that every triad of its component points is a *restricted* triad (§ 90). The points in such an extended linear set may be regarded as forming a circuit. This circuit may be divided into two ordinary straight lines in an infinite number of ways.

156. In the same way, in place of the set of points in ordinary geometric space which compose a plane, *i.e.*, which are such that every tetrad of them is a *flat tetrad* (§ 150), in the extended geometric set we have an extended plane which is such that every tetrad of its component points is a *restricted tetrad.*

157. And generally, in place of the set of points in flat space of infinite dimensions which compose *a flat space of n dimensions, i.e.*, are such that every $(n+2)$-ad of them is a flat $(n+2)$-ad, we have in our extended geometric space a set of points composing a *restricted space of n dimensions, i.e.*, such that every $(n+2)$-ad of them is a restricted $(n+2)$-ad.

158. It would carry me beyond the purpose of this paper to develop further the consequences of thus regarding geometric space of any dimensions as only a half of the more complete and symmetrical space derived by taking into account the obverses of points as well as the points themselves. It seemed right, however, to call attention to the fact that space could be so regarded, as such fact is plainly brought out by the preceding results.

On the Square of Euler's Series.

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Introduction, §§ 1–3.

1. Few results in pure algebra are more curious than Euler's celebrated theorem that the expanded value of the infinite product

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\ \ldots$$

is the series

$$1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-\&c.,$$

the general term being $(-1)^n q^{\frac{1}{2}n(3n\pm 1)}$. The exponents are the pentagonal numbers, and the signs of the terms, after the first, are negative and positive in pairs.

Euler discovered the law of the series by actually multiplying together the factors $1-q$, $1-q^2$, $1-q^3$, &c. He published his result in 1750, and applied it to obtain various remarkable theorems relating to partitions, sums of divisors, &c.* Finally, in 1754, he succeeded in demonstrating the equality of the product and the series.†

2. Among the numerous and wonderful algebraical identities that sprang out from the analysis of Elliptic Functions as Jacobi developed the subject in the historic *Fundamenta Nova*, not only did the expanded value of the Eulerian product present itself, but also a hitherto unsuspected expression of its cube‡. The series thus found by Jacobi was

$$1-3q+5q^3-7q^6+9q^{10}-11q^{15}+13q^{21}-\&c.,$$

the general term being $(-1)^n(2n+1)\,q^{\frac{1}{2}n(n+1)}$. The exponents are the triangular numbers, and the terms are alternately positive and negative.

We thus see that the cube of the Eulerian product

$$(1-q)(1-q^2)(1-q^3)\,\ldots,$$

or of the Eulerian series $\quad 1-q-q^2+q^5+q^7-\&c.,$

is the Jacobian series $\quad 1-3q+5q^3-7q^6+\&c.$

3. The object of the present paper is to consider the law of the coefficients in the series which is equal to the square of the Eulerian product or series. I had no hope that these coefficients would follow any simple law, as in the Eulerian or Jacobian series; for, if such a law existed, it could not fail to have been discovered long ago by observation. Nevertheless, it seemed interesting to examine in some detail the functions of n which form the coefficients of the powers of q.

It will be seen in the following articles that the coefficient of q^n depends upon a function of $12n+1$, which is denoted by $G(12n+1)$. This function (in common with a great many functions connected with the Theory of Numbers) possesses the property that, if p and r are relatively prime, $G(pr)=G(p)\,G(r)$. We are thus enabled to obtain rules by means of which the value of G can be calculated with great facility (§§ 9, 10). It is shown that the numerical value of the coefficient of q^n can always be assigned by means of the *real* divisors of $12n+1$ (§ 13).

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* *Commentationes Arithmeticæ Collectæ*, Vol. I., pp. 91, 151.

† *Ibid.*, Vol. I., p. 234. Full references to the subject of Euler's product are given in a note to Art. 129 of H. J. S. Smith's "Report on the Theory of Numbers" (*Report of the British Association*, 1865, p. 346). See also §§ 33, 34 of the present paper.

‡ *Fundamenta Nova* (1829), p. 185.

In the latter part of the paper, various recurring formulæ, which are available for the calculation of G, are investigated. Two other functions E and H are also considered (§§ 15 and 28).

Squaring the Series, §§ 4–7.

4. Let

$$(1-q-q^2+q^5+q^7-q^{12}-q^{15}+\&c.)^2 = 1+P_1q+P_2q^2+P_3q^3+P_4q^4+\&c$$

Put q^{24} for q, and multiply throughout by q^2; we thus find

$$(q-q^{25}-q^{49}+q^{121}+q^{169}-q^{289}-q^{361}+\&c.)^2$$
$$= q^2+P_1q^{26}+P_2q^{50}+P_3q^{74}+P_4q^{98}+\&c.$$

This equation shows that P_n is equal to the number of compositions of $24n+2$ as the sum of two squares both of which are of the form $(12r\pm1)^2$, or of the form $(12r\pm5)^2$, diminished by the number of compositions in which one square is of one form and the other square of the other form.

5. If a number be expressible as the sum of two uneven squares, it must be $\equiv 2$, mod. 8; and if we put

$$8n+2 = (2r+1)^2+(2s+1)^2,$$

we have $$4n+1 = (r-s)^2+(r+s+1)^2,$$

in which one square is even and the other uneven.

It can also be shown that every composition of $8n+2$ corresponds to four representations of $4n+1$.

6. By considering the separate cases which arise in § 4, it will be found that every composition of $24n+2$ of either of the forms

$$(12r\pm1)^2+(12s\pm1)^2, \quad (12r\pm5)^2+(12s\pm5)^2$$

corresponds to four representations of $12n+1$ of the form

$$(6r)^2+(6s\pm1)^2,$$

and that every composition of the form

$$(12r\pm1)^2+(12s\pm5)^2$$

corresponds to four representations of the form

$$(6r\pm2)^2+(6s\pm3)^2.$$

We are thus enabled to express the value of P_n by means of the representations of $12n+1$, instead of by means of the compositions of

$24n+2$; viz., we have found that $4P_n$ is equal to the number of representations of $12n+1$ as a sum of two squares of the form

$$(6r)^2+(6s\pm 1)^2$$

diminished by the number of representations of the form

$$(6r\pm 2)^2+(6s\pm 3)^2.*$$

7. We therefore put

$$P_n = G(12n+1),$$

where $4G(n)$ denotes the excess of the number of representations of n of the form $(6r)^2+(6s+1)^2$ over those of the form $(6r+2)^2+(6s+3)^2$. This excess may, of course, be negative.

We are only concerned with the function $G(n)$ for values of n which are $\equiv 1$, mod. 12, and in this case the representations of n as a sum of two squares are necessarily of the forms

$$(6r)^2+(6s+1)^2 \text{ or } (6r+2)^2+(6s+3)^2.$$

In what follows, the argument of the function G will always be supposed to be $\equiv 1$, mod. 12.

The Function G, §§ 8–14.

8. If p and q are any two numbers $\equiv 1$, mod. 12, which are prime to each other, then

$$G(pr) = G(p)\,G(r).$$

To prove this, resolve p and r in all possible ways into their conjugate complex factors, so that

$$p = (a_1+ib_1)(a_1-ib_1) = (a_2+ib_2)(a_2-ib_2) = \dots,$$

$$r = (\alpha_1+i\beta_1)(\alpha_1-i\beta_1) = (\alpha_2+i\beta_2)(\alpha_2-i\beta_2) = \dots.$$

Let these equations be written

$$p = p_1p_1' = p_2p_2' = \dots = p_ap_a' = \pi_1\pi_1' = \pi_2\pi_2' = \dots = \pi_\alpha\pi_\alpha',$$

$$r = r_1r_1' = r_2r_2' = \dots = r_br_b' = \rho_1\rho_1' = \rho_2\rho_2' = \dots = \rho_\beta\rho_\beta',$$

where p_1 and p_1', p_2 and p_2', &c. are pairs of conjugate factors; and $p_1, p_2, \dots, p_a, r_1, r_2, \dots r_b$ are of the form $6r+i(6s+1)$, and $\pi_1, \pi_2, \dots \pi_\alpha$, $\rho_1, \rho_2, \dots, \rho_\beta$ of the form $6r+2+i(6s+3)$.

Thus $\qquad G(p) = a-\alpha, \quad G(r) = b-\beta.$

Since p and r are prime to each other, the resolutions of pr into pairs

* We may dispense with the double sign in $(6s\pm 1)^2$, $(6r\pm 2)^2$, $(6s\pm 3)^2$, if we admit negative as well as positive values of the integers r and s. This will be supposed to be the case in future, and the double sign will be omitted throughout the whole of the subsequent portion of the paper.

of conjugate complex factors will be of the four types

$$prp'r', \quad p\rho p'\rho', \quad \pi r\pi' r', \quad \pi\rho\pi'\rho'.$$

Now the product of two numbers of the form $6r+i\,(6s+1)$, or of two numbers of the form $6r+2+i\,(6s+3)$, is of the form $6r+i\,(6s+1)$; and the product of two numbers of which one is of one form, and the other of the other form, is of the form $6r+2+i\,(6s+3)$.

Thus the numbers which are of the types pr and $\pi\rho$ are of the form $6r+i\,(6s+1)$, and those which are of the types $p\rho$, πr are of the form $6r+2+i\,(6s+3)$. Now the numbers of numbers of the forms pr, $p\rho$, πr, $\pi\rho$ are respectively ab, $a\beta$, ab, $a\beta$; therefore

$$\begin{aligned} G\,(pr) &= ab - a\beta - ab + a\beta \\ &= (a-a)(b-\beta) = G\,(p)\,G\,(r).* \end{aligned}$$

9. The values of $G\,(n)$ for all values of n may therefore be derived by simple multiplication from the values of $G\,(n)$ for the cases in which n is a prime or a power of a prime. It is only necessary, therefore, to consider the values of $G\,(n)$ for these values of n.

The only possible factors of a number which is $\equiv 1$, mod. 12, are necessarily $\equiv 11$, 7, 5, or 1, mod. 12. The cases in which n is a prime, or a power of a prime, having one of these forms will now be examined.

