Note on some Properties of a Generalized Brocard Circle. By JOHN GRIFFITHS, M.A.

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Notation.

The notation in the following pages is identical with that employed in a former note on the generalized Brocard circles of a triangle ABC. See *Proc. Lond. Math. Soc.*, Vol. XXV., Nos. 479, 480. A brief explanation of it will be sufficient.

1. The abbreviation $GBC\,(UU'VV'A'OWKa')$ denotes the generalized Brocard circle of the first system drawn through the nine points U, U', ... a', which are all discussed—with the exception of the last—in the note in question. [In fact, a' is a new point of the circle, and the principal theorem in this paper consists in showing that it is derived from A' in the same manner as V, V', and O are from U, U', and W.]

2. The isogonal and trilinear coordinates of a point P are connected by the relations

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}=\frac{\Sigma a\alpha}{\Sigma a\beta\gamma},$$

which give $$\Sigma a\,(x-yz)=0.$$

Hence a linear equation $\lambda x+\mu y+\nu z=\delta$

denotes a circle.

3. The equation of $GBC\,(UU'\ldots a')$ may be written in the two forms

$$x\operatorname{cosec}A\sin B\sin C\sin(2\theta-A)\operatorname{cosec}^2\theta+y\sin B(\cot\theta+\cot B)$$
$$+z\sin C\,(\cot\theta+\cot C)$$
$$=2\operatorname{cosec}A\sin B\sin C\cot\omega;$$

$$x\sin B\sin C\left\{\operatorname{cosec}^2A-(\cot\omega-\cot\Omega)^2\right\}+y\sin B\,(\cot\omega-\cot C)$$
$$+z\sin C\,(\cot\omega-\cot B)$$
$$=2\operatorname{cosec}A\sin B\sin C\cot\omega;$$

where $$\cot\Omega=\Sigma\cot A,$$

and $$\cot\omega=\cot\theta+\cot B+\cot C.$$

Section I.

Common Property of the Points U, U′, A′, and W.

Let any one of the four be denoted by P, and the mid-point of BC by M; also let the circle BCP be met again by AP (produced, if necessary) in P', and by $P'M$ in P''; then P'' is also a point on $GBC\,(U, U', \dots a')$.

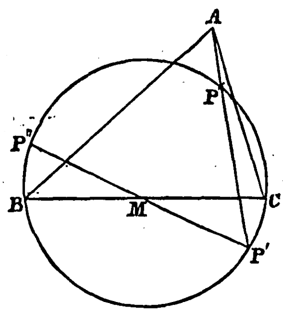


Fig. 1.

There is no difficulty in verifying this construction by the employment of isogonal coordinates. Taking (xyz), $(x'y'z')$, $(x''y''z'')$ to denote P, P', P'' respectively, we have

$$x' = x, \quad y' = -\frac{x}{z}, \quad z' = -\frac{x}{y},$$

and $$x'' = x, \quad y'' = 2\frac{\sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin B} + \frac{x}{y}\,\frac{\sin C}{\sin B},$$

$$z'' = 2\frac{\sin B}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin C} + \frac{x}{z}\,\frac{\sin B}{\sin C}.$$

(See *Proc. Lond. Math. Soc.*, June, 1894.)

Hence, if P and P'' both lie on the circumference of a G.B. circle of the first system, it follows that

$$x \sin B \sin C \left\{\operatorname{cosec}^2 A - (\cot\omega - \cot\Omega)^2\right\} + y \sin B\,(\cot\omega - \cot C)$$
$$+ z \sin C\,(\cot\omega - \cot B)$$
$$= 2 \operatorname{cosec} A \sin B \sin C \cot\omega \quad \dots\dots\dots\dots (1),$$

and $x \sin B \sin C \{\operatorname{cosec}^2 A - (\cot\omega - \cot\Omega)^2\}$

$$+\left(2\frac{\sin B\sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A} + \frac{x}{y}\sin C\right)(\cot\omega - \cot C)$$

$$+\left(2\frac{\sin B\sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A} + \frac{x}{z}\sin B\right)(\cot\omega - \cot B)$$

$$= 2\operatorname{cosec} A \sin B \sin C \cot\omega,$$

$$\frac{2\cot\theta}{x\sin A} + \frac{\cot\omega - \cot C}{y\sin B} + \frac{\cot\omega - \cot B}{z\sin C} = \cot^2\omega + 2\cot A\cot\omega - 1 \qquad \text{..................(2).}$$

Now, since x, y, z are proportional to the trilinear coordinates of P, and $\Sigma x\sin A = \Sigma yz\sin A$, it is easy to see that (2) represents a quartic curve passing through the two circular points at infinity, and, consequently, that P is an intersection of the G.B. circle with this curve.

