De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur.

(Auctore *C. Neumann*, Hallae.)

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§. 1.

Problema proponitur.

Sint puncti mobilis Coordinatae orthogonales x, y, z; sit

$$x^2+y^2+z^2 = 1$$

aequatio globi, in cujus superficie hoc punctum manere coactum est, sit denique $ax^2+by^2+cz^2$, designantibus a, b, c datas Constantes quaslibet positivas vel negativas, functio potentialis virium punctum solicitantium: quaeritur, qualis sit puncti motus.

Semper $a<b<c$ haberi supponimus.

Ut ante oculos habeatur talium virium exemplum, adnotamus, homogeneam massam Ellipsoidis punctum internum x, y, z ita attrahere, ut virium attrahentium functio potentialis formam habeat $ax^2+by^2+cz^2$, siquidem axes principales Ellipsoidis pro axibus Coordinatarum sumti sint. Adnotamus praeterea, si Ellipsoidis aequatio sit

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2} = 1$$

atque semiaxes α, β, γ in relatione $\alpha<\beta<\gamma$ habeantur, etiam haberi $a<b<c$ (quod formulis ad Valores Constantium a, b, c determinandos usitatis faciliter apparet).

§. 2.

De aequatione Hamiltoniana.

Hamilton und ***Jacobi*** theorema posuerunt, quod nostro problemati accommodatum, sic enuntiare convenit.

Propositus sit motus puncti, in data superficie manere coacti, atque a datis viribus sollicitati. Sint x, y, z Coordinatae puncti orthogonales; sit $0=\Pi(x,y,z)$ aequatio superficiei, sit denique $U=f(x,y,z)$ functio potentialis virium. — Loco Variabilium x, y, z introducantur novae Variabiles λ_1, λ_2, quae substitutae aequationi $0 = \Pi(x, y, z)$ sponte ei satisfaciant; formetur aequatio differentialis partialis:

$$(1.) \qquad 0 = \left(2U - \frac{\partial\varphi}{\partial t}\right) \cdot \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} + \begin{vmatrix} 0 & \frac{\partial\varphi}{\partial\lambda_1} & \frac{\partial\varphi}{\partial\lambda_2} \\ \frac{\partial\varphi}{\partial\lambda_1} & u_{11} & u_{12} \\ \frac{\partial\varphi}{\partial\lambda_2} & u_{21} & u_{22} \end{vmatrix}$$

designantibus u_{11}, u_{12}, u_{22} expressiones:

$$u_{11} = \left(\frac{\partial x}{\partial\lambda_1}\right)^2 + \left(\frac{\partial y}{\partial\lambda_1}\right)^2 + \left(\frac{\partial z}{\partial\lambda_1}\right)^2$$

$$u_{22} = \left(\frac{\partial x}{\partial\lambda_2}\right)^2 + \left(\frac{\partial y}{\partial\lambda_2}\right)^2 + \left(\frac{\partial z}{\partial\lambda_2}\right)^2$$

$$u_{12} = \frac{\partial x}{\partial\lambda_1} \cdot \frac{\partial x}{\partial\lambda_2} + \frac{\partial y}{\partial\lambda_1} \cdot \frac{\partial y}{\partial\lambda_2} + \frac{\partial z}{\partial\lambda_1} \cdot \frac{\partial z}{\partial\lambda_2};$$

inveniatur functio φ Variabilium λ_1, λ_2, t, isti aequationi differentiali partiali sufficiens, ac duas Constantes arbitrarias A, B praeter Constantem additionalem involvens: pendebit determinatio motus propositi ab integratione binarum aequationum differentialium primi ordinis inter tres Variabiles λ_1, λ_2, t:

$$(2.) \qquad \begin{cases} \frac{\partial\varphi}{\partial\lambda_1} = u_{11} \frac{d\lambda_1}{dt} + u_{12} \frac{d\lambda_2}{dt}, \\ \frac{\partial\varphi}{\partial\lambda_2} = u_{21} \frac{d\lambda_1}{dt} + u_{22} \frac{d\lambda_2}{dt}. \end{cases}$$

§. 3.

Propositum problema mechanicum ad integralia ultraelliptica primae classis solvenda revocatur.

