot 2 \cdot 3} \Delta^2 u_{x-2} + \&c. \right\} \end{aligned}$$

24. Let us apply these formulæ to examples.

Ex. 1. Let $u_x = e^{\alpha x}$, then

$$\Delta^n u_x = (e^{\frac{\Delta}{\alpha} x} - 1)^n e^{\alpha x} = (e^{\alpha} - 1)^n e^{\alpha x} \text{ (by A.)}$$

Ex. 2. Let $u_x = x$, $n = \frac{1}{2}$, then, formula (2),

$$\begin{aligned} \Delta^{\frac{1}{2}} x &= \sqrt{-1} \left\{ x - \frac{1}{2} (x+1) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \cdot 2} (x+2) - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1 \cdot 2 \cdot 3} (x+3) + \&c. \right\} \\ &= \sqrt{-1} \left\{ x - \frac{1}{2} x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \cdot 2} x - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1 \cdot 2 \cdot 3} x + \&c. \right. \\ &\quad \left. - \frac{1}{2} (1 + \frac{1}{1} + \frac{\frac{1}{2} \cdot \frac{3}{2}}{1 \cdot 2} + \&c.) \right\} \\ &= \sqrt{-1} x (1-1)^{\frac{1}{2}} - \frac{\sqrt{-1}}{2} (1-1)^{-\frac{1}{2}} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \text{Ex. 3. } \Delta^n x &= (-1)^n \left(x - n(x+1) + \frac{n(n-1)}{1 \cdot 2} (x+2) - \&c. \right) \\ &= (-1)^n x (1-1)^n - n(-1)^n \left(1 - \frac{n-1}{1} + \frac{(n-1)(n-2)}{1 \cdot 2} - \&c. \right) \\ &= (-1)^n x (1-1)^n - n(-1)^n (1-1)^{n-1} \end{aligned}$$

which is zero when n is greater than 1, finite only when $n=1$, in which case it is 1; and infinite when n is less than 1.

It is evident that this introduction of ∞ may indicate simply that the form of the expansion is incorrect: for $\Delta \infty x = \infty(x+1) - \infty x = x + \text{const.}$ is the analytical result of the equation $\Delta x = (x+1) - x = 0x + \text{const.}$, by dividing both sides by the symbol 0.

When n is less than 1, therefore, it is necessary to seek some other method of obtaining the n th difference. The following method, analogous to that by which we obtained the n th differential coefficient of a logarithm in Art. 2 appears to be the most simple.

Let x be represented by $\frac{e^p x - e^q x}{p}$, where q is of a higher order than p , and both are 0.

$$\begin{aligned} \Delta^n x &= \frac{(e^p - 1)^n e^p x - (e^q - 1)^n e^q x}{p} \\ &= \frac{(p + \frac{p^2}{1 \cdot 2} + \frac{p^3}{1 \cdot 2 \cdot 3} + \&c.)^n e^p x - (q + \&c.)^n e^q x}{p} \\ &= \frac{(p + \frac{p^2}{1 \cdot 2} + \&c.)^n (1 + px + \&c.) - (q + \frac{q^2}{1 \cdot 2} + \&c.)^n (1 + qx + \&c.)}{p} \\ &= \frac{p^n + \frac{n p^{n+1}}{2} + \frac{n(3n+1)}{24} p^{n+2} + \&c. - q^n - \frac{n q^{n+1}}{2} - \frac{n(3n+1)}{24} q^{n+2} + \&c.}{p} \\ &\quad + \frac{p + \frac{n+1}{2} \frac{n p^{n+2}}{2} + \&c. - q^{n+1} - \frac{n q^{n+2}}{2} - \&c.}{p} x \\ &\quad + \frac{\frac{p^{n+2}}{2} + \&c. - \frac{q^{n+2}}{2} + \&c.}{p} x^2 \\ &\quad + \&c. \end{aligned}$$

If $n > 0 < 1$, every part vanishes except the constant, which is infinite: if $n=1$, $\Delta x=1$; if $n > 1$, every term is zero.

If n is negative, there will still exist the infinite constant which may be regarded as part of the arbitrary constant; there will also exist in some instances infinite functions of x , which, as will easily be seen, may be considered in those cases as part of the arbitrary functions.

Let $n = -1$; then

$$\begin{aligned} \Delta^{-1} x &= \text{const.} - \frac{\frac{1}{2} p + \&c.}{p} x + \frac{1}{2} x^2 \\ &= \text{const.} + \frac{x(x-1)}{2} \end{aligned}$$

Let $n = -2$; and

$$\Delta^{-2} x = \text{const.} + \text{const.} x - \frac{x^2}{2} + \frac{x^3}{6}$$

and so on.

$$\text{Ex. 4.} \quad \Delta^n x^m = (x+n)^m - n(x+n-1)^m + \frac{n(n-1)}{1 \cdot 2} (x+n-2)^m - \&c.$$

by the first formula.

$$\begin{aligned} \text{Ex. 5.} \quad \Delta^n \frac{1}{(a+x)^m} &= (-1)^n \left\{ \frac{1}{(a+x)^m} - \frac{n}{(a+x+1)^m} + \&c. \right\} \\ &= (-1)^n \Delta_{x'}^n \frac{1}{x'^m}; \text{ if } x' = a+x \end{aligned}$$