(1) Values of $G\,(a^s)$ where a is a prime $\equiv 11$ or 7, mod. 12. Primes of these forms have no complex factors. If a be uneven, $G\,(a^s) = 0$; for a^s cannot be expressed as a sum of two squares. If a be even,

* It will be observed that the reasoning in the text admits of a much more general application. For suppose that, connected with a number P, we have a set of quantities $p_1, p_2, \ldots, p_a, \pi_1, \pi_2, \ldots \pi_a$, and connected with a number R we have a similar set $r_1, r_2, \ldots r_b, \rho_1, \rho_2, \ldots \rho_\beta$; and suppose that the corresponding set, connected with the number PR, may be obtained by multiplying all the members of the P-set by those of the R-set. Suppose, further, that the quantities denoted by the Italic letters belong to a class A, and that the quantities denoted by the Greek letters belong to a class B, and that the product of two members of class A, or of class B, is a member of class A, but that the product of one member of class A and one member of class B is a member of class B. Then, if

$$\phi\,(P) = p_1^k + p_2^k + \ldots + p_a^k \pm (\pi_1^k + \pi_2^k + \ldots + \pi_a^k),$$

and

$$\phi\,(R) = r_1^k + r_2^k + \ldots + r_b^k \pm (\rho_1^k + \rho_2^k + \ldots + \rho_\beta^k),$$

it follows that

$$\phi\,(PR) = \phi\,(P)\,\phi\,(R).$$

The function G belongs to the particular case of $k = 0$, the negative sign being taken. As other examples of the application of this reasoning may be mentioned the function $E_k\,(n)$ which expresses the excess of the sum of the k^{th} powers of the divisors of n which $\equiv 1$, mod. 4, over the sum of the k^{th} powers of the divisors which $\equiv 3$, mod. 4; and the function $H_k\,(n)$ which expresses the sum of the k^{th} powers of the divisors of n which $\equiv 1$, mod. 3, over the sum of the k^{th} powers of the divisors which $\equiv 2$, mod. 3. The function $E\,(n)$ and $H\,(n)$ (§§ 15 and 28) are the particular cases corresponding to $k = 0$ of the functions $E_k\,(n)$ and $H_k\,(n)$.

$G(a^\alpha) = 1$; for p^α can be expressed as a sum of two squares in one way, *i.e.*, as $0^2 + (a^{\frac{1}{2}\alpha})^2$. This partition* is of the form $(6r)^2 + (6s+1)^2$; and it gives rise to four representations; therefore the value of G is $+1$.

(2) Values of $G(b^\beta)$ where b is a prime $\equiv 5$, mod. 12.

In this case the two conjugate complex factors of b are of the form $6r + 2 \pm i\,(6s+1)$. Thus $G(b) = 0$. The partitions of b^2 as the sum of two squares consist of $0^2 + b^2$ and a single partition of the form $(6r+2)^2 + (6s+3)^2$. The former gives rise to four representations, the latter to eight. Thus $G(b^2) = 1 - 2 = -1$.

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* In counting the number of compositions of a number as a sum of squares, the order of the squares is to be taken into account. In counting the representations, the signs of the roots of the squares are to be taken into account as well. Each composition of a number as a sum of r squares none of which is zero gives rise, therefore, to 2^r representations. When we are restricting ourselves to uneven squares, it is convenient to consider compositions; but, when even squares are admitted (including zero among them), it is necessary to consider representations instead. There seems to be need of a word to express a partitionment into squares without reference to their order or to the signs of their roots. For this purpose I have in other writings used the word *resolution*; but in this paper I use the word *partition*, which seems to me preferable. Thus, for example, taking two squares, the partition $1^2 + 2^2$ gives rise to the two compositions $1^2 + 2^2$ and $2^2 + 1^2$, and to the eight representations

$$(+1)^2 + (+2)^2,\quad (-1)^2 + (+2)^2,\quad (+1)^2 + (-2)^2,\quad (-1)^2 + (-2)^2,$$
$$(+2)^2 + (+1)^2,\quad (-2)^2 + (+1)^2,\quad (+2)^2 + (-1)^2,\quad (-2)^2 + (-1)^2;$$

the partition $1^2 + 0^2$ gives rise to two compositions, but to only four representations. (The square 0^2 is treated in exactly the same manner as any other square, as far as partitions or compositions are concerned; but, when representations are considered, it differs from all other squares in having one root instead of two.)

It seems to me convenient to regard these meanings of the words *partition* and *composition* as of general application, whenever a number is to be partitioned into a given number of the parts $\alpha, \beta, \gamma, \ldots$. Thus every distinct manner in which a number n can be produced by the addition of the parts $\alpha, \beta, \gamma, \ldots$ is called a *partition* into these parts. If, in addition, we take cognisance of the places occupied by the parts, we use the word *composition*. Thus two partitions are identical if the parts occurring in them are the same; but for two compositions to be the same, it is further necessary that the same part should occupy the same place. (We may, of course, consider partitions in which no limit is placed on the number of times that any part may occur, or in which the same part may not occur twice, or in which any other condition is imposed. When the word *partition* is used without qualification, it is understood that the same part may occur any number of times.) These definitions of partitions and compositions do not conflict in any respect with ordinary usage.

Partitions and compositions have reference solely to the magnitude of the parts by the addition of which the given number is produced. The word *representation* has reference to a different kind of problem, *i.e.*, to the number of possible solutions, in integral numbers, of systems of equations. Partitions and compositions are concerned only with the magnitudes of $\alpha, \beta, \gamma \ldots$, not with their structure; the word representation has the technical meaning assigned to it by Gauss in the Theory of Forms. I may add that the above use of partition corresponds exactly to Gauss's definition of *discerptio*; for he distinguishes between *discerptiones* and *repræsentationes* as follows:—"Discerptiones numerorum (ut formarum binariarum supra) in tria quadrata a repræsentationibus per $xx + yy + zz$ ita distinguimus, ut in illis ad solam quadratorum magnitudinem, in his vero insuper ad ipsorum ordinem radicumque signa respiciamus, adeoque repræsentationes $x = a$, $y = b$, $z = c$ et $x = a'$, $y = b'$, $z = c'$ pro diversis habeamus nisi simul $a = a'$, $b = b'$, $c = c'$; discerptiones autem in $aa + bb + cc$ et in $a'a' + b'b' + c'c'$ pro una, si nullo ordinis respectu habito hæc quadrata illis æqualia sunt" (*Disq. Arith.*, § 292).

In the case of b^3 there is no partition into two squares of the forms considered. Thus $G(b^3) = 0$. For b^4 we have the partitions of b^2 each multiplied by b^2, and a new partition, which is of the form

$$(6r)^2 + (6s+1)^2.$$

Thus $$G(b^4) = 1-2+2 = 1.$$

Proceeding in this manner, we find that, if β be uneven, $G(b^\beta) = 0$, and that, if β be even, $G(b^\beta) = (-1)^{\frac{1}{2}\beta}$.

(3) Values of $G(c^\gamma)$ where c is a prime $\equiv 1$, mod. 12.

The two conjugate complex factors of c may be either of the form $6r \pm i(6s+1)$ or of the form $6r+2 \pm i(6s+3)$. Consider these two cases separately.

(i.) If c be of the form $(6r)^2+(6s+1)^2$, then

$$G(c) = 2, \quad G(c^2) = 3, \quad G(c^3) = 4,$$

and in general $$G(c^\gamma) = \gamma+1.$$

(ii.) If c be of the form $(6r+2)^2+(6s+3)^2$, then

$$G(c) = -2, \quad G(c^2) = 3, \quad G(c^3) = -4,$$

and in general $$G(c^\gamma) = (-1)^\gamma (\gamma+1).$$

10. Thus, on the whole, we find that, if

$$12n+1 = a_1^{a_1} a_2^{a_2} \dots b_1^{\beta_1} b_2^{\beta_2} \dots e_1^{\epsilon_1} e_2^{\epsilon_2} \dots f_1^{\phi_1} f_2^{\phi_2},$$

where

$a_1, a_2, \dots$ are primes $\equiv$ 11 or 7, mod. 12,
$b_1, b_2, \dots$ " " $\equiv$ 5, mod. 12,
$e_1, e_2, \dots$ " " $\equiv$ 1, mod. 12, and of the form $(6r)^2+(6s+1)^2$,
$f_1, f_2, \dots$ " " $\equiv$ 1, mod. 12, and of the form $(6r+2)^2+(6s+3)^2$,

then $$G(12n+1) = 0,$$

unless $a_1, a_2, \dots$, and $\beta_1, \beta_2, \dots$, are all even (including zero as an even number); and that, if these exponents are all even, then

$$G(12n+1) = (-1)^{\frac{1}{2}\beta_1+\frac{1}{2}\beta_2+\dots} \times (\epsilon_1+1)(\epsilon_2+1)\dots \times (-1)^{\phi_1+\phi_2+\dots}(f_1+1)(f_2+1)\dots.$$

For example,

$$P_{14} = G(169) = G(13^2) = +3,$$
$$P_{22} = G(265) = G(5.53) = G(5)\,G(53) = 0,$$
$$P_{27} = G(325) = G(5^2.13) = G(5^2)\,G(13) = -1 \times -2 = +2,$$
$$P_{79} = G(949) = G(13.73) = G(13)\,G(73) = -2 \times -2 = +4.$$

As a more complicated example, we may take

$$G\,(11^4 . 19^2 . 5^2 . 17^4 . 13^3 . 37 . 97^3 . 109^4),$$

which $$= -1 \times 3 \times 2 \times -4 \times 5 = +120.$$

The even powers of the primes 11 and 19 (which are $\equiv 11$ and 7, mod. 12) produce no effect. The same is true of the evenly even power of 17 (which $\equiv 5$, mod. 12), but the unevenly even power of 5 (which also $\equiv 5$, mod. 12) gives rise to the factor -1. The only primes remaining are $\equiv 1$, mod. 12. The first two, 13 and 37, are of the form $(6r)^2+(6s+1)^2$; they give rise to the factors 3 and 2. The last two are of the form $(6r+2)^2+(6s+3)^2$, and give rise to the factors $(-1)^3\,4$ and $(-1)^4\,5$.

11. It will be noticed that for a great many values of n the value of P_n will be zero; for this will happen whenever any of the prime factors of $12n+1$ which $\equiv 11$, 7, or 5, mod. 12, occur with uneven exponents.

It is only in the case of prime factors which are $\equiv 1$, mod. 12, and when these factors occur with uneven exponents, that we have to take into consideration their complex factors, or, which is the same thing, resolve them into the sum of two squares. There is but one such partition for each prime, and upon its nature depends the sign which is to be attributed to the factor to which it gives rise in the value of G.

12. We may conveniently use the term *character* to distinguish between the cases when the prime is of the form $(6r)^2+(6s+1)^2$, and when it is of the form $(6r+2)^2+(6s+3)^2$. In the former case the character will be said to be positive; in the latter, negative.

Thus the character of a prime $\equiv 1$, mod. 12, is positive or negative, according as, when expressed as a sum of two squares, it is the even or the uneven square which is divisible by 3, or, which is the same thing, according as it is the even or uneven term in the complex factors of the prime which is divisible by 3.

We may also extend the idea of character to partitions, compositions, or representations, which will be distinguished as positive or negative, according as it is the even or the uneven square which is divisible by 3.

13. It is singular that the determination of the magnitude of P_n should depend wholly on the real factors of $12n+1$, and that it is only for the sake of the sign that we have to attend to the complex factors. Even for this purpose, it is only occasionally that recourse

to the complex factors is necessary, and, when such is the case, it is only the character of certain prime factors that has to be determined.