Praemissam regulam nostro problemati adhibeamus. Conjungamus Variabiles x, y, z cum novis Variabilibus λ_1, λ_2 per aequationes:

$$\frac{x^2}{\lambda_1 - a} + \frac{y^2}{\lambda_1 - b} + \frac{z^2}{\lambda_1 - c} = 0,$$

$$\frac{x^2}{\lambda_2 - a} + \frac{y^2}{\lambda_2 - b} + \frac{z^2}{\lambda_2 - c} = 0,$$

adjiciamus aequationem conditionalem

$$x^2 + y^2 + z^2 = 1;$$

eruuntur Quantitatum x, y, z; u_{11}, u_{22}, u_{12}; U hae expressiones:

$$(3.)\quad \begin{cases} x = \xi\cdot\sqrt{\dfrac{(a-\lambda_1)(a-\lambda_2)}{(a-b)(a-c)}}, & u_{11} = \frac{1}{4}\dfrac{\lambda_2-\lambda_1}{\varDelta_1}, \\ y = \eta\cdot\sqrt{\dfrac{(b-\lambda_1)(b-\lambda_2)}{(b-c)(b-a)}}, & u_{22} = \frac{1}{4}\dfrac{\lambda_1-\lambda_2}{\varDelta_2}, \\ z = \zeta\cdot\sqrt{\dfrac{(c-\lambda_1)(c-\lambda_2)}{(c-a)(c-b)}}, & u_{12} = 0, \end{cases}$$

$$U = ax^2+by^2+cz^2 = (a+b+c)-(\lambda_1+\lambda_2),$$

ubi ξ, η, ζ Quantitatem abiguam ± 1 repraesentant ac brevitatis causa

$$\varDelta_1 = (\lambda_1-a)(\lambda_1-b)(\lambda_1-c),$$
$$\varDelta_2 = (\lambda_2-a)(\lambda_2-b)(\lambda_2-c)$$

positum est. Itaque aequatio differentialis partialis (1.) hanc induit formam

$$0 = \left(2(a+b+c)-2(\lambda_1+\lambda_2)-\frac{\partial\varphi}{\partial t}\right)\cdot\begin{vmatrix} u_{11} & 0 \\ 0 & u_{22}\end{vmatrix} + \begin{vmatrix} 0 & \dfrac{\partial\varphi}{\partial\lambda_1} & \dfrac{\partial\varphi}{\partial\lambda_2} \\ \dfrac{\partial\varphi}{\partial\lambda_1} & u_{11} & 0 \\ \dfrac{\partial\varphi}{\partial\lambda_2} & 0 & u_{22}\end{vmatrix}$$

vel Determinantia solvendo:

$$0 = (\lambda_1+\lambda_2)+\frac{1}{2}\frac{\partial\varphi}{\partial t}-(a+b+c)+\frac{1}{2u_{11}}\left(\frac{\partial\varphi}{\partial\lambda_1}\right)^2+\frac{1}{2u_{22}}\left(\frac{\partial\varphi}{\partial\lambda_2}\right)^2$$

vel ipsorum u_{11}, u_{22} valores introducendo:

$$0 = (\lambda_1^2-\lambda_2^2)+(\lambda_1-\lambda_2)\left(\frac{1}{2}\frac{\partial\varphi}{\partial t}-(a+b+c)\right)-2\varDelta_1\left(\frac{\partial\varphi}{\partial\lambda_1}\right)^2+2\varDelta_2\left(\frac{\partial\varphi}{\partial\lambda_2}\right)^2 \equiv M.$$

Cujus aequationis dextera pars, brevitas causa $\equiv M$ posita, discerpi potest in hos tres terminos, binis Constantibus Arbitrariis A, B praeditos:

$$\left(\frac{\partial\varphi}{\partial t}-(a+b+c)+A+B\right)(\lambda_1-\lambda_2) \equiv M_0,$$

$$2\varDelta_1\left(\frac{\partial\varphi}{\partial\lambda_1}\right)^2-(\lambda_1-A)(\lambda_1-B) \equiv M_1,$$

$$2\varDelta_2\left(\frac{\partial\varphi}{\partial\lambda_2}\right)^2-(\lambda_2-A)(\lambda_2-B) \equiv M_2,$$

ita ut $M \equiv +M_0-M_1+M_2$ habeatur. Apparet expressionum φ_0, φ_1, φ_2, ita constituarum, ut respective aequationibus $M_0=0$, $M_1=0$, $M_2=0$ satisfaciant, primam φ_0 unius Variabilis t, secundam φ_1 unius Variabilis λ_1, tertiam φ_2 unius Variabilis λ_2 functiones esse. Unde aggregatum $\varphi_0+\varphi_1+\varphi_2$ erit aequatio nostrae $M=0$ solutio completa, duas scilicet Constantes arbitrarias