Ex. 6. To find $\Delta^{\frac{1}{2}} \frac{1}{x}$.

$$\text{By formula (2), } \Delta^{\frac{1}{2}} \frac{1}{x} = \sqrt{-1} \left\{ \frac{1}{x} - \frac{\frac{1}{2}}{x+1} - \frac{1 \cdot 1}{2 \cdot 4} \frac{1}{x+2} - \&c. \right\}$$

$$\text{Let} \quad v = \frac{y^x}{x} - \frac{1}{2} \frac{y^{x+1}}{x+1} - \&c.$$

$$\begin{aligned} \text{then} \quad \frac{dv}{dy} &= y^{x-1} \left(1 - \frac{1}{2} y - \frac{1 \cdot 1}{2 \cdot 4} \cdot y^2 \&c. \right) \\ &= y^{x-1} (1-y)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \Delta^{\frac{1}{2}} \frac{1}{x} &= \sqrt{-1} \int_0^1 y^{x-1} (1-y)^{\frac{1}{2}} dy \\ &= \sqrt{-1} \int_0^1 (x, \frac{3}{2}) = \sqrt{-1} \frac{\sqrt{x} \sqrt{\frac{3}{2}}}{x + \frac{3}{2}} \end{aligned}$$

Ex. 7. To find $\Delta^n \frac{1}{x}$.

$$\Delta^n \frac{1}{x} = (-1)^n \left\{ \frac{1}{x} - \frac{n}{x+1} + \frac{n(n-1)}{1 \cdot 2} \frac{1}{x+2} - \&c. \right\}$$

$$\text{Let} \quad v = \frac{y^x}{x} - \frac{n y^{x+1}}{x+1} + \&c.$$

$$\text{then} \quad \frac{dv}{dy} = y^{x-1} (1-y)^n$$

$$\begin{aligned} \text{and} \quad \Delta^n \frac{1}{x} &= (-1)^n \int_0^1 y^{x-1} (1-y)^n dy \\ &= (-1)^n \int_0^1 (x, n+1) = (-1)^n \frac{\sqrt{x} \sqrt{n+1}}{x+n+1} \end{aligned}$$

COR. 1. If n be a whole number, $\sqrt{x+n+1} = x(x+1) \dots (x+n) \sqrt{x}$

$$\therefore \Delta^n \frac{1}{x} = (-1)^n \frac{1 \cdot 2 \dots n}{x(x+1) \dots (x+n)}$$

$$\text{COR. 2. If } n = \frac{1}{2}, \Delta^{\frac{1}{2}} \frac{1}{x} = \sqrt{-1} \frac{\sqrt{x} \sqrt{\frac{3}{2}}}{x - \frac{3}{2}} = \frac{1}{2} \sqrt{-1} \sqrt{\pi} \frac{\sqrt{x}}{x + \frac{3}{2}}$$

Ex. 8. To find, $\Delta^n \frac{1}{x^m}$, m being any integer.

$$\Delta^n \frac{1}{x^m} = (-1)^n \left\{ \frac{1}{x^m} - n \frac{1}{(x+1)^m} + \frac{n(n-1)}{1 \cdot 2} \frac{1}{(x+2)^m} - \&c. \right\} \text{ by formula (2).}$$

Let
$$v = \frac{y^x}{x^m} - n \frac{y^{x+1}}{(x+1)^m} + \&c.$$

then
$$\left(y \frac{d}{dy}\right)^m v = y^x (1-y)^n$$

$\therefore v = \int_0^1 \frac{dy}{y} \int_0^1 \frac{dy}{y} \dots (m \text{ times}) \dots y^x (1-y)^n.$

and
$$\Delta^n \frac{1}{x^m} = (-1)^n \int_0^1 \frac{dy}{y} \int_0^1 \frac{dy}{y} \int_0^1 \frac{dy}{y} \dots y^x (1-y)^n.$$

Ex. 9. To find $\Delta^n \sin ax$.