14. I give below a table of the values of the coefficient

$$P_n = G(12n+1),$$

for all non-zero values of n up to $n = 100$. The values of the function $E(12n+1)$, which forms the subject of §§ 15–18, are also added.

TABLE I.—Values of $P_n = G(12n+1)$, and of $E(12n+1)$, for all values of n for which P_n is not zero from $n = 1$ to $n = 100$.

n	$12n+1$	$P_n = G(12n+1)$	$E(12n+1)$	n	$12n+1$	$P_n = G(12n+1)$	$E(12n+1)$
1	13	−2	2	48	577	+2	2
2	25	−1	3	50	601	+2	2
3	37	+2	2	51	613	+2	2
4	49	+1	1	52	625	+1	5
5	61	+2	2	53	637	−2	2
6	73	−2	2	55	661	+2	2
8	97	−2	2	56	673	+2	2
9	109	−2	2	59	709	−2	2
10	121	+1	1	61	733	−2	2
13	157	+2	2	63	757	−2	2
14	169	+3	3	64	769	+2	2
15	181	−2	2	66	793	−4	4
16	193	+2	2	69	829	−2	2
19	229	−2	2	70	841	−1	3
20	241	−2	2	71	853	+2	2
23	277	−2	2	73	877	+2	2
24	289	−1	3	77	925	−2	6
26	313	+2	2	78	937	+2	2
27	325	+2	6	79	949	+4	4
28	337	−2	2	80	961	+1	1
29	349	+2	2	83	997	+2	2
30	361	+1	1	84	1009	−2	2
31	373	+2	2	85	1021	+2	2
33	397	+2	2	86	1033	−2	2
34	409	−2	2	89	1069	+2	2
35	421	−2	2	91	1093	−2	2
36	433	+2	2	93	1117	−2	2
38	457	−2	2	94	1129	−2	2
40	481	−4	4	96	1153	−2	2
44	529	+1	1	100	1201	+2	2
45	541	−2	2				

The next table shows the character of all primes $\equiv 1$, mod. 12, up to 12373. By its means we are able to write down at once (§ 10) the values of P_n as far as $n = 1031$, and, unless $12n+1$ be prime, to a very much greater extent. The manner in which the table was obtained is explained in § 19.*

TABLE II. — Showing the positive or negative character of the primes $\equiv 1$, mod. 12, up to 12373.

13 −	997 +	2017 −	3169 +	4297 +	5689 −
37 +	1009 −	2029 −	3181 −	4357 +	5701 −
61 +	1021 +	2053 +	3217 −	4441 +	5737 −
73 −	1033 −	2089 −	3229 −	4513 +	5749 −
97 −	1069 +	2113 −	3253 −	4549 +	5821 −
109 −	1093 −	2137 +	3301 +	4561 +	5857 −
157 +	1117 −	2161 −	3313 −	4597 +	5869 −
181 −	1129 −	2221 −	3361 −	4621 +	5881 −
193 +	1153 −	2269 +	3373 −	4657 −	5953 −
229 −	1201 +	2281 −	3433 −	4729 −	6037 +
241 −	1213 −	2293 +	3457 −	4789 +	6073 +
277 −	1237 −	2341 −	3469 −	4801 +	6121 −
313 +	1249 −	2377 −	3517 +	4813 +	6133 +
337 −	1297 +	2389 +	3529 +	4861 −	6217 −
349 +	1321 +	2437 +	3541 +	4909 −	6229 +
373 +	1381 −	2473 +	3613 +	4933 −	6277 +
397 +	1429 +	2521 +	3637 −	4957 −	6301 −
409 −	1453 −	2557 −	3673 +	4969 +	6337 +
421 −	1489 −	2593 +	3697 +	4993 −	6361 −
433 +	1549 +	2617 −	3709 +	5077 +	6373 +
457 −	1597 −	2677 −	3733 −	5101 −	6397 +
541 −	1609 −	2689 −	3769 +	5113 +	6421 −
577 +	1621 −	2713 −	3793 −	5197 +	6469 −
601 +	1657 +	2749 +	3853 −	5209 +	6481 −
613 +	1669 −	2797 −	3877 +	5233 +	6529 +
661 +	1693 +	2833 +	3889 +	5281 +	6553 +
673 +	1741 +	2857 −	4021 −	5413 −	6577 −
709 −	1753 −	2917 +	4057 +	5437 −	6637 +
733 −	1777 −	2953 +	4093 −	5449 +	6661 −
757 −	1789 +	3001 −	4129 +	5521 +	6673 −
769 +	1801 +	3037 +	4153 +	5557 −	6709 +
829 −	1861 +	3049 −	4177 −	5569 −	6733 −
853 +	1873 −	3061 +	4201 −	5581 +	6781 −
877 +	1933 +	3109 +	4261 +	5641 −	6793 +
937 +	1993 +	3121 −	4273 −	5653 +	6829 +

* Barlow's Tables of 1814 contain the complete resolutions of all numbers into their prime factors up to 10,000. Chernac (*Cribrum Arithmeticum*, 1811) gives all the prime factors of numbers up to 1,020,000. The Tables of Burckhardt, Dase, and J. Glaisher give the least factors of numbers up to 9,000,000.

6841 −	7681 +	8581 +	9433 −	10429 +	11497 +
6949 −	7717 −	8629 +	9601 +	10453 +	11593 +
6961 −	7741 −	8641 +	9613 −	10477 −	11617 +
6997 −	7753 −	8677 −	9649 −	10501 +	11677 −
7057 +	7789 +	8689 −	9661 −	10513 +	11689 +
7069 −	7873 −	8713 −	9697 −	10597 +	11701 −
7129 −	7933 +	8737 +	9721 −	10657 −	11821 +
7177 +	7993 +	8761 −	9733 +	10729 −	11833 +
7213 +	8017 +	8821 +	9769 −	10753 +	11941 +
7237 −	8053 −	8893 +	9781 +	10789 +	11953 +
7297 −	8089 +	8929 +	9817 −	10837 +	12037 −
7309 +	8101 +	8941 +	9829 −	10861 −	12049 −
7321 +	8161 −	9001 −	9901 −	10909 +	12073 +
7333 −	8209 +	9013 −	9949 +	10957 −	12097 +
7369 +	8221 +	9049 −	9973 −	10993 −	12109 −
7393 +	8233 +	9109 +	10009 −	11113 +	12157 −
7417 +	8269 +	9133 −	10069 −	11149 −	12241 +
7477 −	8293 +	9157 +	10093 −	11161 −	12253 +
7489 −	8317 +	9181 +	10141 +	11173 +	12277 +
7537 +	8329 −	9241 +	10177 +	11197 +	12289 +
7549 +	8353 −	9277 −	10273 −	11257 −	12301 −
7561 −	8377 −	9337 +	10321 +	11317 −	12373 +
7573 −	8389 +	9349 +	10333 −	11329 +	
7621 −	8461 +	9397 +	10357 +	11353 −	
7669 −	8521 +	9421 −	10369 −	11437 −	

Connexion with the Function $E(n)$, §§ 15–18.

15. The function $E(n)$, which may be defined as the excess of the number of divisors of n which are $\equiv 1$, mod. 4, over the number of divisors which are $\equiv 3$, mod. 4, has been considered in Vol, xv., pp. 104–122.* If n be uneven, $E(n)$ is equal to the number of primary complex numbers having n as their norm; and for all values of n, $4E(n)$ is equal to the number of representations of n as a sum of two squares.

Thus, when $n \equiv 1$, mod. 12, $4E(n)$ is equal to the sum of the numbers of positive and negative representations of n, while $4G(n)$ is numerically equal to their difference. When, therefore, all the compositions of n are of the same character, $G(n)$ is numerically equal to $E(n)$. This will evidently be the case when n is a prime, and, as will be shown in the next article, it will happen also whenever n contains no prime factor which $\equiv 5$, mod. 12.

* "On the Function which denotes the difference between the number of $(4m+1)$-divisors, and the number of $(4m+3)$-divisors of a Number," read February 14th, 1884.

The function $E(n)$ satisfies the condition that, if p and r are prime to each other, $E(pr) = E(p)\,E(r)$; and, in general, if

$$n = a_1^{a_1} a_2^{a_2} \dots b_1^{\beta_1} b_2^{\beta_2} \dots,$$

where $\qquad a_1, a_2, \dots$ are primes $\equiv 3$, mod. 4,

and $\qquad b_1, b_2, \dots \qquad$,, $\qquad \equiv 1$, mod. 4,

then $E(n) = 0$ unless $a_1, a_2, \dots$ are all even (including zero as an even number), and if these exponents are all even,

$$E(n) = (\beta_1+1)(\beta_2+1)\dots.$$

16. In the calculation, therefore, of E and G, in cases in which they do not both vanish, prime factors which are $\equiv 11$ or 7, mod. 12, give rise to the factor unity, that is to say, they produce no effect; prime factors which are $\equiv 1$, mod. 12, give rise to the same factors in E and G, though they may differ in sign. But in the case of prime factors which are $\equiv 5$, mod. 12, there is a difference of numerical value; for, if b be a prime of this form, the factor b^{β} occurring in n gives rise to the factor $\beta+1$ in E, but in G it gives rise to the factors -1 or 0, according as β is even or uneven.

Thus (supposing n, as always, to be $\equiv 1$, mod. 12) the numerical values of $G(n)$ and $E(n)$ are the same whenever n contains no prime factor $\equiv 5$, mod. 12. If any such factor occurs raised to an uneven power, it reduces G to zero; if raised to an even power, it merely produces a change of sign. In E, it gives rise in each case to a finite factor greater than unity.

The greater number of zero values of $G(n)$ than of $E(n)$ is due to the presence of uneven powers of primes $\equiv 5$, mod. 12; viz., any one of the factors $5, 5^3, 5^5, \dots, 17, 17^3, \dots 29, 29^3, \dots$ &c. reduces G to zero. For non-vanishing values of $G(n)$, we have seen that the only case in which a difference of numerical value occurs is when these factors present themselves with even exponents, and if $b_1, b_2, \dots$ denote the prime factors of this form, and $\beta_1, \beta_2, \dots$ are their exponents (supposed to be all even), then

$$G(n) = \pm \frac{E(n)}{(\beta_1+1)(\beta_2+1)\dots}.$$

17. In Table I. (p. 190) the values of $E(n)$ were given in an additional column for the sake of comparison. Within the limits of that table there are seventeen arguments for which E remains finite while

G vanishes. These are—

$$\begin{array}{lll} 85 = 5 \cdot 17, & 505 = 5 \cdot 101, & 901 = 17 \cdot 53, \\ 145 = 5 \cdot 29, & 565 = 5 \cdot 113, & 985 = 5 \cdot 197, \\ 205 = 5 \cdot 41, & 685 = 5 \cdot 137, & 1105 = 5 \cdot 13 \cdot 17, \\ 265 = 5 \cdot 53, & 697 = 17 \cdot 41, & 1165 = 5 \cdot 233, \\ 445 = 5 \cdot 89, & 745 = 5 \cdot 149, & 1189 = 29 \cdot 41. \\ 493 = 17 \cdot 29, & 865 = 5 \cdot 173, & \end{array}$$

For the argument 1105 the value of E is 8; in all the other cases it is 4.*

There are six cases in which G does not vanish, but in which its numerical value differs from E. These are—

	Value of G.	Value of E.
$25 = 5^2$,	-1,	3,
$289 = 17^2$,	-1,	3,
$325 = 5^2 . 13$,	2,	6,
$625 = 5^4$,	1,	5,
$841 = 29^2$,	-1,	3,
$925 = 5^2 . 37$,	-2,	6.