It thus appears that, excluding the pair of circular points at infinity, the construction gives six positions of P for each of which the companion point P'' is also on the circle. But, obviously, P'' coincides with P in two special cases, viz., when P is the intersection of the G.B. circle with the median AM. Hence there remain four positions of P to be considered, and it is found that they coincide with U, U', A', and W. The companion points are V, V', a', and O.

Section II.

Verification of the above Construction.

1. Let P coincide with U, whose coordinates are

$$x = \frac{\sin C}{\sin B};$$

$$y\sin B = (\cot\omega - \cot C)^{-1}, \quad z = \sin C(\cot\omega - \cot B).$$

U will therefore lie on the quartic represented by the equation

$$\frac{2(\cot\omega - \cot B - \cot C)}{x\sin A} + \frac{\cot\omega - \cot C}{y\sin B} + \frac{\cot\omega - \cot B}{z\sin C}$$

$$= \cot^2\omega + 2\cot A\cot\omega - 1,$$

if $\;2(\cot\omega - \cot B - \cot C)(\cot A + \cot C) + (\cot\omega - \cot C)^2 + \operatorname{cosec}^2 C$

$$= \cot^2\omega + 2\cot A\cot\omega - 1,$$

i.e., if $\operatorname{cosec}^2 C = (\cot A + \cot C)(\cot B + \cot C)$,

or $\Sigma \cot B \cot C = 1$,

an identity.

In this case P'' is given by

$$x'' = \frac{\sin C}{\sin B} = x, \quad y'' = 2\frac{\sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin B} + \frac{x}{y}\,\frac{\sin C}{\sin B}.$$

Here the fundamental relation

$$\Sigma x \sin A = \Sigma yz \sin A$$

takes the form
$$\frac{x}{y} = z - \frac{\sin(B-C)}{\sin B},$$

since
$$x = \frac{\sin C}{\sin B};$$

we therefore have

$$y'' = 2\frac{\sin C}{\sin A} - \frac{\sin C}{\sin B}\,\frac{\sin^2 B + \sin^2 C}{\sin A \sin B} + \left\{ z - \frac{\sin(B-C)}{\sin B} \right\} \frac{\sin C}{\sin B} = xz,$$

$$z'' = \frac{1}{z} + \frac{\sin(B-C)}{\sin B}.$$

In other words, P'' becomes V. See *Proceedings*, quoted *supra.*

2. Let P take the position U' given by

$$x = \frac{\sin B}{\sin C}, \quad y^{-1}\sin B = (\cot\omega - \cot C)^{-1}, \quad z^{-1} = \sin C\,(\cot\omega - \cot B).$$

It is then found that P will lie on the quartic if

$$\Sigma \cot B \cot C = 1,$$

as before. In this case P'' coincides with V', whose coordinates are

$$x'' = \frac{\sin B}{\sin C}, \quad y'' = \frac{1}{y} + \frac{\sin(C-B)}{\sin C}, \quad z'' = xy.$$

3. Let (x, y, z) coincide with A'. Here

$$x = \frac{\sin 2\omega}{\sin(2\omega + A)}, \quad y = x\sin C\,(\cot\omega - \cot C), \quad z = x\sin B\,(\cot\omega - \cot B).$$