$$\begin{aligned} \Delta^n \sin ax &= \frac{1}{2\sqrt{-1}} (e^{\frac{d}{dx}} - 1)^n (e^{ax\sqrt{-1}} - e^{-ax\sqrt{-1}}) \\ &= \frac{1}{2\sqrt{-1}} \left\{ (e^{ax\sqrt{-1}} - 1)^n e^{ax\sqrt{-1}} - (e^{-ax\sqrt{-1}} - 1)^n e^{-ax\sqrt{-1}} \right\} \\ &= \frac{1}{2\sqrt{-1}} \left\{ \left(e^{\frac{a}{2}\sqrt{-1}} - e^{-\frac{a}{2}\sqrt{-1}} \right)^n e^{\frac{na}{2}\sqrt{-1} + ax\sqrt{-1}} \right. \\ &\quad \left. - (-1)^n \left(e^{\frac{a}{2}\sqrt{-1}} - e^{-\frac{a}{2}\sqrt{-1}} \right)^n e^{-\frac{na}{2}\sqrt{-1} + ax\sqrt{-1}} \right\} \\ &= \frac{1}{2\sqrt{-1}} \left(2\sqrt{-1} \sin \frac{a}{2} \right)^n \left\{ \left(\cos ax + \frac{na}{2} + \sqrt{-1} \sin ax + \frac{na}{2} \right) \right. \\ &\quad \left. - (\cos 2\lambda + 1 n\pi - \sqrt{-1} \sin 2\lambda + 1 n\pi) \left(\cos ax + \frac{na}{2} - \sqrt{-1} \sin ax + \frac{na}{2} \right) \right\} \\ &= (2\sqrt{-1})^{n-1} \sin \frac{na}{2} \left\{ \cos \left(ax + \frac{na}{2} \right) + \sqrt{-1} \sin \left(ax + \frac{na}{2} \right) \right. \\ &\quad \left. - \cos \left(2\lambda + 1 n\pi + ax + \frac{na}{2} \right) + \sqrt{-1} \sin \left(2\lambda + 1 n\pi + ax + \frac{na}{2} \right) \right\} \\ &= 2^n (\sqrt{-1})^{n-1} \sin^n \frac{a}{2} \sin \left(ax + \frac{na}{2} + \frac{2\lambda+1}{2} n\pi \right) \left(\sin 2\lambda + 1 \frac{n\pi}{2} \&c. \right. \\ &= 2^n (\cos nr\pi - \sqrt{-1} \sin nr\pi) \sin^n \frac{a}{2} \sin \left(ax + \frac{na}{2} + \frac{(2\lambda+1)}{2} n\pi \right); \end{aligned}$$

r and λ being any integers.

Ex. 10. To find $\Delta^n e^{mx} \sin ax$.

$$\begin{aligned} \Delta^n e^{mx} \sin ax &= \frac{1}{2\sqrt{-1}} (e^{\frac{d}{dx}} - 1)^n (e^{mx+ax\sqrt{-1}} - e^{mx-ax\sqrt{-1}}) \\ &= \frac{1}{2\sqrt{-1}} \left\{ (e^{m+ax\sqrt{-1}} - 1)^n e^{mx+ax\sqrt{-1}} - (e^{m-ax\sqrt{-1}} - 1)^n e^{mx-ax\sqrt{-1}} \right\} \\ &= \frac{e^{mx}}{2\sqrt{-1}} \left\{ (e^m \cos a + e^m \sqrt{-1} \sin a - 1)^n (\cos ax + \sqrt{-1} \sin ax) \right. \end{aligned}$$

$$-(e^m \cos a - e^m \sqrt{-1} \sin a - 1)^n (\cos ax - \sqrt{-1} \sin ax)\}$$

Let $e^m \cos a - 1 = P \cos \theta$, $e^m \sin a = P \sin \theta$;

then $P^2 = e^{2m} - 2e^m \cos a + 1$, and $\tan \theta = \frac{e^m \sin a}{e^m \cos a - 1}$

$$\begin{aligned} \therefore \Delta^n e^{mx} \sin ax &= \frac{e^{mx}}{2\sqrt{-1}} P^n \{(\cos n\sqrt{2\lambda\pi} + \theta + \sqrt{-1} \sin n\sqrt{2\lambda\pi} + \theta) (\cos ax + \sqrt{-1} \sin ax) \\ &\quad - (\cos n\sqrt{2\lambda'\pi} + \theta - \sqrt{-1} \sin n\sqrt{2\lambda'\pi} + \theta) (\cos ax - \sqrt{-1} \sin ax)\} \\ &= \frac{e^{mx} P^n}{\sqrt{-1}} \{\sqrt{-1} \cos n(\lambda' - \lambda)\pi + \sin n(\lambda' - \lambda)\pi\} \sin(ax + n\theta + n\sqrt{\lambda + \lambda'}\pi) \\ &= e^{mx} P^n (\cos r n \pi - \sqrt{-1} \sin r n \pi) \sin(ax + n\theta + n\lambda\pi) \end{aligned}$$

r and λ being any integers.

Similar expressions may be obtained for the n th differences of $\cos ax$ and of $e^{mx} \cos ax$.

25. We shall now proceed to the demonstration of certain theorems analogous to those in the ordinary calculus of differences.

PROP. 1.
$$\frac{1}{v_{x+n}} = \frac{1}{v_x} - \frac{nb}{v_x v_{x+1}} + \frac{n(n-1)b^2}{v_x v_{x+1} v_{x+2}} - \&c.$$

where $v_x = a + bx$ or $\Delta v_x = b$.

By formula (6); putting $\frac{1}{v_x}$ for u_x

$$\frac{1}{v_{x+n}} = \frac{1}{v_x} + n \Delta \frac{1}{v_x} + \frac{n(n-1)}{1 \cdot 2} \Delta^2 \frac{1}{v_x} + \&c.$$

Now
$$\Delta \frac{1}{v_x} = -\frac{\Delta v_x}{v_x v_{x+1}} = -\frac{b}{v_x v_{x+1}}, \quad \Delta^2 \frac{1}{v_x} = \frac{1 \cdot 2 \cdot b^2}{v_x v_{x+1} v_{x+2}} \&c, \&c.$$

$$\therefore \frac{1}{v_{x+n}} = \frac{1}{v_x} - \frac{nb}{v_x v_{x+1}} + \frac{n(n-1)b^2}{v_x v_{x+1} v_{x+2}} \&c.$$

PROP. 2. A similar result may be obtained from formula (8).