A, B involvens; prodeuntque aequationes:

$$\varphi \equiv \varphi_0 + \varphi_1 + \varphi_2, \quad \frac{\partial\varphi}{\partial\lambda_1} \equiv \frac{\partial\varphi_1}{\partial\lambda_1}$$
$$\frac{\partial\varphi}{\partial\lambda_2} \equiv \frac{\partial\varphi_2}{\partial\lambda_2},$$

vel, valoribus quotientium differentialium $\frac{\partial\varphi_1}{\partial\lambda_1}$, $\frac{\partial\varphi_2}{\partial\lambda_2}$ ope aequationum $M_1 = 0$, $M_2 = 0$ determinatis,

$$\frac{\partial\varphi}{\partial\lambda_1} = -\varepsilon_1 \sqrt{\frac{(\lambda_1 - A)(\lambda_1 - B)}{2\varLambda_1}},$$
$$\frac{\partial\varphi}{\partial\lambda_2} = -\varepsilon_2 \sqrt{\frac{(\lambda_2 - A)(\lambda_2 - B)}{2\varLambda_2}},$$

ubi ε_1, ε_2 quantitatem ambiguam ± 1 significant. Jam aequationes differentiales (2.) evadunt sequentes:

$$-\varepsilon_1 \sqrt{\frac{(\lambda_1 - A)(\lambda_1 - B)}{2\varLambda_1}} = \frac{1}{4} \frac{\lambda_2 - \lambda_1}{\varLambda_1} \frac{d\lambda_1}{dt},$$
$$-\varepsilon_2 \sqrt{\frac{(\lambda_2 - A)(\lambda_2 - B)}{2\varLambda_2}} = \frac{1}{4} \frac{\lambda_1 - \lambda_2}{\varLambda_2} \frac{d\lambda_2}{dt}.$$

Quae aequationes, brevitatis causa posito:

$$\sqrt{8\varLambda_1(\lambda_1 - A)(\lambda_1 - B)} = \sqrt{8(\lambda_1 - a)(\lambda_1 - b)(\lambda_1 - c)(\lambda_1 - A)(\lambda_1 - B)} \equiv L_1,$$
$$\sqrt{8\varLambda_2(\lambda_2 - A)(\lambda_2 - B)} = \sqrt{8(\lambda_2 - a)(\lambda_2 - b)(\lambda_2 - c)(\lambda_2 - A)(\lambda_2 - B)} \equiv L_2,$$

induunt hanc simplicem formam:

$$(4.) \quad \begin{cases} \varepsilon_1 L_1 = (\lambda_1 - \lambda_2)\dfrac{d\lambda_1}{dt}, \\[2mm] \varepsilon_2 L_2 = (\lambda_2 - \lambda_1)\dfrac{d\lambda_2}{dt}, \end{cases}$$

vel etiam hanc:

$$(5.) \quad \begin{cases} \dfrac{d\lambda_1}{\varepsilon_1 L_1} + \dfrac{d\lambda_2}{\varepsilon_2 L_2} = 0, \\[2mm] \dfrac{\lambda_1 d\lambda_1}{\varepsilon_1 L_1} + \dfrac{\lambda_2 d\lambda_2}{\varepsilon_2 L_2} = dt. \end{cases}$$

Sint l_1, l_2 Variabilium λ_1, λ_2 valores conjugati temporis valori $t = 0$, ex istis aequationibus differentialibus intra limites $t = 0$ et $t = t$ summatis prodeunt aequationes integrales:

$$(6.) \quad \begin{cases} \varepsilon_1 \displaystyle\int_{l_1}^{\lambda_1} \frac{d\lambda}{L} + \varepsilon_2 \int_{l_2}^{\lambda_2} \frac{d\lambda}{L} = 0 \\[2mm] \varepsilon_1 \displaystyle\int_{l_1}^{\lambda_1} \frac{\lambda d\lambda}{L} + \varepsilon_2 \int_{l_2}^{\lambda_2} \frac{\lambda d\lambda}{L} = t. \end{cases}$$

En formulae, quibus Coordinatae ellipticae λ_1, λ_2 a tempore t pendent, affectae quatuor Constantibus arbitrariis l_1, l_2, A, B, quarum valores ex dato statu initiali repetendi sunt.