This point will therefore lie on the quartic

$$\frac{2(\cot\omega-\cot B-\cot C)}{x\sin A}+\frac{\cot\omega-\cot C}{y\sin B}+\frac{\cot\omega-\cot B}{z\sin C}$$

$$=\cot^2\omega+2\cot A\cot\omega-1=\frac{\sin(2\omega+A)}{\sin A\sin^2\omega},$$

if $$\frac{2(\cot\omega-\cot B-\cot C)}{\sin A}+\frac{2}{\sin B\sin C}=\frac{2\cot\omega}{\sin A},$$

or $$\cot B+\cot C=\frac{\sin A}{\sin B\sin C}.$$

The corresponding position of P'', which I call a', is defined by

$$x''=\frac{\sin 2\omega}{\sin(2\omega+A)},$$

$$y''=\frac{2\sin B\sin C\cos 2\omega-\sin A\sin 2\omega}{\sin B\sin(2\omega+A)}+\frac{1}{\sin B(\cot\omega-\cot C)},$$

$$z''=\frac{2\sin B\sin C\cos 2\omega-\sin A\sin 2\omega}{\sin C\sin(2\omega+A)}+\frac{1}{\sin C(\cot\omega-\cot B)}.$$

4. Let P take the position W, whose coordinates are

$$x=\frac{\sin 2\theta}{\sin(2\theta-A)},\quad y=\frac{\sin\theta}{\sin(\theta+B)},\quad z=\frac{\sin\theta}{\sin(\theta+C)},$$

where $$\cot\theta=\cot\omega-\cot B-\cot C.$$

By writing these in the form

$$x\sin A=2\cot\theta\left\{\operatorname{cosec}^2 A-(\cot\omega-\cot\Omega)^2\right\}^{-1},$$

$$y\sin B=(\cot\omega-\cot C)^{-1},\quad z\sin C=(\cot\omega-\cot B)^{-1},$$

it is seen that W will lie on the quartic if

$$\operatorname{cosec}^2 A-(\cot\omega-\cot\Omega)^2+(\cot\omega-\cot C)^2+(\cot\omega-\cot B)^2$$

$$=\cot^2\omega+2\cot A\cot\omega-1,$$

or $$\Sigma\cot A=\cot\Omega,\quad \Sigma\operatorname{cosec}^2 A=\operatorname{cosec}^2\Omega.$$

The corresponding position of P'' is defined by

$$x''=\frac{\sin 2\theta}{\sin(2\theta-A)},\quad y''=-\frac{2\sin\theta\cos(\theta+C)}{\sin(2\theta-A)},$$

and $$z'' = -\frac{2\sin\theta\cos(\theta+B)}{\sin(2\theta-A)}.$$

In other words, P'' coincides with O.

It may be here remarked that the four pairs of points UV, $U'V'$, $A'a'$, and WO form an involution on the G.B. circle, since the chords UV, &c., all meet in a point on BC. More generally, the join of the two points of intersection of a given circle with a circle through B and C passes through a given point on BC, viz., the intersection of this side with the radical axis of the given circle and the circumcircle ABC. This follows at once from the fact that the radical axes of three circles taken in pairs are concurrent.

Hence, since (1) W, O are concyclic with B and C, (2) W, U with C and A, and (3) W, U' with A and B, the intersections of WO, BC; WU, CA; and WU', AB lie on the radical axis of $GBC\,(UU'WO)$ and the circumcircle ABC. There is nodifficulty in arriving at the relations which connect the isogonal coordinates of the two points of intersection of a generalized Brocard circle with a circle drawn through two vertices of the triangle of reference. For instance, if (xyz), $(x'y'z')$ denote two points on $GBC\,(UU'WO)$ which are concyclic with B and C, we have

$$x' = x,$$

$$y' = \frac{\sin C}{\sin A}\,\frac{\cot B-\cot C}{\cot\omega-\cot C} + x\,\frac{\sin^2 C\,(\cot\omega-\cot B)-\sin^2 B\,(\cot\omega-\cot C)}{\sin A\sin B\,(\cot\omega-\cot C)}$$
$$+z\,\frac{\sin C}{\sin B}\,\frac{\cot\omega-\cot B}{\cot\omega-\cot C},$$

$$z' = \frac{\sin B}{\sin A}\,\frac{\cot C-\cot B}{\cot\omega-\cot B} + x\,\frac{\sin^2 B\,(\cot\omega-\cot C)-\sin^2 C\,(\cot\omega-\cot B)}{\sin A\sin C\,(\cot\omega-\cot B)}$$
$$+y\,\frac{\sin B}{\sin C}\,\frac{\cot\omega-\cot C}{\cot\omega-\cot B}.$$

These transformations give

$$y'\sin B\,(\cot\omega-\cot C)+z'\sin C\,(\cot\omega-\cot B)$$
$$= y\sin B\,(\cot\omega-\cot C)+z\sin C\,(\cot\omega-\cot B).$$

(See equation of generalized Brocard circle, *supra*.)