For
$$\Delta \frac{1}{v_{x-1}} = -\frac{b}{v_x v_{x-1}}, \quad \Delta^2 \frac{1}{v_{x-2}} = \frac{1 \cdot 2 \cdot b^2}{v_x v_{x-1} v_{x-2}} \&c.$$

$$\therefore \frac{1}{v_{x+n}} = \frac{1}{v_x} - \frac{nb}{v_x v_{x-1}} + \frac{n(n+1)b^2}{v_x v_{x-1} v_{x-2}} - \&c.$$

PROP. 3.
$$\Delta^n u_x v_x = v_x \Delta^n u_x + n \Delta v_x \Delta^{n-1} u_{x+1} + \&c.$$

For $\{(1 + \Delta)(1 + \Delta') - 1\} u_x v_x$ being an operation on $u_x v_x$ may be repeated according to any law, consequently

$\Delta^n u_x v_x = \{(1 + \Delta)(1 + \Delta') - 1\}^n u_x v_x$: and every step in the demonstration is the same as when n is a whole number.

COR. The same is true of the formula for the n th difference of $u_{x,y}$: for

$$\begin{aligned}\Delta u_{x,y} &= \{ (1 + \Delta_x) (1 + \Delta_y) - 1 \} u_{x,y} \\ \therefore \Delta^n u_{x,y} &= \{ \overline{1 + \Delta_x} \overline{1 + \Delta_y} - 1 \}^n u_{x,y} \\ \text{or} \quad &= (e^{\frac{d}{dx} + \frac{d}{dy}} - 1)^n u_{x,y}\end{aligned}$$

and the same results are produced as when n is a whole number.

PROP. 4. $F(\Delta) e^{rx} f(x) = e^{rx} F(e^r \overline{1 + \Delta} - 1) f(x)$

Let $u_x = e^{rx}$; and $v_x = f(x)$ in Prop. 3.

$$\begin{aligned}\therefore \Delta^n e^{rx} f(x) &= f(x) \Delta^n e^{rx} + n \Delta f(x) \Delta^{n-1} e^{rx+r} + \&c. \\ &= f(x) (e^r - 1)^n e^{rx} + n \Delta f(x) (e^r - 1)^{n-1} e^{rx+r} + \&c. \\ &= e^{rx} (e^r - 1 + e^r \Delta)^n f(x) \\ &= e^{rx} (e^r \overline{1 + \Delta} - 1)^n f(x)\end{aligned}$$

which being true for all values of n , shews that the following theorem is also true:

$$F(\Delta) \cdot e^{rx} f(x) = e^{rx} F(e^r \overline{1 + \Delta} - 1) \cdot f(x)$$

$$\text{PROP. 5. } \Delta u_x = \left(\left(1 + \frac{1}{x} \right)^D - 1 \right) u_\theta = \left\{ (1 + \Delta_\theta)^{\log \left(1 + \frac{1}{x} \right)} - 1 \right\} u_\theta$$

Let $x = e^\theta$, and let u_x be represented by u_θ when e^θ is written for x , let also Δ_θ be the symbol of difference $u_{\theta+1} - u_\theta$. Then by (C), when n is an integer,

$$\begin{aligned}x^n \frac{d^n u_x}{dx^n} &= D(D-1)(D-2) \dots (D-n+1) u_\theta \\ \therefore \Delta u_x &= (e^{\frac{d}{dx}} - 1) u_x = \left(\frac{d}{dx} + \frac{1}{1 \cdot 2} \left(\frac{d}{dx} \right)^2 + \frac{1}{1 \cdot 2 \cdot 3} \left(\frac{d}{dx} \right)^3 + \&c. \right) u_x \\ &= \left(\frac{1}{x} D + \frac{1}{1 \cdot 2} \frac{1}{x^2} D(D-1) + \frac{1}{1 \cdot 2 \cdot 3} \frac{1}{x^3} D(D-1)(D-2) + \&c. \right) u_\theta \\ &= \left(\left(1 + \frac{1}{x} \right)^D - 1 \right) u_\theta\end{aligned}$$

$$\text{But } \Delta_\theta u = (e^D - 1) u_\theta \quad \therefore e^D u_\theta = (1 + \Delta_\theta) u_\theta$$

$$\text{or } D = \log(1 + \Delta_\theta).$$

$$\text{Hence } \left(1 + \frac{1}{x} \right)^D = \left(1 + \frac{1}{x} \right)^{\log(1 + \Delta_\theta)} = (1 + \Delta_\theta)^{\log \left(1 + \frac{1}{x} \right)}$$

$$\text{and } \Delta u_x = \left(\left(1 + \frac{1}{x} \right)^D - 1 \right) u_\theta = \left\{ (1 + \Delta_\theta)^{\log \left(1 + \frac{1}{x} \right)} - 1 \right\} u_\theta$$

$$\text{COR. } u_{x+n} = (1 + \Delta_\theta)^{\log \left(1 + \frac{1}{x} \right)} u_\theta$$

PROP. 6. $u_{x+n} = (1 + \Delta_\theta)^{\log \left(1 + \frac{n}{x}\right)} u_\theta$