It will be noticed that the corresponding values of G and E are connected by the relation given at the end of the preceding article.

18. The values of $G(n)$, in Table I., were originally calculated by finding the partitions of $24n+2$ into two uneven squares (§ 4), before I had obtained the method described in §§ 6–10. The corresponding values of $E(n)$ were taken from the paper in Vol. xv. of the *Proceedings*, already cited in the note to § 15. The table of $E(n)$ contained in that paper gives the values of $E(n)$ for all values of n up to $n = 1000$ for which $E(n)$ does not vanish.†

The Function $\chi(n)$, § 19.

19. In Vol. xx. (1884) of the *Quarterly Journal*‡ I have considered

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* It is evident that, when G vanishes and E does not, the value of E must necessarily be an evenly even number, for there must be an equal number of positive and negative compositions, each of which counts in E as $+2$.

† The definition of $E(n)$ applies to all positive integral values of n. The definition of $G(n)$ only applies to numbers $\equiv 1$, mod. 12.

‡ "On the Function $\chi(n)$," pp. 97–167.

the function $\chi(n)$, which is defined as the sum of the primary numbers having n as their norm.

If n be a prime $\equiv 1$, mod. 12, then either

$$n = (6r+1)^2+6s^2$$

or

$$n = (6r+3)^2+(6s+2)^2.$$

In the former case the value of $\chi(n)$ is

$$(-1)^{3(r+s)}(12r+2),$$

and in the latter

$$(-1)^{3(r+s)}(12r+6).$$

Therefore, when the character of a prime is positive, $\chi(n)$ is not divisible by 3; and when the character is negative, $\chi(n)$ is divisible by 3.

In the paper on $\chi(n)$ to which reference has just been made, I have given a table of the values of $\chi(n)$ for all primes $\equiv 1$, mod. 4, up to $n = 12377$. Table II. was derived from this table by selecting from it the primes $\equiv 1$, mod. 12, and affixing the sign $-$ or $+$ according as $\chi(n)$ was, or was not, divisible by 3.

Analytical Formulæ connected with the Functions E and G, §§ 20–27.

20. It can be shown by elliptic functions that, if n denote any number (and therefore $2n$ any even number), and m any uneven number,

$$\left\{\Sigma_{-\infty}^{\infty} q^{n^2}\right\}^2 = 1+4\,\Sigma_1^{\infty} E(n)\, q^n,$$

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{4n^2} = 2\,\Sigma_0^{\infty} E(4n+1)\, q^{4n+1},$$

$$\left\{\Sigma_{-\infty}^{\infty} q^{m^2}\right\}^2 = 4\,\Sigma_0^{\infty} E(4n+1)\, q^{8n+2}.$$

The second formula shows that the number of representations of $4n+1$ as the sum of an even and an uneven square is equal to $4E(4n+1)$, and the third formula shows that the number of representations of $8n+2$ as a sum of two uneven squares is also equal to $4E(4n+1)$. It follows, therefore, that if $n \equiv 1$, mod. 4, the number of representations of $2n$ as a sum of two squares is equal to the number of representations of n as a sum of two squares. This theorem, of which it is easy to give an arithmetical proof, has been referred to in § 5.

21. Evidently

$$(6r+1)^2+(6s)^2 \equiv 1, \text{ mod. } 12,$$

$$(6r+3)^2+(6s+2)^2 \equiv 1, \text{ mod. } 12,$$

$$(6r+1)^2+(6s+2)^2 \equiv 5, \text{ mod. } 12,$$

$$(6r+3)^2+(6s)^2 \equiv 9, \text{ mod. } 36.$$

By separating the terms whose exponents are $\equiv 1$ from those which are $\equiv 5$, mod. 12, in the second formula of the preceding article, we obtain the following results :—

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} + \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \Sigma_0^{\infty} E\,(12n+1)\, q^{12n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \tfrac{1}{2}\Sigma_0^{\infty} E\,(12n+5)\, q^{12n+5}.$$

We find also

$$\Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} = 2\,\Sigma_0^{\infty} E\,(36n+9)\, q^{36n+9};$$

but this formula is equivalent to the original equation, for, by replacing q by $q^{\frac{1}{9}}$, it becomes

$$\Sigma_{-\infty}^{\infty} q^{(2n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(2n)^2} = 2\,\Sigma_0^{\infty} E\,(36n+9)\, q^{4n+1}.$$

It is easy to see that, for all values of n,

$$E\,(36n+9) = E\,(4n+1)\,;$$

for, if the highest power of 3 which occurs in $4n+1$ as a factor be uneven, E vanishes for both arguments, and if the highest power is even, $= 3^{2a}$ (including the case $a=0$), then

$$E\,(4n+1) = E\,\{3^{2a}\,(4r+1)\} = E\,(3^{2a})\,E\,(4r+1) = E\,(4r+1)\,;$$

and, similarly,

$$E\,(36n+9) = E\,\{3^{2a+2}\,(4r+1)\} = E\,(4r+1).$$

22. Treating in the same manner the third formula of § 20, by means of the congruences

$$(6r+1)^2+(6s+1)^2 \equiv 2, \quad \text{mod. } 24,$$

$$(6r+1)^2+(6s+3)^2 \equiv 10, \quad \text{mod. } 24,$$

$$(6r+3)^2+(6s+3)^2 \equiv 18, \quad \text{mod. } 36,$$

we obtain the formulæ

$$\{\Sigma_{-\infty}^{\infty} q^{(6n+1)^2}\}^2 = \Sigma_0^{\infty} E\,(12n+1)\, q^{24n+2},$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} = \Sigma_0^{\infty} E\,(12n+5)\, q^{24n+10}.$$

We find also

$$\{\Sigma_{-\infty}^{\infty} q^{(6n+3)^2}\}^2 = 4\Sigma_0^{\infty} E\,(36n+9)\, q^{72n+18},$$

which is, however, only a repetition of the original formula.

23. It is worth while to notice the identical relations to which the formulæ in the two preceding articles give rise, viz.,

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} + \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(6n+1)^2}\right\}^2,$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \tfrac{1}{2}\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(6n+3)^2}.$$

The original formulæ in § 20 give rise to the identity

$$\Sigma_{-\infty}^{\infty} q^{(2n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(2n)^2} = \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(2n+1)^2}\right\}^2,$$

from which, of course, the two preceding identities might be directly obtained by the method employed in the two preceding articles.

24. Passing now to the function G, we have, from § 4,

$$\left\{\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} G(12n+1)\, q^{24n+2},$$

and from § 5 it follows that

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} - \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \Sigma_0^{\infty} G(12n+1)\, q^{12n+1}.$$

These formulæ correspond to

$$\left\{\Sigma_{-\infty}^{\infty} q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} E(12n+1)\, q^{24n+2}$$

and

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} + \Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \Sigma_0^{\infty} E(12n+1)\, q^{12n+1},$$

which were given in §§ 22 and 21.

25. Combining, by addition and subtraction, the first and third formulæ, we find

$$\left\{\Sigma_{-\infty}^{\infty} q^{(12n+1)^2}\right\}^2 + \left\{\Sigma_{-\infty}^{\infty} q^{(12n+5)^2}\right\}^2 = \tfrac{1}{2}\Sigma_0^{\infty} \left\{E(12n+1) + G(12n+1)\right\} q^{24n+2}$$

$$\Sigma_{-\infty}^{\infty} q^{(12n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(12n+5)^2} = \tfrac{1}{4}\Sigma_0^{\infty} \left\{E(12n+1) - G(12n+1)\right\} q^{24n+2}.$$

These equations express the theorems:

(i.) The number of compositions of a number p (necessarily $\equiv 2$, mod. 24) as the sum of two squares, which are both of the form $(12n+1)^2$, or both of the form $(12n+5)^2$, is equal to

$$\tfrac{1}{2}\left\{E(p) + G(p)\right\}.$$

(ii.) The number of compositions of a number p (necessarily $\equiv 2$, mod. 24) as the sum of two squares, of which one is of the form $(12n+1)^2$ and the other of the form $(12n+5)^2$, is equal to

$$\tfrac{1}{2}\left\{E(p) - G(p)\right\}.$$

26. Combining in the same manner the second and fourth formulæ of § 24, we find

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n)^2} = \tfrac{1}{2}\Sigma_0^{\infty} \left\{E(12n+1)+G(12n+1)\right\} q^{12n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = \tfrac{1}{2}\Sigma_0^{\infty} \left\{E(12n+1)-G(12n+1)\right\} q^{12n+1},$$

which express the theorems:

(i.) The number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $(6r+1)^2+(6s)^2$ is equal to

$$2\left\{E(p)+G(p)\right\}.$$

(ii.) The number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $(6r+3)^2+(6s+2)^2$ is equal to

$$2\left\{E(p)-G(p)\right\}.*$$

27. It will be noticed that the formulæ in the last two articles lead to the identities

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty}{}' q^{(6n)^2} = \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(12n+1)^2}\right\}^2 + \left\{\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(12n+5)^2}\right\}^2,$$

$$\Sigma_{-\infty}^{\infty} q^{(6n+3)^2} \times \Sigma_{-\infty}^{\infty} q^{(6n+2)^2} = 2\Sigma_{-\infty} q^{\frac{1}{2}(12n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{\frac{1}{2}(12n+5)^2}.$$

The Function $H(n)$, §§ 28, 29.

28. It can be shown by the Theory of Numbers that the number of representations of any uneven number m by the form x^2+3y^2 is equal to four times the excess of the number of divisors of m which $\equiv 1$, mod. 3, over the number of divisors which $\equiv 2$, mod. 3.

Let $H(n)$ denote the excess of the number of divisors of n which $\equiv 1$, mod. 3, over those which $\equiv 2$, mod. 3.

If an uneven number be represented by the form x^2+3y^2, either x or y must be even and the other uneven; observing that

$$(2r+1)^2+3(2s)^2 \equiv 1, \text{ mod. } 4,$$

$$(2r)^2+3(2s+1)^2 \equiv 3, \text{ mod. } 4,$$

we thus obtain the analytical theorems,

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = 2\Sigma_0^{\infty} H(4n+1)\, q^{4n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{4n^2} \times \Sigma_{-\infty}^{\infty} q^{3m^2} = 2\Sigma_0^{\infty} H(4n+3)\, q^{4n+3},$$

where n denotes any number, and m any uneven number.

