THE INFLUENCE OF VISCOSITY ON THE OSCILLATIONS OF SUPERPOSED FLUIDS

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IN this paper two problems of hydrodynamics are attacked with a view to discovering the influence of viscosity, the solutions being known for the case of non-viscous fluids.

The first is the case of two fluids of infinite depth, where it is found to a first approximation that the modulus of decay is of the order $1/\sqrt{\nu}$.

The second is the case of a fluid of finite depth superposed on a fluid of infinite depth. There are two modes; in the first the modulus of decay is of the order $1/\nu$, and in the second it is of the order $1/\sqrt{\nu}$.

In the problems dealt with in this paper the fluids are at rest, except for the wave-motion. In a subsequent paper I shall publish some results dealing with wave-motion at the surface of a stream of viscous fluid.

Waves at the Interface between Two Viscous Fluids of Infinite Depth.

1. Take the origin in the undisturbed surface of separation, and the axis of y vertically upwards.

For the lower fluid we have

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y},$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x},$$

where

$$\phi = A e^{ky} e^{\iota kx + at},$$

$$\psi = H e^{\lambda y} e^{\iota kx + at},$$

and

$$\lambda^2 = k^2 + \frac{a}{\nu}.*$$

* See Lamb's *Hydrodynamics*, p. 564.

For the upper fluid
$$u' = \frac{\partial\phi'}{\partial x} + \frac{\partial\psi'}{\partial y},$$
$$v' = \frac{\partial\phi'}{\partial y} - \frac{\partial\psi'}{\partial x},$$
where
$$\phi' = A'e^{-ky}\,e^{ikx+at},$$
$$\psi' = H'e^{-\lambda' y}\,e^{ikx+at},$$
and
$$\lambda'^2 = k^2 + \frac{a}{\nu'}\,.$$

The kinematical conditions at the interface are
$$u = u', \qquad v = v'\,;$$
these give
$$ikA + \lambda H = \;\; ikA' - \lambda' H',$$
$$kA - ikH = -\,kA' - ikH'\,;$$
since
$$u = ikAe^{ky} + \lambda He^{\lambda y},$$
$$v = kAe^{ky} - ikHe^{\lambda y},$$
$$u' = \;\; ikA'e^{-ky} - \lambda' H'e^{-\lambda' y},$$
$$v' = -\,kA'e^{-ky} - ikH'e^{-\lambda' y}.$$

We have tacitly dropped the factor e^{ikx+at}.

The dynamical conditions are the continuity of p_{xy}, p_{yy} across the interface.

Now
$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - g\eta,$$
where η is the elevation of the interface; and therefore
$$\frac{\partial\eta}{\partial t} = v_{(y=0)}\,;$$
therefore
$$\eta = \frac{1}{a}(kA - ikH) = \frac{1}{a}(-kA' - ikH').$$

Hence $$\left(\frac{p_{yy}}{\rho}\right)_{y=0}=-\frac{p}{\rho}+2\nu\frac{\partial v}{\partial y}$$

$$=aA+\frac{g}{a}(+kA-\iota kH)+2\nu\,(k^2A-\iota k\lambda H),$$

$$\left(\frac{p'_{yy}}{\rho'}\right)_{y=0}=aA'+\frac{g}{a}(-kA'-\iota kH')+2\nu'\,(k^2A'+\iota k\lambda' H'),$$

$$\left(\frac{p_{xy}}{\rho\nu}\right)_{y=0}=\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{y=0}$$

$$=\quad 2\iota k^2A+(\lambda^2+k^2)\,H,$$

$$\left(\frac{p'_{xy}}{\rho'\nu'}\right)_{y=0}=-2\iota k^2A'+(\lambda'^2+k^2)\,H'.$$

Hence we obtain the remaining conditions

$$\rho\left\{aA+\frac{g}{a}(kA-\iota kH)+2\nu(k^2A-\iota k\lambda H)\right\}$$
$$=\rho'\left\{aA'+\frac{g}{a}(-kA'-\iota kH')+2\nu'\,(k^2A'+\iota k\lambda' H')\right\},$$

and $$\rho\nu\left\{2\iota k^2A+(\lambda^2+k^2)\,H\right\}=\rho'\nu'\left\{-2\iota k^2A'+(\lambda'^2+k^2)\,H'\right\}.$$