For $u_{x+n} = e^{n \frac{d}{dx}} u_x = \left(1 + n \frac{d}{dx} + \frac{n^2 \left(\frac{d}{dx}\right)^2}{1 \cdot 2} + \&c.\right) u_\theta$

$$= \left(1 + \frac{n}{x} D + \frac{1}{1 \cdot 2} \frac{n^2}{x^2} D(D-1) + \&c.\right) u_\theta$$

$$= \left(\left(1 + \frac{n}{x}\right)^D - 1\right) u_\theta$$

$$= (1 + \Delta_\theta)^{\log \left(1 + \frac{n}{x}\right)} u_\theta$$

It must be observed that x is considered constant with respect to Δ_θ in the formula $\left(1 + \frac{1}{x}\right)^{\log(1 + \Delta_\theta)}$. Had we supposed it otherwise, we must have taken account of the differential coefficients of $\frac{1}{x}$ itself. This would have given the following theorem.

PROP. 7. $\Delta u_x = \{(1 - e^{-\theta})^{-(D+1)} - 1\} u_\theta$

$$= \{(1 + \Delta_\theta)^{-\log(1 - e^{-\theta})} (1 - e^{-\theta})^{-1} - 1\} u_\theta$$

For $\Delta u_x = (e^{\frac{d}{dx}} - 1) u_x = \left(\frac{d}{dx} + \frac{1}{1 \cdot 2} \left(\frac{d}{dx}\right)^2 + \&c.\right) u_x$

$$= \left(\frac{1}{x} D + \frac{1}{1 \cdot 2} \frac{1}{x^2} D(D-1) + \&c.\right) u_\theta$$

$$= \left\{(D+1)e^{-\theta} + \frac{1}{1 \cdot 2} (D+2)(D+1)e^{-2\theta} + \&c.\right\} u_\theta \quad (B)$$

$$= \{(1 - e^{-\theta})^{-(D+1)} - 1\} u_\theta$$

$$= \{(1 - e^{-\theta})^{-\log(1 + \Delta_\theta)} - 1\} u_\theta$$

$$= \{(1 + \Delta_\theta)^{-\log(1 - e^{-\theta})} (1 - e^{-\theta})^{-1} - 1\} u_\theta$$

PROP. 8. $u_{x+n} = e^{n \frac{d}{dx}} u_x = \left(1 + n \frac{d}{dx} + \frac{n^2 \left(\frac{d}{dx}\right)^2}{1 \cdot 2} + \&c.\right) u_x$

$$= \left(1 + n(D+1)e^{-\theta} + \frac{n^2}{1 \cdot 2} (D+2)(D+1)e^{-2\theta} + \&c.\right) u_\theta$$

$$= (1 - n e^{-\theta})^{-(D+1)} u_\theta$$

$$= (1 + \Delta_\theta)^{-\log(1 - n e^{-\theta})} (1 - n e^{-\theta})^{-1} u_\theta$$

It is manifest that these formulæ do not follow the distributive law. They cannot, consequently, be applied with any great advantage to the solution of equations of differences. We shall exhibit their application only in one instance.

EQUATIONS OF DIFFERENCES.

26. As the method of solving equations of differences of the second and higher orders, by treating the symbols of operation as symbols of quantity, and reducing the resulting fraction by decomposing it into partial fractions, has been little, if at all, employed, we shall commence with an example or two in ordinary equations of differences.

Ex. 1. $u_{x+3} + a u_{x+2} + b u_{x+1} + c u_x = X.$

This may be written

$$\{(1+\Delta)^3 + a(1+\Delta)^2 + b(1+\Delta) + c\}u_x = X.$$

If we write Δ , for $1+\Delta$, and suppose a, β, γ the roots of the equation $\Delta^3 + a\Delta^2 + b\Delta + c = 0$; we get $(\Delta - a)(\Delta - \beta)(\Delta - \gamma)u_x = X$, or

$$u_x = \frac{1}{(\Delta - a)(\Delta - \beta)(\Delta - \gamma)} \cdot (X + 0).$$

This equation is reduced, by the decomposition of the fraction of operation into its equivalent partial fractions, to

$$u_x = \frac{1}{(a - \beta)(a - \gamma)} \frac{1}{(\Delta - a)} (X + 0) + \frac{1}{(\beta - a)(\beta - \gamma)} \frac{1}{(\Delta - \beta)} (X + 0) \\ + \frac{1}{(\gamma - a)(\gamma - \beta)} \frac{1}{(\Delta - \gamma)} (X + 0).$$

Now $\frac{1}{\Delta - a}(X + 0)$ is the solution of the equation $v_{x+1} - a v_x = X + 0$;

hence it is equal to $a^x \left(A + \sum \frac{X}{a^{x+1}} \right)$; and similarly of the others.