§. 4.

Quatuor casus status initialis discernuntur.

Variabiles λ_1, λ_2, quippe quae radices sint aequationis quadraticae:

$$\frac{x^2}{\lambda-a}+\frac{y^2}{\lambda-b}+\frac{z^2}{\lambda-c}=0,$$

semper quantitates reales esse, ac, posito $a<b<c$ et $\lambda_1<\lambda_2$, etiam fieri:

$$(7.)\qquad a<\lambda_1<b<\lambda_2<c,$$

satis notum est. Habetur igitur:

$$\Delta_1\equiv(\lambda_1-a)(\lambda_1-b)(\lambda_1-c)=\text{pos.}$$
$$\Delta_2\equiv(\lambda_2-a)(\lambda_2-b)(\lambda_2-c)=\text{neg.},$$

quae formulae, quoniam aequationes (4.) vetant expressiones L_1, L_2 valores induere imaginarios, suppeditant

$$\frac{L_1^2}{8\Delta_1}\equiv(\lambda_1-A)(\lambda_1-B)=\text{pos.}$$
$$\frac{L_2^2}{8\Delta_2}\equiv(\lambda_2-A)(\lambda_2-B)=\text{neg.}$$

Unde sequitur, Constantes A, B semper reales esse, ac, posito $A<B$, etiam fieri

$$(8.)\qquad \lambda_1<A<\lambda_2<B.$$

Formulae (7.) et (8.) comparatae demonstrant, inter Quantitates λ_1, λ_2, a, b, c, A, B hosce quatuor ordines

$$\text{I.}\quad a<\lambda_1<A<b<\lambda_2<B<c$$
$$\text{II.}\quad a<\lambda_1<A<b<\lambda_2<c<B$$
$$\text{III.}\quad a<\lambda_1<b<A<\lambda_2<B<c$$
$$\text{IV.}\quad a<\lambda_1<b<A<\lambda_2<c<B$$

possibiles, impossibilem alium ordinem esse. Qui ordines quatuor, quorum quisque Variabilem λ_1 in secundo loco et Variabilem λ_2 in quinto loco habet, prodeunt e quatuor Constantium a, b, c, A, B ordinibus

$$\text{I.}\quad a<A<b<B<c$$
$$\text{II.}\quad a<A<b<c<B$$
$$\text{III.}\quad a<b<A<B<c$$
$$\text{IV.}\quad a<b<A<c<B.$$

Constantes a, b, c viribus Mobile sollicitantibus pendent, Constantes A, B ab initiali statu Mobilis. Dum igitur vires Mobile sollicitantes eaedem manent, aliis initialibus statibus alii Constantium A, B valores, aliique Constantium a, b, c, A, B ordines prodibunt. Jam quatuor casus status initialis discerni licet. Ponimus in primo casu status initiales ita constitutos, ut primus istarum Constantium ordo evadat; ponimus in secundo casu status initiales ita constitutos, ut secundus istarum Constantium ordo evadat; etc.

§. 5.

Ex status initialis quatuor casibus provenire quatuor genera motus.

Status initialis casibus aliis, aliae naturae esse orbitam Mobilis ostendamus. Ac primum quidem conos, contentos aequatione

$$\frac{x^2}{\lambda-a}+\frac{y^2}{\lambda-b}+\frac{z^2}{\lambda-c}=0 \tag{9.}$$

vel curvas, quibus globus ab istis conis secatur, consideremus.