Section III.

Illustration of the Construction by a Diagram.

The figure shows that a G.B. circle can be constructed when its O-point is known.

M and μ are taken to be the mid-points of the side BC and the perpendicular AD, respectively; O is a given point on the line OMA_1'

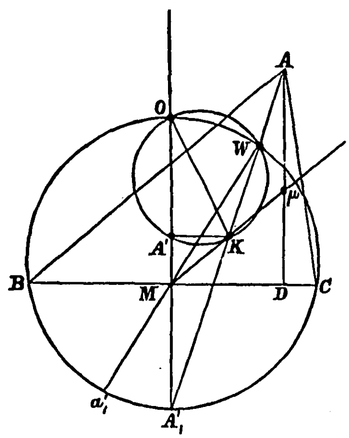


Fig. 2.

drawn parallel to AD; the circle BOC meets OMA_1' again in A_1' and AA_1' in W, AWA_1' meets $M\mu$ in K, and KA' is parallel to BC. The circle $A'KWO$, described upon OK as diameter, is a G.B. circle of the first system.

Hence we have the following properties, viz.,

(1) B, C, W, O are concyclic.

(2) The sides AB, AC subtend equal angles at W.

(3) The join of K and W passes through the vertex A of the triangle ABC.

Section IV.

The Intersection of the Chord Wa' and the Median AM lies on the Radical Axis of GBC (OWa') and the Circumcircle ABC.

This theorem is immediately suggested by the form of the expressions for the coordinates of a', viz.,

$$X = \alpha + x, \quad Y = \beta + y, \quad Z = \gamma + z,$$

where (XYZ), (xyz) denote a' and W, and

$$a = \frac{\sin 2\omega}{\sin(2\omega + A)} - \frac{\sin 2\theta}{\sin(2\theta - A)},$$

$$\beta \sin B = (2 \sin B \sin C \cos 2\omega - \sin A \sin 2\omega) \div \sin(2\omega + A) = \gamma \sin C.$$

(See Section II.)

Hence a', W, and the point, r, say, whose trilinear coordinates are proportional to a, β, γ are collinear. Now, since

$$\beta \sin B = \gamma \sin C,$$

r must lie on the median AM. It also lies on the radical axis in question. For, if

$$\lambda\xi + \mu\eta + \nu\zeta = \delta$$

be the equation of $GBC\,(OWa')$, we have

$$\lambda X + \mu Y + \nu Z = \delta = \lambda x + \mu y + \nu z, \quad \text{or} \quad \Sigma\lambda\,(X - x) = 0,$$

and consequently $$\Sigma\lambda a = 0.$$

In other words, r is a point on the radical axis.

The theorem is thus proved.

Section V.

The four positions of P' corresponding to U, U', A', and W (see Fig. 1) are concyclic. In fact, they are on the circumference of another G.B. circle of the first system, viz., the inverse with respect to M of $GBC\,(UU'A'W)$, the constant of inversion being

$$\rho^2 = -\tfrac{1}{4}(BC)^2.$$

For $$MP' \cdot MP'' = MB \cdot MC = -\tfrac{1}{4}a^2.$$

(See a former note, *Proc. Lond. Math. Soc.*, June, 1894.) [In Fig. 2, the points A_1' and a_1', the projections through M of O and W on the circle $BOWC$ become the A' and a' points of the new or inverse G.B. circle.]

Section VI.

An Application of the Theory of Inversion with respect to the Point M.

Theorem.—If the coordinates of any point P on a G.B. circle are known, those of a certain other point on the circle, say Q, are also known.