After eliminating A, A', H, H' from these four equations, we obtain the period equation

$$4k^3(\nu\rho-\nu'\rho')(k-\lambda)(k-\lambda')+4k^2a(\rho\nu-\rho'\nu')\left[\rho(k-\lambda')-\rho'(k-\lambda)\right]$$
$$+\rho^2(a^2+gk)(k-\lambda')+\rho'^2(a^2-gk)(k-\lambda)$$
$$-\rho\rho'\left[2ak^2+a^2(\lambda+\lambda')+gk(\lambda-\lambda')\right]=0.$$

In this equation a is still implicitly contained in λ and λ'; after rationalisation the equation is found to be of the tenth degree in a.

If we take ν and ν' to be small, and include only the most important terms, we obtain

$$(\rho\lambda'+\rho'\lambda)\left[(\rho+\rho')\,a^2+gk\,(\rho-\rho')\right]=0,$$

or $$a^2=-\frac{gk(\rho-\rho')}{\rho+\rho'}.$$

This is the known result for non-viscous fluids, as we should have expected.*

* See § 2.

To proceed to a higher approximation we write

$$a = a_0+\beta,$$

where
$$a_0 = \pm i\sqrt{\frac{gk(\rho-\rho')}{\rho+\rho'}},$$

in the equation

$$4k^3(\nu\rho-\nu'\rho')\lambda\lambda'+\rho^2(a^2+gk)(k-\lambda')$$
$$+\rho'^2(a^2-gk)(k-\lambda)-\rho\rho'[2ak^2+a^2(\lambda+\lambda')+gk(\lambda-\lambda')] = 0.$$

We find
$$\beta = -\left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{4}} \frac{2k\rho\rho'\sqrt{\nu\nu'}}{(\rho+\rho')\{\rho\sqrt{\nu}+\rho'\sqrt{\nu'}\}} \frac{1\pm i}{\sqrt{2}}.$$

We see that both the change in the velocity due to viscosity and the reciprocal of the modulus of decay depend on terms of the order $\sqrt{\nu}$ to a first approximation. When there is only one fluid the modulus of decay of the amplitude is of the order $1/\nu$, and the change in the velocity is of the order ν^2. The difference is due to this fact. When there is wave motion at the interface between two non-viscous fluids, the tangential velocities at the interface are different; in the viscous motion they must be the same. Hence, for dynamical reasons, we should expect a change of the nature obtained above. We have the result that, in general, wave motion at the interface between two fluids dies away much more rapidly than in the case of a single fluid. The difference is especially marked for great wave-lengths. Nevertheless the change in the velocity is small compared with the velocity.

To proceed to a still higher approximation, we write

$$a = a_0+\beta+\gamma$$

in terms chosen suitably from the period equation.

We easily find
$$\gamma = -\frac{2k^2(\nu^2\rho^3+\nu'^2\rho'^3)}{(\rho+\rho')\{\rho\sqrt{\nu}+\rho'\sqrt{\nu'}\}^2}.$$

Hence to this order we have, as our final value for a,

$$a = \pm i\left[\left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{2}} - \left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{4}} \frac{\sqrt{2}\,k\rho\rho'\sqrt{\nu\nu'}}{(\rho+\rho')(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})}\right]$$
$$-\left[\left\{\frac{gk(\rho-\rho')}{\rho+\rho'}\right\}^{\frac{1}{4}} \frac{\sqrt{2}\,k\rho\rho'\sqrt{\nu\nu'}}{(\rho+\rho')(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})} + 2k^2\frac{\nu^2\rho^3+\nu'^2\rho'^3}{(\rho+\rho')(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})^2}\right].$$

When $\rho' = 0$, $\nu' = 0$, we have

$$a = \pm i\sqrt{gk}-2\nu k^2,$$

which is known to be the result for a single fluid of infinite depth.