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* In (i.) the number p is expressed as a sum of two squares, of which the even square is divisible by 3, and the uneven square is not; in (ii.) p is expressed as a sum of two squares, of which the uneven square is divisible by 3, and the even square is not.

Taking the first formula, since

$$(6r+1)^2+3\,(2s)^2 \equiv 1, \text{ mod. } 12,$$

$$(6r+3)^2+3\,(2s)^2 \equiv 9, \text{ mod. } 12,$$

we find, by similar reasoning to that employed in § 21,

$$\Sigma_{-\infty}^{\infty}\, q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty}\, q^{12n^2} = \Sigma_0^{\infty}\, H\,(12n+1)\, q^{12n+1}.$$

29. The function $H\,(n)$ possesses the property that, if p and q are prime to each other,

$$H\,(pq) = H\,(p)\, H\,(q).$$

This may be proved by exactly the same methods as in the case of the corresponding theorem

$$E\,(pq) = E\,(p)\, E\,(q)\,;$$

i.e., analytically, as in *Proc. Lond. Math. Soc.*, Vol. xv., pp. 104, 105, or by general reasoning, as in the note to § 8 of the present paper.

The former method shows that $H\,(n)$ vanishes unless every prime divisor of n which $\equiv 2$, mod. 3, is raised to an even power, and that if, in general, $n = 3^p uv^2$, where all the prime factors of u are $\equiv 1$, mod. 2, and all the prime factors of v are $\equiv 2$, mod. 3, then

$$H\,(n) = H\,(u) = \nu\,(u),$$

$\nu\,(u)$ denoting the number of divisors of u.

The Functions G, E, H, §§ 30–32.

30. We may obtain an equation similar in form to the last equation in § 28, but in which G is involved instead of H, by the following method.

$$\begin{aligned}\Sigma_1^{\infty}\, P_n q^n &= \Pi_1^{\infty}\,(1-q^n)^2 \\ &= \Pi_1^{\infty}\,(1-q^{2n})\, \Pi_1^{\infty}\left(\frac{1-q^n}{1+q^n}\right) \\ &= \Sigma_{-\infty}^{\infty}\,(-1)^n\, q^{n(3n+1)} \times \Sigma_{-\infty}^{\infty}\,(-1)^n\, q^{n^2}.\end{aligned}$$

Putting q^{12} for q, this equation becomes

$$\Sigma_1^{\infty}\, P_n q^{12n} = \Sigma_{-\infty}^{\infty}\,(-1)^n\, q^{12n(3n+1)} \times \Sigma_{-\infty}^{\infty}\,(-1)^n\, q^{12n^2}\,;$$

whence, replacing P_n by $G\,(12n+1)$, and multiplying throughout by q, we find

$$\Sigma_{-\infty}^{\infty}\,(-1)^n\, q^{(6n+1^2)} \times \Sigma_{-\infty}^{\infty}\,(-1)^n\, q^{12n^2} = \Sigma_0^{\infty}\,(-1)^n\, G\,(12n+1)\, q^{12n+1}.$$

31. It was found, in §§ 22 and 24, that

$$\left\{\Sigma_{-\infty}^{\infty} q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} E(12n+1)\, q^{24n+2},$$

$$\left\{\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2}\right\}^2 = \Sigma_0^{\infty} G(12n+1)\, q^{24n+2},$$

and, in §§ 28 and 30, that

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty} H(12n+1)\, q^{12n+1},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} (-1)^n q^{12n^2} = \Sigma_0^{\infty} G(12n+1)\, q^{12n+1}.$$

Writing these formulæ at full length, they become

$$E(1)\, q^2 + E(13)\, q^{26} + E(25)\, q^{50} + E(37)\, q^{74} + \&c.$$
$$= (q + q^{25} + q^{49} + q^{121} + q^{169} + \&c.)^2,$$

$$G(1)\, q^2 + G(13)\, q^{26} + G(25)\, q^{50} + G(37)\, q^{74} + \&c.$$
$$= (q - q^{25} - q^{49} + q^{121} + q^{169} - \&c.)^2,$$

and

$$H(1)\, q + H(13)\, q^{13} + H(25)\, q^{25} + H(37)\, q^{37} + \&c.$$
$$= (q + q^{25} + q^{49} + q^{121} + q^{169} + \&c.)(1 + 2q^{12} + 2q^{48} + 2q^{108} + 2q^{192} + \&c.),$$

$$G(1)\, q + G(13)\, q^{13} + G(25)\, q^{25} + G(37)\, q^{37} + \&c.$$
$$= (q - q^{25} - q^{49} + q^{121} + q^{169} - \&c.)(1 - 2q^{12} + 2q^{48} - 2q^{108} + 2q^{192} - \&c.).$$

The first pair of formulæ show that $G(12n+1)$ must vanish when $E(12n+1)$ vanishes; for $4E(12n+1)$ is equal to the total number of representations of $12n+1$ as a sum of two squares neither of which is divisible by 3, so that, when $E(12n+1)$ vanishes, $12n+1$ does not admit of being so expressed.

Similarly, from the second pair of formulæ we see that $G(12n+1)$ must vanish when $H(12n+1)$ vanishes; for $4H(12n+1)$ is equal to the total number of representations of $12n+1$ as the sum of an uneven square not divisible by 3 and of the triple of an even square.

Now $E(n)$ vanishes unless all the prime factors of n which are $\equiv 3$, mod. 4, are raised to even powers; and $H(n)$ vanishes unless all the prime factors of n which are $\equiv 2$, mod. 3, are raised to even powers.

Thus either $E(n)$ or $H(n)$ vanishes (or both vanish) unless all the prime factors of n which are $\equiv 3, 5, 7$, or 11, mod. 12, are raised to even powers. It follows, therefore, that $G(12n+1)$ vanishes, unless all the prime factors of $12n+1$ which are $\equiv 3, 5, 7$, or 11, mod. 12, are raised to even powers. This result is included in the arithmetical investigation of § 9.

32. Five years ago (in 1884) I found, by the analytical process of the preceding article, that P_n was always zero unless all the prime factors of $12n+1$ which were $\equiv 3, 5, 7$, or 11, mod. 12, were raised to even powers. The process did not show, however, that it might not happen that P_n should vanish, even when this condition was satisfied, and I did not then attempt an arithmetical investigation. It appears from § 9, however, that P_n can never vanish when all the prime factors of the above forms occur with even exponents. I have thought the results contained in the four preceding articles worth giving, partly because of the analytical proof which we thus obtain of the theorem that P_n vanishes unless the prime factors of $12n+1$ which are of a certain form occur with even exponents, and partly for the sake of introducing the function H, which belongs to the same class of coefficients as E and G.

Linear relations connecting the values of P_n, §§ 33–44.

33. When Euler had obtained the formula

$$(1-q)(1-q^2)(1-q^3)\ldots = 1-q-q^2+q^5+q^7-\&c.,$$

he applied it in the following manner to obtain results connected with the Theory of Numbers:—

(i.) If $P(n)$ denote the number of partitions of the number n into the parts 1, 2, 3, ..., repetitions not excluded, we have

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\ldots} = 1+P(1)\,q+P(2)\,q^2+P(3)\,q^3+\&c.,$$

whence it follows that

$$\{1+P(1)\,q+P(2)\,q^2+P(3)\,q^3+\&c.\}\,\{1-q-q^2+q^5+q^7-\&c.\} = 1.$$

Equating the coefficients of q^n, we find, for all values of n,

$$P(n)-P(n-1)-P(n-2)+P(n-5)+P(n-7)-\&c. = 0,$$

the series being continued so long as the arguments remain positive.* The value of $P(0)$, when it occurs, is supposed to be unity.

(ii.) By taking logarithms and differentiating, we deduce from Euler's series the equation

$$\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\&c. = \frac{q+2q^2-5q^5-7q^7+\&c.}{1-q-q^2+q^5+q^7-\&c.}.$$

* *Commentationes Arithmeticæ Collectæ*, Vol. I., p. 91.

Denoting by $\sigma(n)$ the sum of the divisors of n, this equation may be written

$$\{\sigma(1)q+\sigma(2)q^2+\sigma(3)q^3+\&c.\}\{1-q-q^2+q^5+q^7-\&c.\}$$
$$=q+2q^2-5q^5-7q^7+\&c.;$$

whence, by equating coefficients, we obtain the relation

$$\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\&c.=0,^*$$

if we adopt the convention that $\sigma(0)$, *i.e.*, $\sigma(n-n)$, when it occurs, is to have assigned to it the value n.

34. It may be remarked that, by writing the equation obtained in the preceding section in the form

$$\frac{q+2q^2-5q^5-7q^7+\&c.}{(1-q)(1-q^2)(1-q^3)\dots}=\sigma(1)q+\sigma(2)q^2+\sigma(3)q^3+\&c.,$$

we find, by equating the coefficients of q^n on each side,

$$P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)+\&c.=\sigma(n).\dagger$$

This result, combined with Euler's second theorem, shows that the values of

$$\sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)+\&c.$$

and

$$P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)+\&c.$$

are equal.

35. We may apply Euler's second method to deduce from Jacobi's formula,

$$\{(1-q)(1-q^2)(1-q^3)\dots\}^3=1-3q+5q^3-7q^6+9q^{10}-\&c.,$$

a corresponding property of the function $\sigma(n)$.

For we thus find

$$\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\&c.=\frac{q-5q^3+14q^6-30q^{10}+\&c.}{1-3q+5q^3-7q^6+9q^{10}-\&c.};$$

from which, by equating coefficients, we have

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+9\sigma(n-10)-\&c.=0,\ddagger$$

if we assign to $\sigma(0)$, *i.e.* to $\sigma(n-n)$, the meaning $\frac{1}{3}n$.

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* *Ibid.*, pp. 151–154.

† *Messenger of Mathematics*, Vol. XII., p. 170. This paper also contains other theorems connecting partitions and sums of divisors.

‡ *Quarterly Journal*, Vol. XIX., p. 220. It is so obvious, however, that Euler's method is exactly applicable to Jacobi's formula, that the result must have been noticed before. See also *Proc. Camb. Phil. Soc.*, Vol. V., p. 109.

36. Passing now to the formula

$$\{(1-q)(1-q^2)(1-q^3)\ldots\}^2 = 1+P_1q+P_2q^2+P_3q^3+\&c.,$$

I proceed to consider the properties of the coefficient P_n that can be obtained by methods of a similar character to those employed in the three preceding sections.