Hence the complete solution of the given equation is

$$u_x = \frac{1}{(a - \beta)(a - \gamma)} a^x \left(A + \sum \frac{X}{a^{x+1}} \right) + \frac{1}{(\beta - a)(\beta - \gamma)} \beta^x \left(B + \sum \frac{X}{\beta^{x+1}} \right) \\ + \frac{1}{(\gamma - a)(\gamma - \beta)} \gamma^x \left(C + \sum \frac{X}{\gamma^{x+1}} \right)$$

COR. If $a = \beta$, we must, as in similar cases, put $a + c$ for β , and expand in terms of c . The result is

$$u_x = -\frac{1}{c(a - \gamma)} a^x \left(A + \sum \frac{X}{a^{x+1}} \right) + \\ \frac{1}{a - \gamma} \left(\frac{a^x}{c} + x a^{x-1} - \frac{a^x}{a - \gamma} \right) \left(B + \sum \frac{X}{a^{x+1}} - c \sum \frac{X(x+1)}{a^{x+2}} \right) \\ + \frac{1}{(\gamma - a)^2} \gamma^x \left(C + \sum \frac{X}{\gamma^{x+1}} \right)$$

$$= \frac{A, a^x}{a-\gamma} + \frac{B x a^{x-1}}{a-\gamma} + \left(\frac{x a^{x-1}}{a-\gamma} - \frac{a^x}{(a-\gamma)^2} \right) \Sigma \frac{X}{a^{x+1}} - \frac{a^x}{a-\gamma} \Sigma \frac{X(x+1)}{a^{x+2}} \\ - \frac{1}{(\gamma-a)^2} \gamma^x \left(C + \Sigma \frac{X}{\gamma_{x+1}} \right).$$

In precisely the same manner we may integrate the general equation with constant coefficients.

Let us apply the formula of Prop. 6 to the following example.

Ex. 2. $u_x - 3(x+1) u_{x+1} + 2(x+1)(x+2) u_{x+2} = 0.$

Calling $\left(1 + \frac{1}{x}\right)^l$ we get

$$u_\theta - 3(e^\theta + 1)(1 + \Delta_\theta)^l u_\theta + 2(e^\theta + 1)(e^\theta + 2)(1 + \Delta_\theta)^{\left(1 + \frac{2}{x}\right)} u_\theta = 0$$

Now since $e^{r\theta}(1 + \Delta_\theta)^l u_\theta = e^{-r}{}^l(1 + \Delta_\theta)^l e^{r\theta} u_\theta$
we have

$$u_\theta - 3(1 + e^{-\theta}) e^{-l}(1 + \Delta_\theta)^l e^\theta u_\theta + 2(e^\theta + 1)(1 + 2e^{-\theta}) e^{-\left(1 + \frac{2}{x}\right)}(1 + \Delta_\theta)^{\left(1 + \frac{2}{x}\right)} e^\theta u_\theta = 0.$$

or $u_\theta - 3(1 + \Delta_\theta)^l e^\theta u_\theta + 2(e^\theta + 1)(1 + \Delta_\theta)^{\left(1 + \frac{2}{x}\right)} e^\theta u_\theta = 0.$

Put $(1 + \Delta_\theta)^l \cdot (1 + \Delta_\theta)^l$ for $(1 + \Delta_\theta)^{\left(1 + \frac{2}{x}\right)}$
where Δ_θ in the former operates on the x in the latter.

$$\therefore u_\theta - 3(1 + \Delta_\theta)^l e^\theta u_\theta + 2(1 + e^{-\theta}) e^\theta (1 + \Delta_\theta)^l \cdot (1 + \Delta_\theta)^l e^\theta u_\theta = 0$$

or $u_\theta - 3(1 + \Delta_\theta)^l e^\theta u_\theta + 2(1 + \Delta_\theta)^l e^\theta \cdot (1 + \Delta_\theta)^l e^\theta u_\theta = 0$

or $u_\theta - 3(1 + \Delta_\theta)^l e^\theta u_\theta + 2(1 + \Delta_\theta)^l e^{\theta^2} u_\theta = 0$

which can be resolved into the two

$$\{1 - (1 + \Delta_\theta)^l e^\theta\} u_\theta = 0 \text{ and } \{1 - 2(1 + \Delta_\theta)^l e^\theta\} u_\theta = 0$$

or $u_x - (x+1) u_{x+1} = 0$ and $u_x - 2(x+1) u_{x+1} = 0$

where $u_x = \frac{A}{x+1}$ or $u_x = \frac{B}{2^x(x+1)}$

and therefore generally, $u_x = \frac{A + B 2^{-x}}{x+1}$, which is the complete solution of the equation.

It is evident that the process employed in Example 1, applies equally in this Example, when a function of x appears on the right-hand side of the equation. Hence

Ex. 3. $u_x - 3(x+1) u_{x+1} + 2(x+1)(x+2) u_{x+2} = X$ gives

$$u_\theta = -\frac{X+0}{1-(1+\Delta_\theta)^l e^\theta} + 2\frac{X+0}{1-2(1+\Delta_\theta)^l e^\theta}$$

$$= \frac{A+B \cdot 2^{-x}}{\sqrt{x+1}} + \frac{1}{\sqrt{x+1}} \Sigma \sqrt{x+1} X - \frac{2^{-x+1}}{\sqrt{x+1}} \Sigma 2^x \sqrt{x+1} X$$

COR. In the same manner may the more general equation

$$u_x - a(x+1)^r u_{x+1} + b(x+1)^r (x+2)^r u_{x+2} + \&c. = X, \text{ be solved.}$$

It is not necessary to solve such equations as $u_{x+\frac{1}{2}} + a u_x + \&c. = X$, since it is evident that, by putting $x = \frac{1}{2} x'$, this form of equation is reduced to $v_{x'} + a v_{x'+1} + \&c. = X$, which has been already solved.