Since the equation of a circle of the first system may be written as follows, viz.,

$$x\sin B\sin C\left\{\operatorname{cosec}^2 A-(\xi-\cot\Omega)^2\right\}$$
$$+y\sin B\,(\xi-\cot C)+z\sin C\,(\xi-\cot B)=2\,\frac{\sin B\sin C}{\sin A}\,\xi,$$

where $$\xi=\cot\omega,$$

we may assume that the coordinates x, y, z of a point P on the circumference are functions of ξ, and certain constants dependent on the form of the triangle ABC. Hence we may take

$$x=f(\xi),\quad y=\phi(\xi),\quad\text{and}\quad z=\psi(\xi).$$

This being so, the coordinates X, Y, Z of Q are

$$X=f(\cot B+\cot C-\xi),$$
$$Y=2\,\frac{\sin C}{\sin A}-\frac{\sin^2 B+\sin^2 C}{\sin A\sin B}\,f(\cot B+\cot C-\xi)-\frac{\sin C}{\sin B}\,\psi(\cot B+\cot C-\xi),$$
$$Z=2\,\frac{\sin B}{\sin A}-\frac{\sin^2 B+\sin^2 C}{\sin A\sin C}\,f(\cot B+\cot C-\xi)-\frac{\sin B}{\sin C}\,\phi(\cot B+\cot C-\xi).$$

Ex.—Let P be U, given by

$$x=\frac{\sin C}{\sin B}=f(\xi),$$
$$y=\operatorname{cosec}B\,(\xi-\cot C)^{-1}=\phi(\xi),$$
$$z=\sin C\,(\xi-\cot B)=\psi(\xi).$$

Hence we have for Q

$$X=\frac{\sin C}{\sin B},$$
$$Y=2\,\frac{\sin C}{\sin A}-\frac{\sin^2 B+\sin^2 C}{\sin A\sin B}\,\frac{\sin C}{\sin B}-\frac{\sin C}{\sin B}\sin C\,(\cot C-\xi),$$
$$Z=\frac{\sin B}{\sin A}-\frac{\sin^2 B+\sin^2 C}{\sin A\sin C}\,\frac{\sin C}{\sin B}-\frac{\sin B}{\sin C}\operatorname{cosec}B(\cot B-\xi)^{-};$$

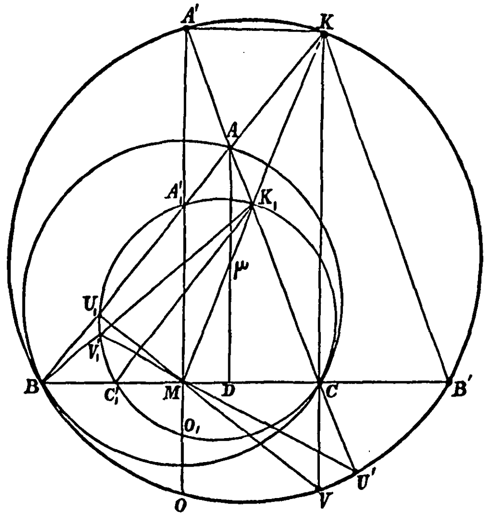
or $$X=x,\quad Y=xz,\quad Z=\frac{1}{z}+\frac{\sin(B-C)}{\sin B}.$$

In other words, Q is V.

[The coordinates of A', a' can, in like manner, be derived from those of O and W respectively.]

Section VII.

Construction of the G.B. Circles of the First System corresponding to $\omega = B$ and $\omega = C$, in illustration of the Method of Inversion.



[$MA'.MO_1 = MB.MC = MU_1.MV = MU'.MV_1'$, &c. KB', K_1C_1' are parallel to CA, AB, respectively.]

For the circle corresponding to $\omega = C$, we have the following properties, viz.,

(1) The points U, W, V' and a' coincide with B.

(2) A' lies on the side CA, and K on BA.

(3) The circle touches the circumcircle ABC at B.

(4) The triangle $A'B'B$ is inversely similar to ABC, and belongs to the system of similar in-triangles whose centre of similitude is the point on the Brocard circle of ABC given by the coordinates

$$x = \frac{c}{b}, \quad y = \frac{c}{a}, \quad z = 2\cos C.$$

Hence the circle $A'B'B$ has double contact with the conic which touches the sides of ABC and has the centre of similitude in question for a focus.