2. We have said above that the result, that to a first approximation the period is the same as that for the motion of non-viscous fluids of the same type, is according to expectation. This was reasoned not from the nature of the present problem, but from the known results of similar problems. An objection might be raised to this conclusion on the ground that the boundary conditions are totally different from those employed in the non-viscous motion, and are apparently contradictory if $\nu = \nu' = 0$. The answer to the latter objection is that, even if H and H' become zero, λ and λ' become infinite and the equations indeterminate. A similar objection to the first would apply to the work on page 571 of Lamb's *Hydrodynamics*, where he discusses the effect of oil on water waves, and also to the work of Basset in treating the case of a fluid of finite depth. In both these cases the boundary conditions are different, and yet to a first approximation the period is the same as for non-viscous motion of the same type. We are not questioning the physical truth of the assumption that there is no slipping at the interface, but the correctness of the result on this assumption. The final court of appeal is the analysis itself. But a physical answer may perhaps be given along the following lines. In the non-viscous motion there is a vortex sheet at the interface of strength $-2kc\beta \cos kx$ (Lamb's *Hydrodynamics*, p. 354), where β is the amplitude of the surface waves. This vortex sheet does not exist in the viscous motion. Now this difference between the two motions may be made as small as we please by sufficiently diminishing β, without at the same time affecting the average tangential velocity. Hence, unless the period of the motion of the viscous fluids is to depend on the amplitude, even when squares of the amplitude are neglected, it must be the same as the period for the non-viscous fluids, when the viscosity is neglected, dynamically but not kinematically. A more rigid formulation could be given, but in a general way probably this will suffice.

It is interesting to notice that in the work mentioned above, on the effect of oil on water waves, the modulus of decay depends on $1/\sqrt{\nu}$ as in the present case.

3. We can include the effect of capillarity at the interface by writing $g(\rho-\rho')+k^2T$ instead of $g(\rho-\rho')$ in our results. When the wave-length is small capillarity has a very great effect in causing the decay of the motion. However, when the wave-length is small our approximations are not sufficiently good, as they would, if continued, be in the form of a series of ascending powers of k. When the wave-length is small the effect of capillarity in causing decay of the motion would be more evident from the

term arising from the next approximation to that which we have already written down.

In the table given below the results are shown for the case of air over water. The c.g.s. system of units is used, and the following data: $\rho = 1$, $\rho' = \cdot00129$; $\nu = \cdot0109$, $\nu' = \cdot139$, for water and air at 17° C. respectively; $T = 74$.

Wave-length.	1 cm.	10	100	1000
v_0	12·48	39·46	124·79	394·62
v_c	24·90	40·05	124·81	394·62
v	24·89	40·04	124·81	394·62
τ_0	1·162″	1′ 56·2″	3 hrs. 12′ 39·4″	321 hrs. 5′ 40″
τ	1·125″	1′ 34·1″	1 hr. 21′ 40·6″	36 hrs. 50′ 36″
τ_c	1·106″	1′ 34·0″	1 hr. 21′ 40·3″	36 hrs. 50′ 34″

v_0 is the wave-velocity in centimetres per second without viscosity and capillarity, v_c the velocity with capillarity only, v the velocity with both.

τ_0 is the modulus of decay of the water alone; τ, of water and air without capillarity; τ_c, of water and air with capillarity.

In the above table we notice the great influence of the air in damping waves of great wave-length.

Fluid of Finite Depth Superposed on a Fluid of Infinite Depth.