(i.) Applying Euler's first process (§ 33), we have

$$\{1+P_1q+P_2q^2+P_3q^3+\&c.\}\{1+P(1)\,q+P(2)\,q^2+P(3)\,q^3+\&c.\}$$
$$= 1-q-q^2+q^5+q^7-\&c.;$$

whence it follows that

$$P_n+P(1)\,P_{n-1}+P(2)\,P_{n-2}+\ldots+P(n-1)\,P(1)+P(n)$$
$$= 0 \text{ or } (-1)^r,$$

according as n is not of the form $\frac{1}{2}r\,(3r\pm 1)$, or is equal to $\frac{1}{2}r\,(3r\pm 1)$.

(ii.) Applying Euler's second method, we have

$$2\left\{\frac{q}{1-q}+\frac{2q^2}{1-q^2}+\frac{3q^3}{1-q^3}+\&c.\right\} = -\frac{P_1q+2P_2q^2+3P_3q^3+\&c.}{1+P\,q+P_2q^2+P_3q^3+\&c.},$$

which gives the relation

$$-\tfrac{1}{2}nP_n = \sigma(1)\,P_{n-1}+\sigma(2)\,P_{n-2}+\ldots+\sigma(n-1)\,P_1+\sigma(n).$$

37. Since

$$\{1+P(1)\,q+P(2)\,q^2+P(3)\,q^3+\&c.\}\{1-3q+5q^3-7q^6+9q^{10}-\&c.\}$$
$$= 1+P_1q+P_2q^2+P_3q^3+\&c.,$$

and $$\{1+P_1q+P_2q^2+P_3q^3+\&c.\}\{1-q-q^2+q^5+q^7-\&c.\}$$
$$= 1-3q+5q^3-7q^6+9q^{10}-\&c.,$$

we find, by equating coefficients,

$$P(n)-3P(n-1)+5P(n-3)-7P(n-6)+\&c. = P_n,$$

and $$P_n-P_{n-1}-P_{n-2}+P_{n-5}+P_{n-7}-\&c.$$
$$= 0 \text{ or } (-1)^r(2r+1),$$

according as n is not of the form $\frac{1}{2}r\,(r+1)$, or is equal to $\frac{1}{2}r\,(r+1)$. The value of P_0 is supposed to be unity.

38. Since

$$(1-q-q^2+q^5+q^7-\&c.)^3 = 1-3q+5q^3-7q^6+9q^{10}-\&c.,$$

we find, by differentiating,

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(q+2q^2-5q^5-7q^7+\&c.)$$
$$= q-5q^3+14q^6-30q^{10}+\&c.;$$

whence, by equating coefficients,

$$P_{n-1}+2P_{n-2}-5P_{n-5}-7P_{n-7}+\&c. = 0 \quad \text{or} \quad (-1)^{r-1}\frac{r(r+1)(2r+1)}{6},$$

according as n is not of the form $\frac{1}{2}r(r+1)$, or is equal to $\frac{1}{2}r(r+1)$.*

In §§ 34, 35, and 36, three fractional formulæ have been obtained for the series $\Sigma_1^\infty \sigma(n)\, q^n$. We do not obtain new formulæ by equating these fractional expressions. For example, from §§ 35 and 36 we have

$$-\tfrac{1}{2}(P_1q+2P_2q^2+3P_3q^3+\&c.)(1-q-q^2+q^5+q^7-\&c.)$$
$$= q-5q^3+14q^6-30q^{10}+\&c.,$$

giving

$$nP_n-(n-1)P_{n-1}-(n-2)P_{n-2}+(n-5)P_{n-5}+(n-7)P_{n-7}-\&c.$$
$$= 0 \quad \text{or} \quad (-1)^r\frac{r(r+1)(2r+1)}{3},$$

according as n is not of the form $\frac{1}{2}r(r+1)$, or is equal to $\frac{1}{2}r(r+1)$.

This formula, however, is easily deducible from the expressions for $P_{n-1}+2P_{n-2}-5P_{n-5}-\&c.$, and $P_n-P_{n-1}-P_{n-2}-\&c.$, which have been already obtained.

In the six following articles, various formulæ involving the functions P_n, and connecting them with other functions, are obtained by equating coefficients.

39. From § 30 we have

$$1+P_1q+P_2q^2+P_3q^3+\&c.$$
$$= (1-q^2-q^4+q^{10}+q^{14}-\&c.)(1-2q+2q^4-2q^9+2q^{16}-\&c.),$$

whence

$$\frac{1+P_1q+P_2q^2+P_3q^3+\&c.}{1-P_1q+P_2q^2-P_3q^3+\&c.} = \frac{1-2q+2q^4-2q^9+2q^{16}-\&c.}{1+2q+2q^4+2q^9+2q^{16}+\&c.}.$$

* In the case of the function $P(n)$, we have the theorem

$$P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)+\&c. = \sigma(n),$$

which, however, only holds good when n is uneven (*Messenger*, Vol. XII., p. 170).

Multiplying up and equating the coefficients of q^n, we find that, if n be any uneven number,

$$P_n+2P_{n-1}+2P_{n-4}+2P_{n-9}+2P_{n-16}+\&c. = 0.^*$$

40. By multiplying the first equation in the preceding article by $(1-q^2)(1-q^4)(1-q^6)\ldots$, we have

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1-q^2-q^4+q^{10}+q^{14}-\&c.)$$
$$= (1+P_1q^2+P_2q^4+P_3q^6+\&c.)(1-2q+2q^4-2q^9+2q^{16}-\&c.);$$

whence, by equating coefficients, we find, for all values of n,

$$P_{2n}-P_{2n-2}-P_{2n-4}+P_{2n-10}+P_{2n-14}-\&c.$$
$$= P_n+2P_{n-2}+2P_{n-8}+2P_{n-18}+\&c.,$$

and $$P_m-P_{m-2}-P_{m-4}+P_{m-10}+P_{m-14}-\&c.$$
$$= -2P_n-2P_{n-4}-2P_{n-12}-2P_{n-24}-\&c.,$$

where, in the second formula, m denotes $2n+1$.

In the right-hand member of the first formula the suffix in the general term is $n-2r^2$, and in the second formula it is $n-2r^2-2r$.

41. If we denote by $Q(n)$ the number of partitions of n into the elements 1, 2, 3, ... in which no part is repeated in the same partition, so that

$$(1+q)(1+q^2)(1+q^3)\ldots$$
$$= 1+Q(1)\,q+Q(2)\,q^2+Q(3)\,q^3+Q(4)\,q^4+\&c.,$$

then we have

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1-q^2-q^4+q^{10}+q^{14}-\&c.)$$
$$= \{1+Q(1)q+Q(2)q^2+Q(3)\,q^3+\&c.\}\,(1-3q+5q^3-7q^6+9q^{10}-\&c.);$$

whence, by equating coefficients,

$$P_n-P_{n-2}-P_{n-4}+P_{n-10}+P_{n-14}-\&c.$$
$$= Q(n)-3Q(n-1)+5Q(n-3)-7Q(n-6)+\&c.,$$

for all values of n; the value of $Q(0)$ being supposed to be unity.

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* Similar formulæ relating to the functions χ, E, and σ are given in the *Quarterly Journal*, Vol. xx., pp. 120, 121. A formula of the same kind for H is given in § 48.

By combining this result with those of the preceding article, we see that

$$P_{2n}-P_{2n-2}-P_{2n-4}+P_{2n-10}+P_{2n-14}-\&c.,$$

$$P_n+2P_{n-2}+2P_{n-8}+2P_{n-18}+2P_{n-32}+\&c.,$$

$$Q(2n)-3Q(2n-1)+5Q(2n-3)-7Q(2n-6)+\&c.$$

are all three equal; and that

$$P_m-P_{m-2}-P_{m-4}+P_{m-10}+P_{m-14}-\&c.,$$

$$-2P_n-2P_{n-4}-2P_{n-12}-2P_{n-24}-2P_{n-40}-\&c.,$$

$$Q(m)-3Q(m-1)+5Q(m-3)-7Q(m-6)+\&c.,$$

where $m = 2n+1$, are all three equal.

42. From the equation

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1+q+q^3+q^6+q^{10}+\&c.)$$
$$= (1+P_1q^2+P_2q^4+P_3q^6+\&c.)(1-q-q^2+q^5+q^7-\&c.)$$

we may deduce the formulæ

$$P_{2n}+P_{2n-1}+P_{2n-3}+P_{2n-6}+\&c.$$
$$= P_n-P_{n-1}-P_{n-6}+P_{n-11}+P_{n-13}-P_{n-20}-\&c.$$

and
$$P_m+P_{m-1}+P_{m-3}+P_{m-6}+\&c.$$
$$= -P_n+P_{n-2}+P_{n-3}-P_{n-7}-P_{n-17}+P_{n-25}+\&c.,$$

where $n = 2n+1$.

The numbers 1, 6, 11, ... which occur in the first formula are the halves of the even pentagonal numbers. The corresponding numbers 2, 3, 7, ... in the second formula are the halves of the uneven pentagonal numbers diminished by unity.

Similarly, from the equation

$$\{1+Q(1)q+Q(2)q^2+Q(3)q^3+\&c.\}(1-3q^2+5q^6-7q^{12}+9q^{20}-\&c.)$$
$$= (1+P_1q^2+P_2q^4+P_3q^6+\&c.)(1+q+q^3+q^6+q^{10}+\&c.),$$

we find

$$Q(2n)-3Q(2n-2)+5Q(2n-6)-7Q(2n-12)+\&c.$$
$$= P_n+P_{n-3}+P_{n-5}+P_{n-14}+P_{n-18}+P_{n-33}+\&c.$$

and $Q(m)-3Q(m-2)+5Q(m-6)-7Q(m-12)+\&c.$

$$= P_n+P_{n-1}+P_{n-7}+P_{n-10}+P_{n-22}+P_{n-27}+\&c.,$$

where $m = 2n+1$.

The numbers 3, 5, 14, ... which occur in the first formula are the halves of the even triangular numbers. The corresponding numbers 1, 7, 10, ... in the second formula are the halves of the uneven triangular numbers diminished by unity.

43. It can be shown that

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)\{1+P(1)\,q^2+P(2)\,q^4+P(3)\,q^6+\&c.\}$$
$$= 1-2q+2q^4-2q^9+2q^{16}-\&c.,$$

and

$$(1+P_1q^2+P_2q^4+P_3q^6+\&c.)\{1+P(1)\,q+P(2)\,q^2+P(3)\,q^3+\&c.\}$$
$$= 1+q+q^3+q^6+q^{10}+\&c.$$

By equating the coefficients of q^n in these equations, we find that

$$P_n+P(1)\,P_{n-2}+P(2)\,P_{n-4}+P(3)\,P_{n-6}+\&c. = 0,\ \ +2,\ \text{ or } -2,$$

according as n is not a square, is an even square, or is an uneven square; and that

$$P(n)+P_1P(n-2)+P_2P(n-4)+P_3P(n-6)+\&c. = 0 \text{ or } 1,$$

according as n is not, or is, a triangular number.*

44. We have also the formula

$$(1+P_1q+P_2q^2+P_3q^3+\&c.)(1+P_1q^2+P_2q^4+P_3q^6+\&c.)$$
$$= 1+\chi(5)\,q+\chi(9)\,q^2+\chi(13)\,q^3+\&c.,$$

where $\chi(n)$ is the function considered in § 19.