We proceed then to the solution of equations involving fractional differences. Here we must, at present, confine ourselves to very simple examples.

Ex. 4. $\Delta^{\frac{1}{2}} u_x - a u_x = 0.$

Since $\Delta^{\frac{1}{2}} e^{m x} = (e^m - 1)^{\frac{1}{2}} e^{m x}$

It is evident that if $m = \log(a^2 + 1)$ the solution of the equation is $u_x = A e^{m x}$,

$$= A e^{x \log(a^2 + 1)} = A (a^2 + 1)^x$$

Ex. 5. $\Delta^{\frac{1}{2}} u_x - a u_x = c e^{a x}$

$$u_x = A (a^2 + 1)^x + \frac{c}{(e^a - 1)^{\frac{1}{2}} - a} e^{a x}$$

or, if $e^a = 1 + b^2$, $u_x = A (a^2 + 1)^x + \frac{c}{b - a} (b^2 + 1)^x$

COR. If $b = a$, this solution fails. Put $b = a + \beta$ as in similar cases in Differential Equations :

then $\frac{c}{b-a} (b^2 + 1)^x = \frac{c}{\beta} (a^2 + 1 + 2 a \beta)^x = \frac{c}{\beta} (a^2 + 1)^x + \frac{c}{\beta} 2 a \beta x (a^2 + 1)^{x-1}$

$$u_x = A_1 (a^2 + 1)^x + 2 a c x (a^2 + 1)^{x-1}$$

A_1 being an arbitrary constant.

It may be interesting to verify this solution.

$$\Delta^{\frac{1}{2}} (a^2 + 1)^x - a \cdot (a^2 + 1)^x = 0 \text{ evidently,}$$

and $\Delta^{\frac{1}{2}} 2 a c x (a^2 + 1)^{x-1} = 2 a c x a (a^2 + 1)^{x-1} + \frac{1}{2} 2 a c (a^2 + 1)^x \frac{1}{a}$ by Prop. 3.

$$= 2 a^2 c x (a^2 + 1)^{x-1} + c (a^2 + 1)^x$$

$\therefore \Delta^{\frac{1}{2}} \cdot 2 a c x (a^2 + 1)^{x-1} - a \cdot 2 a c x (a^2 + 1)^{x-1} = c (a^2 + 1)^x = c e^{a x}$ as it ought.

Ex. 6. Generally, let $\Delta^{\frac{1}{2}} u_x - a u_x = X$

then $u_x = A (a^2 + 1)^x + (\Delta^{\frac{1}{2}} - a)^{-1} X$

$$= A(a^2 + 1)^x + \frac{\Delta^{\frac{1}{2}} + a}{\Delta - a^2} X$$

$$= A(a^2 + 1)^x + (\Delta^{\frac{1}{2}} + a)(a^2 + 1)^x \Sigma \frac{X}{(a^2 + 1)^{x+1}}$$

COR. 1. Let $X = bx$; then the solution of the equation $\Delta^{\frac{1}{2}} u_x - a u_x = bx$ is

$$u_x = A(a^2 + 1)^x - (\Delta^{\frac{1}{2}} + a) \cdot \left(\frac{bx}{a^2} + \frac{b}{a^4} \right)$$

COR. 2. Let $X = b \frac{a^2 x + a^2 + 1}{x(x+1)}$

then the value of $(\Delta - a^2)^{-1} X$ is $-\frac{b}{x}$

therefore the solution of the equation $\Delta^{\frac{1}{2}} u_x - a u_x = b \frac{a^2 x + a^2 + 1}{x(x+1)}$

is $u_x = A(a^2 + 1)^x - (\Delta^{\frac{1}{2}} + a) \frac{b}{x}$

$$= A(a^2 + 1)^x - \frac{ab}{x} - b \sqrt{-1} \frac{\sqrt{x}^{\frac{3}{2}}}{x + \frac{3}{2}} \quad (\text{by Ex. 6, Art. 24.})$$

EX. 7. $\Delta u_x + a \Delta^{\frac{1}{2}} u_x + b u_x = 0.$

This equation may be written

$$(\Delta + a \Delta^{\frac{1}{2}} + b) u_x = 0.$$

Let α, β be the roots of the equation $z^2 + az + b = 0$, then

$$(\Delta^{\frac{1}{2}} - \alpha)(\Delta^{\frac{1}{2}} - \beta) u_x = 0$$

$$u_x = A(\Delta^{\frac{1}{2}} - \alpha)^{-1} \cdot 0 + B(\Delta^{\frac{1}{2}} - \beta)^{-1} \cdot 0$$

$$= A(1 + \alpha^2)^x + B(1 + \beta^2)^x \quad (\text{Ex. 4.})$$

COR. If $\alpha = \beta$, we obtain, by the usual process, $u_x = (A + Bx)(1 + \alpha^2)^x.$