4. We suppose the upper fluid to have a free upper surface.

For the lower fluid we assume

$$\phi = Ae^{ky}e^{\iota kx+at},$$

$$\psi = iHe^{\lambda y}e^{\iota kx+at},$$

where

$$\lambda^2 = k^2 + \frac{a}{\nu}.$$

For the upper fluid we assume

$$\phi' = (B\cosh ky + C\sinh ky)\,e^{\iota kx+at},$$

$$\psi' = i\,(K\cosh\lambda' y + L\sinh\lambda' y)\,e^{\iota kx+at},$$

where

$$\lambda'^2 = k^2 + \frac{a}{\nu'}.$$

From these we derive the velocities by the formulæ

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y},$$

$$v = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}.$$

The kinematical conditions are the continuity of the component velocities across the interface. The dynamical conditions are the continuity of pressures and tractions across the interface and the free surface. Writing these down, and eliminating the constants, we obtain the period equation

$$\begin{vmatrix} P, & p, & -(\lambda'^2+k^2)\rho'\nu', & -\frac{gk}{a}\rho, & 0, & -1 \\ Q, & q, & 0, & -2\nu'\rho'k\lambda', & -\lambda', & 0 \\ R, & r, & 0, & -\rho'a-2\nu'k^2\rho', & -k, & 0 \\ S, & s, & -2k^2\rho'\nu', & -\frac{gk}{a}\rho', & 0, & -1 \\ 0, & 0, & 2k^2\rho\nu, & \rho\left(a+2\nu k^2+\frac{gk}{a}\right), & k, & 1 \\ 0, & 0, & (\lambda^2+k^2)\rho\nu, & \frac{gk}{a}\rho+2\nu\rho k\lambda, & \lambda, & 1 \end{vmatrix} = 0,$$

where

$$P = (\lambda'^2+k^2)\cosh\lambda' h,$$

$$Q = (\lambda'^2+k^2)\sinh\lambda' h,$$

$$R = 2k^2\sinh kh,$$

$$S = 2k^2\cosh kh,$$

$$p = \frac{kg}{a}\cosh\lambda' h+2\nu' k\lambda'\sinh\lambda' h,$$

$$q = \frac{kg}{a}\sinh\lambda' h+2\nu' k\lambda'\cosh\lambda' h,$$

$$r = a\cosh kh+\frac{gk}{a}\sinh kh+2\nu' k^2\cosh kh,$$

$$s = a\sinh kh+\frac{gk}{a}\cosh kh+2\nu' k^2\sinh kh,$$

and h is the depth of the upper fluid.

When the viscosity is small the terms of greatest importance are those of the type $\lambda'^3 \cosh \lambda' h$. Including only terms of this order, in which we put $\tanh \lambda' h = 1$, we have the period equation

$$a^4 [\rho \cosh kh + \rho' \sinh kh] + a^2 gk\rho [\cosh kh + \sinh kh] + g^2 k^2 (\rho - \rho') \sinh kh = 0,$$

as in the absence of viscosity.

Thus there are two modes, for one

$$a^2 + gk = 0,$$

and for the other

$$a^2 (\rho \cosh kh + \rho' \sinh kh) + gk(\rho - \rho') \sinh kh = 0.$$

In the first mode the tangential velocity is continuous across the interface. Hence in this mode we should expect that the first approximation to the dissipation terms in a would be of the order ν, and that the change in the velocity of propagation would be of the order ν^2; in the second mode we should expect both of these approximations to be of the order $\sqrt{\nu}$. This will be seen to be the case.

The terms of next importance are those of the type $\lambda'^2 \cosh \lambda' h$. If we put $a^2 + gk = 0$ in these, we find them vanish identically. Hence to obtain the next approximation to the first mode we have to take the terms of order $\lambda' \cosh \lambda' h$. We obtain

$$a = \pm i \sqrt{gk} - 2k^2 \frac{\rho\nu \cosh kh + 2(\rho'\nu' - \rho\nu) \sinh kh}{\rho \cosh kh + (2\rho' - \rho) \sinh kh}.$$

We notice that when $\rho = \rho'$, $\nu = \nu'$, then

$$a = \pm i \sqrt{gk} - 2k^2 \nu,$$

as is known.