The circle corresponding to $\omega = B$ may be constructed in like manner. The pair are inverse to each other with respect to M.

It may also be remarked that there is another pair of G.B. circles of the first system which touch the circle ABC, and are inverse to one another with regard to the mid-point of BC.

In general four circles in each of the three systems can be drawn to touch a given circle.

Thursday, February 14th, 1895.

Mr. A. B. KEMPE, F.R.S., Vice-President, in the Chair.

The Chairman announced the decease, since the January meeting, of Professor Cayley and Sir James Cockle, and stated that the Society had been represented at the funeral of the former by the President, himself, and Professors Elliott and Henrici.

Messrs. Walker, Glaisher, and Elliott paid tributes to the memory of the deceased members. A resolution was passed unanimously that the President should be requested to convey, in such form as he should think fit, votes of condolence from the Society to Mrs. Cayley and Lady Cockle.

The following papers were read:—

On certain Differential Operators, and their use to form a Complete System of Seminvariants of any Degree, or any Weight: Prof. Elliott.

On the Electrification of a Circular Disc in any Field of Force Symmetrical with respect to its Plane: Mr. H. M. Macdonald.

Notes on the Theory of Groups of Finite Order, iii. and iv.: Prof. W. Burnside.

The following presents were received:—

"Proceedings of the Royal Society," Vol. LVII., Nos. 340–341.

"Vierteljahrsschrift der Naturforschenden Gesellschaft zu Zürich," Jahrgang 39, Heft 3–4; Zurich, 1894.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 12; Bd. XIX., St. 1; Leipzig, 1894.

"Proceedings of the Royal Irish Academy," Series 3, Vol. III., No. 3; December, 1894.

"Revue Semestrielle des Publications Mathématiques," Tome III., Partie 1ère; Amsterdam, 1895.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. IX., No. 1; 1894–5.

"Proceedings of the Physical Society of London," Vol. XIII., Pts. 2, 3; Jan.–Feb., 1895.

"Nautical Almanack for 1898."

"Nieuw Archief voor Wiskunde," Reeks 2, Deel I. 1; Amsterdam, 1894.

"Wiskundige Opgaven met de Oplossingen," Deel 6, St. 4; Amsterdam, 1895.

Van den Berg, F. J.—"Over Co-ördinaten Stelsels voor Cirkels in het Platte vlak en door Bollen in der Ruimte," pamphlet, 8vo; Amsterdam, 1894.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. I., No. 4.

"Nyt Tidsskrift for Matematik," A. Femte Aargang, Nos. 6, 7–8; B. Femte Aargang, Nos. 3, 4; Copenhagen.

"Bulletin des Sciences Mathématiques," Tome XIX., Jan.–Fev., 1895; Paris.

Bierens de Haan, D.—"Bouwstoffen voor de Geschiedenis der Wis- en Natuurkundige Wetenschappen in de Nederlanden," Deel 2, No. 1, 1ste Sectie; Amsterdam, 1893.

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"Educational Times," February, 1895.

"Journal of the College of Science, Imperial University, Japan," Vol. VII., Pts. 2, 3.

McClintock, E.—"Theorems in the Calculus of Enlargement," 4to pamphlet.

"Annali di Matematica," Tomo XXIII., Fasc. 1; Milano, 1895.

"Journal für die reine und angewandte Mathematik," Bd. CXIV., Heft 3; Berlin, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Vol. VIII., Fasc. 11–12; 1894.

"Acta Mathematica," XVIII., 4; Stockholm.

"Indian Engineering," Vol. XVI., Nos. 25–26; Vol. XVII., Nos. 1, 2, 3.

Reale Istituto Lombardo—"Rendiconti," Serie 2, Vols. XXV., XXVI.; "Indice Generale all' anno 1888;" "Memorie," Vols. XVII., XVIII. della Serie 3, Fasc. 2, 3; Milano, 1892–3.

"American Journal of Mathematics," Vol. XVII., No. 1; Baltimore, 1895.