EX. 8. $\Delta u_x + a \Delta^{\frac{1}{2}} u_x + b u_x = c(1 + e^2)^x$

The solution is

$$u_x = \frac{1}{a - \beta} (\Delta^{\frac{1}{2}} - \alpha)^{-1} (0 + c \sqrt{1 + e^2})^x - \frac{1}{a - \beta} (\Delta^{\frac{1}{2}} - \beta)^{-1} (0 + c \sqrt{1 + e^2})^x$$

$$= A(1 + \alpha^2)^x + B(1 + \beta^2)^x + \frac{c(1 + e^2)^x}{(a - \beta)(e - a)} - \frac{c(1 + e^2)^x}{(a - \beta)(e - \beta)}$$

$$= A(1 + \alpha^2)^x + B(1 + \beta^2)^x + \frac{c(1 + e^2)^x}{e^2 + ae + b}$$

COR. If $e = a$, we must proceed as in Cor. 3, Ex. 4, Class 1, of Differential Equations, and we shall obtain

$$u_x = A(1 + \alpha^2)^x + B(1 + \beta^2)^x + \frac{cx(1 + \alpha^2)^{x-1}}{2a + a}$$

Ex. 9. $\Delta u_x + a \Delta^{\frac{1}{2}} u_x + b u_x = X$

$$\begin{aligned} u_x &= \frac{1}{a-\beta} (\Delta^{\frac{1}{2}} - a)^{-1} (0 + X) - \frac{1}{a-\beta} (\Delta^{\frac{1}{2}} - \beta)^{-1} (0 + X) \\ &= A (1 + a^2)^x + B (1 + \beta^2)^x + \frac{1}{a-\beta} (\Delta^{\frac{1}{2}} - a)^{-1} \cdot X - \frac{1}{a-\beta} (\Delta^{\frac{1}{2}} - \beta)^{-1} X \\ &= A (1 + a^2)^x + B (1 + \beta^2)^x + \frac{1}{a-\beta} (\Delta^{\frac{1}{2}} + a) (1 + a^2)^x \Sigma \frac{X}{(1 + a^2)^{x+1}} \\ &\quad - \frac{1}{a-\beta} (\Delta^{\frac{1}{2}} + \beta) (1 + \beta^2)^x \Sigma \frac{X}{(1 + \beta^2)^{x+1}} \end{aligned}$$

Ex. 10. $u_x + a x \Delta^{\frac{1}{2}} u_x = X.$

This equation gives $u_x = \frac{1}{1 + a x \Delta^{\frac{1}{2}}} X$

$$\begin{aligned} &= \frac{1 - a x \Delta^{\frac{1}{2}}}{1 - a^2 x \Delta^{\frac{1}{2}} x \Delta^{\frac{1}{2}}} X \\ &= (1 - a x \Delta^{\frac{1}{2}}) v_x, \text{ where } v_x \text{ is determined by the equation} \end{aligned}$$

$$v_x - a^2 x \Delta^{\frac{1}{2}} x \Delta^{\frac{1}{2}} v_x = X$$

Now $\Delta^{\frac{1}{2}} x \Delta^{\frac{1}{2}} v_x = x \Delta v_x + \frac{1}{2} v_{x+1}$ (Prop. 3.)

$\therefore v_x - a^2 x^2 \Delta v_x - \frac{a^2}{2} x v_{x+1} = X$

or $v_{x+1} - \frac{2}{a^2} \frac{1 + a^2 x^2}{x + 2 x^2} v_x = - \frac{2 X}{a^2 (x + 2 x^2)}$

which being solved by the ordinary method, v_x and therefore u_x (provided $\Delta^{\frac{1}{2}} v_x$ can be found) is known.

Equations of Differences with two independent variables are not capable of solution to any great extent. An example or two will suffice to illustrate our process.

Ex. 11. $\Delta_x u_{x,y} - \Delta_y u_{x,y} = b.$

The solution is $u_{x,y} = \frac{1}{\Delta_x - \Delta_y} b$

Treat Δ_y as a constant c , then the solution of $(\Delta_x - c) u_{x,y} = b$ is

$$\begin{aligned} u_{x,y} &= (c+1)^x \Delta - \frac{b}{c} \\ u_{x,y} &= (\Delta_y + 1)^x v_y - \Delta_y - 1 \cdot b \\ &= \left(e^{\frac{d}{dy}} \right)^x v_y - b y \\ &= v_{y+x} - b y \end{aligned}$$

v_{y+x} being an arbitrary function of $y+x$.

Ex. 12. $\Delta_x^{\frac{1}{2}} u_{x,y} - \Delta_y^{\frac{1}{2}} u_{x,y} = b$

This equation gives
$$u_{x,y} = \frac{1}{\Delta_x^{\frac{1}{2}} - \Delta_y^{\frac{1}{2}}} b$$

$$= \frac{\Delta_x^{\frac{1}{2}} + \Delta_y^{\frac{1}{2}}}{\Delta_x - \Delta_y} b$$

$$= (\Delta_x^{\frac{1}{2}} + \Delta_y^{\frac{1}{2}}) (v_{y+x} - b y) \quad (\text{Ex. 11.})$$

We have thus succeeded in solving equations with fractional indices of all forms corresponding with the ordinary forms of Linear Differential Equations, whether total or partial,—whether solitary or simultaneous. We have also placed the Calculus of Fractional Differences on the same footing with respect to the ordinary Calculus of Fractional Differences as that which the Calculus of General Differentiation occupies relatively to the ordinary Differential Calculus.

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EDINBURGH, *October 10, 1846.*