Proceeding to the next approximation in the case of the second mode, we obtain

$$a = a_0 + \frac{k\sqrt{\nu\nu'}(\rho - \rho')}{(\rho\sqrt{\nu} + \rho'\sqrt{\nu'})\, a_0^{\frac{1}{2}}}$$
$$\times \frac{\{a^4(\rho - \rho') \sinh kh + gka^2\rho(\sinh kh + \cosh kh) + g^2k^2(\rho \cosh kh + \rho' \sinh kh)\}}{\{4a_0^3(\rho \cosh kh + \rho' \sinh kh) + 2gk\rho a_0 (\cosh kh + \sinh kh)\}},$$

where $$a_0^2 = -\frac{gk(\rho - \rho') \sinh kh}{\rho \cosh kh + \rho' \sinh kh}.$$

Hence $$a = \pm i\sqrt{\frac{gk(\rho-\rho')\sinh kh}{\rho\cosh kh+\rho'\sinh kh}}$$

$$\mp i\frac{k\sqrt{\nu\nu'}\,\{gk(\rho-\rho')\}^{\frac14}}{\sqrt{2}\,(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})}\left[\frac{\rho\cosh kh+\rho'\sinh kh}{\sinh kh}\right]^{\frac34}\times P$$

$$-\frac{k\sqrt{\nu\nu'}\,\{gk(\rho-\rho')\}^{\frac14}}{\sqrt{2}\,(\rho\sqrt{\nu}+\rho'\sqrt{\nu'})}\left[\frac{\rho\cosh kh+\rho'\sinh kh}{\sinh kh}\right]^{\frac34}\times P,$$

where

$$P = \frac{[\rho^3(\cosh kh-\sinh kh)+4\rho\rho'\sinh kh\,\{\rho+\rho'(\sinh^2 kh+\cosh^2 kh)\}]}{[-4(\rho-\rho')\sinh kh+2\rho(\cosh kh+\sinh kh)]\,(\rho\cosh kh+\rho'\sinh kh)^2}.$$

When h is infinite, we obtain our former results.

5. In the first mode when kh is small, *i.e.*, when the wave-length is large compared with the depth of the upper fluid, we have

$$a = \pm i\sqrt{gk}-2k^2\nu.$$

The modulus of decay is thus the same as that of the lower fluid alone.

When kh is large, $$a = \pm i\sqrt{gk}-2k^2\nu'.$$

The modulus of decay is thus the same as that of the upper fluid alone.

In the second mode, when kh is small, the modulus of decay depends on $\sinh^{\frac34} kh/k^{\frac14}$, or on $h^{\frac34}/k^{\frac12}$; and therefore it increases much less rapidly with the wave-length than in general.

When kh is large we obtain our former results for two fluids of infinite depth.

6. In the second mode the upper surface will in general be disturbed less than the common interface, and the waves set up by any disturbance will be of this type to a predominating extent, particularly if the difference between ρ and ρ' be small. Such waves as these are those referred to by Ekman in the *Scientific Results of the Norwegian North Polar Expedition.* He remarks that a ship moving in the Norwegian Fiords experiences great resistance owing to considerable waves being set up at the common interface of the layer of fresh water and the sea-water. Such waves would be very quickly damped, and would therefore drain a great amount of energy from the ship.

We append some numerical results illustrating the case of fresh water over sea-water of infinite depth. We take $\rho'/\rho = \frac{35}{36}$, and $\nu = \nu'$. The fact that we take $\nu = \nu'$ makes no important difference.

Wave-length.	$\frac{1}{10}$	1	10	100	1000 cms.
τ	·013″	1·27″	2′ 7″	3 hrs. 31′ 2″	351 hrs. 43′ 36″
τ_∞	·083″	1·5″	26″	7′ 46″	2 hrs. 18′ 15″
τ_1	·083″	1·4″	16″	1′ 18″	6′ 28″
τ_{10}	·083″	1·5″	26″	4′ 48″	23′ 15″
τ_{100}	·083″	1·5″	26″	7′ 45″	1 hr. 25′ 12″

τ is the modulus of decay for the first mode,

τ_∞ that for the second mode when kh is large,

τ_1 ,, ,, ,, $h = 1$ cm.,

τ_{10} ,, ,, ,, $h = 10$ cms.,

τ_{100} ,, ,, ,, $h = 100$ cms.

The very rapid decay when kh is small is very striking, even for the large wave-lengths.