By equating coefficients, we find

$$P_n+P_1P_{n-2}+P_2P_{n-4}+P_3P_{n-6}+\&c. = \chi(4n+1).$$

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* The value of the expression

$$P_n+P(1)\,P_{n-1}+P(2)\,P_{n-2}+\ldots+P(n-1)\,P_1+P$$

was given in § 36.

Elliptic Function Expressions for Series involving the Function G, §§ 45–47.

45. Denoting $\frac{2K}{\pi}$ by ρ, we have, in Elliptic Functions, the following formulæ giving the values of the series which have been principally considered in this paper :—

$$k^{\frac{1}{6}}k'^{\frac{1}{6}}\rho^{\frac{1}{2}} = 2^{\frac{1}{3}}q^{\frac{1}{12}}(1-q^2-q^4+q^{10}+q^{14}-\&c.),$$

$$k^{\frac{1}{3}}k'^{\frac{1}{3}}\rho = 2^{\frac{2}{3}}q^{\frac{1}{6}}(1+P_1q^2+P_2q^4+P_3q^6+\&c.),$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2q^{\frac{1}{4}}(1-3q^2+5q^6-7q^{12}+9q^{20}-\&c.).$$

These formulæ may be also written

$$k^{\frac{1}{6}}k'^{\frac{1}{6}}\rho^{\frac{1}{2}} = 2^{\frac{1}{3}}(q^{\frac{1}{12}}-q^{\frac{25}{12}}-q^{\frac{49}{12}}+q^{\frac{121}{12}}+q^{\frac{169}{12}}-\&c.),$$

$$k^{\frac{1}{3}}k'^{\frac{1}{3}}\rho = 2^{\frac{2}{3}}(q^{\frac{1}{6}}+P_1q^{\frac{13}{6}}+P_2q^{\frac{25}{6}}+P_3q^{\frac{37}{6}}+\&c.),$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2(q^{\frac{1}{4}}-3q^{\frac{9}{4}}+5q^{\frac{25}{4}}-7q^{\frac{49}{4}}+9q^{81}-\&c.);$$

or, by expressing only the general term of the series and using $G(12n+1)$ in place of P_n in the second formula,

$$k^{\frac{1}{6}}k'^{\frac{1}{6}}\rho^{\frac{1}{2}} = 2^{\frac{1}{3}}\Sigma_1^{\infty}\left(\frac{3}{m}\right)q^{\frac{1}{12}m^2},$$

$$k^{\frac{1}{3}}k'^{\frac{1}{3}}\rho = 2^{\frac{2}{3}}\Sigma_0^{\infty}G(12n+1)\,q^{\frac{1}{6}(12n+1)},$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2\Sigma_1^{\infty}\left(\frac{-1}{m}\right)mq^{\frac{1}{4}m^2},$$

where n represents any number, and m any uneven number. The coefficients $\left(\frac{3}{m}\right)$ and $\left(\frac{-1}{m}\right)$ are Legendre's symbols (as extended by Jacobi) expressing the quadratic character of 3 and of -1 with respect to m, so that $\left(\frac{-1}{m}\right) = (-1)^{\frac{1}{2}(m-1)}$.

46. By substituting in this group of formulæ q^2, $q^{\frac{1}{2}}$ and $-q^{\frac{1}{2}}$ for q, we obtain also the following groups :—

$$k^{\frac{1}{3}}k'^{\frac{1}{12}}\rho^{\frac{1}{2}} = 2^{\frac{2}{3}}\Sigma_1^{\infty}\left(\frac{3}{m}\right)q^{\frac{1}{6}m^2},$$

$$k^{\frac{2}{3}}k'^{\frac{1}{6}}\rho = 2^{\frac{4}{3}}\Sigma_0^{\infty}G(12n+1)\,q^{\frac{1}{3}(12n+1)},$$

$$kk'^{\frac{1}{4}}\rho^{\frac{3}{2}} = 4\Sigma_1^{\infty}\left(\frac{-1}{m}\right)mq^{m^2};$$

$$k^{\frac{1}{12}}k'^{\frac{1}{3}}\rho^{\frac{1}{2}} = 2^{\frac{1}{6}}\,\Sigma_1^\infty\left(\frac{3}{m}\right)q^{\frac{1}{24}m^2},$$

$$k^{\frac{1}{6}}k'^{\frac{2}{3}}\rho \;= 2^{\frac{1}{3}}\,\Sigma_0^\infty\, G\,(12n+1)\, q^{\frac{1}{12}(12n+1)},$$

$$k^{\frac{1}{4}}k'\rho^{\frac{3}{2}} \;= 2^{\frac{1}{2}}\,\Sigma_1^\infty\left(\frac{-1}{m}\right)mq^{\frac{1}{8}m^2};$$

$$k^{\frac{1}{24}}k'^{\frac{1}{12}}\rho^{\frac{1}{2}} = 2^{\frac{1}{6}}\,\Sigma_1^\infty\left(\frac{6}{m}\right)q^{\frac{1}{24}m^2},$$

$$k^{\frac{1}{12}}k'^{\frac{1}{6}}\rho \;= 2^{\frac{1}{3}}\,\Sigma_0^\infty\,(-1)^n\, G(12n+1)\, q^{\frac{1}{12}(12n+1)},$$

$$k^{\frac{1}{8}}k'^{\frac{1}{4}}\rho^{\frac{3}{2}} \;= 2^{\frac{1}{2}}\,\Sigma_1^\infty\left(\frac{-2}{m}\right)mq^{\frac{1}{8}m^2}.$$

The symbol $\left(\frac{-2}{m}\right)$ is equal to $(-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)}$. The symbols $\left(\frac{3}{m}\right)$ and $\left(\frac{6}{m}\right)$ are supposed to be zero whenever m is not prime to 3 or 6 respectively.

It may be remarked that $\left(\frac{3}{m}\right) = (-1)^{\frac{1}{6}(m\pm1)}$, where in the exponent that sign is to be taken which makes $m\pm1$ divisible by 6; m being supposed not to be a multiple of 3, as in that case $\left(\frac{3}{m}\right)$ is zero.

Thus, also, $\left(\frac{6}{m}\right) = \left(\frac{3}{m}\right)\left(\frac{2}{m}\right) = (-1)^{\frac{1}{6}(m\pm1)+\frac{1}{8}(m^2-1)}$.

47. The following formulæ may also be added:—

$$H\,(\tfrac{2}{3}K) = 3^{\frac{1}{2}}\,\Sigma_1^\infty\left(\frac{3}{m}\right)q^{\frac{1}{4}m^2},$$

$$H^2\,(\tfrac{2}{3}K) = 3\;\Sigma_0^\infty\, G\,(12n+1)\,q^{\frac{1}{2}(12n+1)},$$

$$H^3\,(\tfrac{2}{3}K) = 3^{\frac{3}{2}}\,\Sigma_1^\infty\left(\frac{-1}{m}\right)mq^{\frac{3}{4}m^2};$$

where the function H is the same as in the *Fundamenta Nova*.

Linear Relations involving $H(n)$, and connecting $H(n)$ and $E(n)$, §§ 48, 49.

48. We may obtain various formulæ, of the kind considered in §§ 33–44, relating to the function $H(n)$ which was defined in § 28

as the excess of the number of divisors of n which $\equiv 1$, mod. 3, over those which $\equiv 2$, mod. 3. We obtain also, by the same methods, relations involving both $H(n)$ and $E(n)$.

From § 28,

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = 2\Sigma_0^{\infty} H(4n+1)\, q^{4n+1},$$

$$\Sigma_{-\infty}^{\infty} q^{4n^2} \times \Sigma_{-\infty}^{\infty} q^{3m^2} = 2\Sigma_0^{\infty} H(4n+3)\, q^{4n+3},$$

where n and m denote, as before, any number and any uneven number respectively.

From these formulæ we may deduce

$$\frac{H(1)+H(9)\,q^2+H(17)\,q^4+\&c.}{H(5)\,q+H(13)\,q^3+H(21)\,q^5+\&c.} = \frac{1+2q^{12}+2q^{48}+2q^{108}+\&c.}{2q^3+2q^{27}+2q^{75}+\&c.},$$

$$\frac{H(3)+H(11)\,q^2+H(19)\,q^4+\&c.}{H(7)\,q+H(15)\,q^3+H(23)\,q^5+\&c.} = \frac{1+2q^4+2q^{16}+2q^{36}+\&c.}{2q+2q^9+2q^{25}+\&c.},$$

whence, by equating coefficients, we find that, if $n \equiv 5$, mod. 8,

$$H(n)-2H(n-12)+2H(n-48)-2H(n-108)+\&c. = 0,$$

and that, if $n \equiv 7$, mod. 8,

$$H(n)-2H(n-4)+2H(n-16)-2H(n-36)+\&c. = 0.$$

49. From § 20,

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{4n^2} = 2\Sigma_0^{\infty} E(4n+1)\, q^{4n+1},$$

$$\{\Sigma_{-\infty}^{\infty} q^{m^2}\}^2 = 4\Sigma_0^{\infty} E(4n+1)\, q^{8n+2},$$

$$\{\Sigma_{-\infty}^{\infty} q^{4n^2}\}^2 = 1+4\Sigma_1^{\infty} E(n)\, q^{4n};$$

and by combining these identities with those involving H in the preceding article, and equating coefficients, we may obtain the four following formulæ :—

(i.) If $n \equiv 1$, mod. 4,

$$H(n)+2H(n-4)+2H(n-16)+2H(n-36)+\&c.$$

$$= E(n)+2E(n-12)+2E(n-48)+2E(n-108)+\&c.$$

In the first line the numbers 4, 16, 36, ... are the even squares; the corresponding numbers in the second line are the triples of these squares.

(ii.) If $p = 4n+1, \quad s = 8n+1,$

where n is any number, then

$$H(s)+H(s-8)+H(s-24)+H(s-48)+\&c.$$
$$= E(p)+2E(p-24)+2E(p-96)+2E(p-216)+\&c.$$

The numbers 8, 24, 48, ... are the uneven squares diminished by unity. The numbers 24, 96, 216, ... are the squares multiplied by 24.

(iii.) If $l = 4n-3, \quad t = 8n+5,$

where n is any number, then

$$H(t)+H(t-8)+H(t-24)+H(t-48)+\&c.$$
$$= 2E(l)+2E(l-48)+2E(l-144)+2E(l-288)+\&c.$$

The numbers 8, 24, 48, ... are as above. The numbers 48, 144, 288 are these numbers multiplied by 6.

(iv.) If $r = 4n+3,$

$$H(r)+2H(r-4)+2H(r-16)+2H(r-36)+\&c.$$
$$= 4E(n)+4E(n-6)+4E(n-18)+4E(n-36)+\&c.$$

The numbers 6, 18, 36, ... are the triangular numbers multiplied by 6. The quantity $E(0)$, when it occurs, is to have the value $\frac{1}{4}$.

It may be remarked that we find also that, if $p = 4n+1$,

$$E(p)+2E(p-4)+2E(p-16)+2E(p-36)+\&c.$$
$$= 4E(n)+4E(n-2)+4E(n-6)+4E(n-12)+\&c.,$$

where, as in (iv.), $E(0)$ is to have the value $\frac{1}{4}$.

The Functions G and H, §§ 50–56.

50. Taking the second pair of formulæ in § 31, and changing the sign of q^{12} in the second of them, we have

$$\Sigma_{-\infty}^{\infty} q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty} H(12n+1)\, q^{12n+1},$$
$$\Sigma_{-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \times \Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty} (-1)^n G(12n+1)\, q^{12n+1}.$$

These equations express the theorems:—

(i.) If a^2 denote an even square, and b^2 an uneven square not divisible by 3, then the number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $3a^2+b^2$ is equal to $4H(p)$.

(ii.) If a^2 denote an even square, c^2 a square of the form $(12n+1)^2$, and d^2 a square of the form $(12n+5)^2$, then the excess of the number of representations of a number p (necessarily $\equiv 1$, mod. 12) by the form $3a^2+c^2$ over the number of representations by the form $3a^2+d^2$ is equal to $4(-1)^{\frac{1}{12}(p-1)}G(p)$.

Combining these two theorems, we see that (a^2, b^2, c^2 having the same meanings as above) the number of representations of p (necessarily $\equiv 1$, mod. 12) by the form $3a^2+c^2$ is equal to

$$2\left\{H(p)+(-1)^{\frac{1}{12}(p-1)}G(p)\right\},$$

and by the form $3a^2+d^2$ is equal to

$$2\left\{H(p)-(-1)^{\frac{1}{12}(p-1)}G(p)\right\}.$$

51. From the formulæ in the preceding section, it follows that

$$\frac{\Sigma_{-\infty}^{\infty}\, q^{(6n+1)^2}}{\Sigma_{-\infty}^{\infty}\,(-1)^n\, q^{(6n+1)^2}} = \frac{\Sigma_0^{\infty}\, H(12n+1)\, q^{12n+1}}{\Sigma_0^{\infty}\,(-1)^n\, G(12n+1)\, q^{12n+1}},$$

whence

$$\frac{1+H(13)\,q+H(25)\,q^2+H(37)\,q^3+\&c.}{1-G(13)\,q+G(25)\,q^2-G(37)\,q^3+\&c.} = \frac{1+q^2+q^4+q^{10}+q^{14}+\&c.}{1-q^2-q^4+q^{10}+q^{14}-\&c.}.$$

Equating coefficients, we find that, if $n\equiv 1$, mod. 12,

$$G(n)+G(n-24)+G(n-48)+G(n-120)+G(n-168)+\&c.$$
$$=(-1)^{\frac{1}{12}(n-1)}\left\{H(n)-H(n-24)-H(n-48)+H(n-120)\right.$$
$$\left.+H(n-168)-\&c.)\right\}.$$

The numbers 24, 48, 120, ... are the uneven squares which are not divisible by 3 diminished by unity.

52. It can be shown that

$$1+P_1q^2+P_2q^4+P_3q^6+\&c.$$
$$=(1-q-q^2+q^5+q^7-\&c.)(1+q+q^3+q^6+q^{12}+\&c.),$$

whence we deduce

$$\Sigma_{-\infty}^{\infty}\,(-1)^n q^{(6n+1)^2}\times\Sigma_{-\infty}^{\infty}\, q^{8m^2} = 2\Sigma_0^{\infty}\, G(12n+1)\, q^{48n+4},$$

where m denotes any uneven number.

This equation expresses the theorem:

If m^2 denote an uneven square, c^2 a square of the form $(12n+1)^2$, and d^2 a square of the form $(12n+5)^2$, then the excess of the number of representations of any number p (necessarily $\equiv 2$, mod. 24) by the form $3m^2+c^2$ over the number of representations by the form $3m^2+d^2$ is equal to zero, if $p\equiv 2$, mod. 48, and is equal to $8G\left(\tfrac{1}{4}p\right)$ if $p\equiv 4$, mod. 48.

53. From the formulæ,

$$\Sigma_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2}\times\Sigma_1^{\infty} q^{3m^2} = \Sigma_0^{\infty} G(12n+1)\, q^{48n+4},$$

$$\Sigma_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2}\times\Sigma_{-\infty}^{\infty} q^{12n^2} = \Sigma_0^{\infty}(-1)^n G(12n+1)\, q^{12n+1},$$

we obtain, by division,

$$\frac{1+P_1q^4+P_2q^8+P_3q^{12}+\&c.}{1-P_1q+P_2q^2-P_3q^3+\&c.}=\frac{1+q^2+q^6+q^{12}+q^{20}+\&c.}{1+2q+2q^4+2q^9+2q^{16}+\&c.};$$

whence, by equating coefficients, we find that

$$P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.$$

is equal to zero, if $n\equiv 2$ or 3, mod. 4; is equal to

$$P_u+2P_{u-1}+2P_{u-4}+2P_{u-9}+\&c.,$$

if $n\equiv 0$, mod. 4, where $u=\tfrac{1}{4}n$; and is equal to

$$-2P_v-2P_{v-2}-2P_{v-6}-2P_{v-12}-\&c.,$$

if $n\equiv 1$, mod. 4, where $v=\tfrac{1}{4}(n-1)$.

54. The second of the above results shows that, for all values of n,

$$P_n+2P_{n-1}+2P_{n-4}+2P_{n-9}+\&c. = P_{4n}+P_{4n-2}+P_{4n-6}+P_{4n-12}+\&c.$$

Now, in § 39 it was shown that

$$P_n+2P_{n-1}+2P_{n-4}+2P_{n-9}+\&c.$$

is equal to zero for all uneven values of n. It follows, therefore, that

$$P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.$$

is equal to zero whenever n is the quadruple of an uneven number, that is to say, whenever $n\equiv 4$, mod. 8.

The third result shows that, if $p=4n+1$, then, for all values of n,

$$P_p+P_{p-2}+P_{p-6}+P_{p-12}+\&c. = -2\left\{P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.\right\}.$$

Similarly, if $r=16n+5$,

$$P_r+P_{r-2}+P_{r-6}+P_{r-12}+\&c. = 4\left\{P_n+P_{n-2}+P_{n-6}+P_{n-12}+\&c.\right\};$$

and in general, if

$$w = 4^s n + \tfrac{1}{3}(4^s - 1),$$

$$P_w + P_{w-2} + P_{w-6} + P_{w-12} + \&c. = (-2)^s \{P_n + P_{n-2} + P_{n-6} + P_{n-12} + \&c.\}.$$

As a particular case, putting $n = 0$, we see that, if $a = \tfrac{1}{3}(4^s - 1)$,

$$P_a + P_{a-2} + P_{a-6} + P_{a-12} + \&c. = (-2)^s.$$

55. It has been shown in the two preceding articles that

$$P_n + P_{n-2} + P_{n-6} + P_{n-12} + \&c.$$

is equal to zero, if $n \equiv 2$ or 3, mod. 4, or $\equiv 4$, mod. 8; and it follows, therefore, from the theorem just proved, that it is also equal to zero, if

$$n = 4^s c + \tfrac{1}{3}(4^s - 1),$$

where c is any number $\equiv 2$ or 3, mod. 4, or $\equiv 4$, mod. 8; or, which is the same thing, $\equiv 2, 3, 4, 6$, or 7, mod. 8. For example, putting $s = 1$, and $c \equiv 2$ or 3, mod. 4, we see that it is equal to zero when $n \equiv 9$ or 13, mod. 16.

We have found, therefore, that the expression

$$P_n + P_{n-2} + P_{n-6} + P_{n-12} + \&c.$$

is equal to zero, if $n \equiv 2, 3, 4, 6$, or 7, mod. 8; or if $n \equiv 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 14$, or 15, mod. 16; and so on. In other words, if the expression does not vanish, n must necessarily $\equiv 0, 1$, or 5, mod. 8; or $\equiv 0, 1, 5, 8$, mod. 16; and so on.

56. The class of functions to which $G(n)$ belongs possesses two distinct kinds of properties, both of which are available for their calculation. The one kind, depending upon the divisors of n, is practically contained in the theorem

$$\phi(pq) = \phi(p)\,\phi(q),$$

where p and q are relatively prime; the other consists of the various recurring formulæ (such as those considered in §§ 33–44) in which $\phi(n)$ is expressed as a finite series of ϕ's of arguments less than n, and separated from n by numbers of special forms, such as squares, pentagonal numbers, &c. Properties of the former kind depend, as it were, on the arithmetical structure of the function $\phi(n)$, and cannot, so far as I see, be directly derived from the series in which the functions $\phi(n)$ appear as coefficients. On the other hand, the recurring formulæ connecting the values of $\phi(n)$ are readily obtained by equating co-efficients; but the kind of consideration by which such a theorem as

$$\phi(pq) = \phi(p)\,\phi(q)$$

is proved, appears to afford no clue to the discovery of these formulæ, nor to their demonstration when found. Amongst functions of this class [*i.e.*, which possess the property $\phi(pq) = \phi(p)\,\phi(q)$, and satisfy also various recurring formulæ] may be mentioned the functions E, χ, H, σ, referred to in this paper, and ζ, λ (*Proc. Lond. Math. Soc.*, Vol. xv., p. 109, and *Quart. Journ.*, Vol. xx., p. 145).

Complex Multiplication Moduli of Elliptic Functions for the Determinants −53 *and* −61. *By* Professor G. B. Mathews.

[*Read Dec.* 12*th*, 1889.]

The following note contains the solution of two cases of complex multiplication referred to by Mr. Greenhill in his paper on the subject (*Proceedings*, Vol. xix.) as being hitherto unsolved: viz., those in which $\Delta = 53, 61$, respectively.

As my results are merely supplementary to the paper just quoted, and the method of procedure (which is essentially Hermite's) is there sufficiently explained and illustrated (see, for instance, pp. 326–328), I have not thought it necessary to do more than give the actual algebraical work.

Applying Hermite's method to the modular equation for $n = 31$, the values of

$$2n-\rho^2 \quad (\rho = 1, 3, 5, 7)$$

are $\qquad 61, \;\; 53, \;\; 37, \;\; 13.$

We have to put $\qquad \kappa\lambda = w^4, \quad \kappa'\lambda' = 2iw^2,$

and then in Russell's notation

$$P = w + \sqrt{(1+i)\,w} + 1,$$

$$Q = w\sqrt{(1+i)\,w} + \sqrt{(1+i)\,w} + w,$$

$$R = w\sqrt{(1+i)\,w}.$$