Construantur ambae Rectae, in x, z plano sitae, contentae aequatione

$$\frac{x^2}{b-a}=\frac{z^2}{c-b},$$

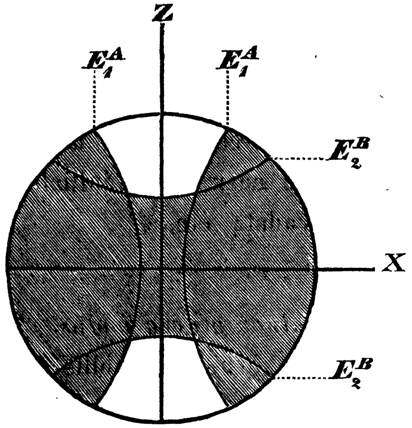
punctaque quatuor, quibus istae Rectae globum secant, pro focis Ellipsium sphaericarum sumantur. Duo inde genera Ellipsium sphaericarum oriuntur, quorum alterum x-axi circumductum, alterum z-axi circumductum est, quoniam $a<b<c$ haberi statuimus. Prius genus signo E_1, posterius genus signo E_2 designamus. Jam ex aequatione nostra (9.), posito $a<\lambda<b$, omnes generis E_1 Ellipses prodire, et, posito $b<\lambda<c$, omnes generis E_2 Ellipses prodire notum est. Signo E_1^k denotamus eas binas generis E_1 Ellipses, quarum $\lambda=k$ est, ac signo E_2^i denotamus eas binas generis E_2 Ellipses, quarum $\lambda=i$ est.

Invenimus

posito primo ordine: $a<A<b<B<c$

Mobilis Coordinatarum ellipticarum alteram λ_1 in intervallo $a\dots A$, alteram λ_2 in intervallo $b\dots B$ manere coactas esse. Unde sequitur, Mobile neque in puncta, quorum λ_1 extra intervallum $a\dots A$ situm sit, neque in puncta, quorum λ_2 extra intervallum $b\dots B$ situm sit, ullo tempore deferri posse. Puncta, quorum λ_1 extra intervallum $a\dots A$ vel (quod idem est) intra intervallum $A\dots b$ situm est, omnia areis binarum Ellipsium E_1^A continentur. Porro puncta, quorum λ_2 extra intervallum $b\dots B$ vel (quod idem est) intra intervallum $B\dots c$ situm est, omnia zona, inclusa a binis Ellipsibus E_2^B, continentur. Jam nulla superficiei globosae portio motui Mobilis relinquitur nisi

Fig. I.



bina Quadrilinea, sita circum bina puncta, quibus globus a z-axi secatur, inclusa a binis Ellipsibus E_1^A et a binis Ellipsibus E_2^B (Figura I. repraesentat dimidium superficiei globosae, in qua bina Ellipses E_1^A et binae Ellipses E_2^B ductae sunt. Ab istis Ellipsibus inclusa Quadrilinea, in quibus punctum moveri debet, alba sunt, cetera vero superficiei globosae portio lineis sparsa est).

Eadem ratione concludimus:
posito secundo ordine $a<A<b<c<B$, zonam, inclusam a binis Ellipsibus E_1^A,
posito tertio ordine $a<b<A<B<c$, binas zonas, inclusas a binis Ellipsibus E_2^A et a binis Ellipsibus E_2^B,
posito quarto ordine $a<b<A<c<B$, zonam, inclusam a binis Ellipsibus E_2^A,
esse eas portiones superficei globosae, in quibus Mobile moveri coactum sit. (In secundo, tertio, quarto casu Mobile semper manebit in iis superficiei globosae portionibus, quae respective in figuris II. III. IV. albae sunt.)

Fig. II.

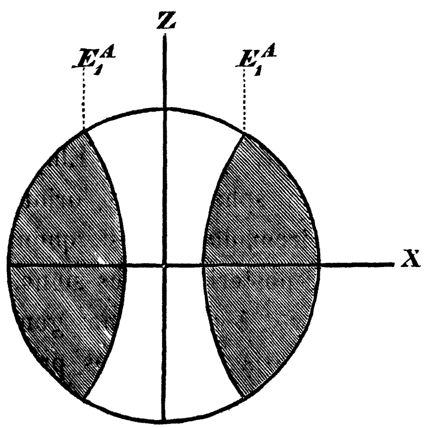


Fig. III.

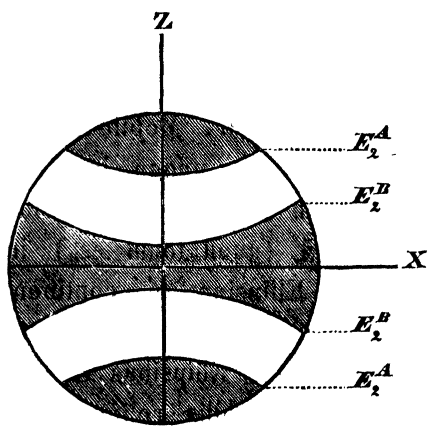
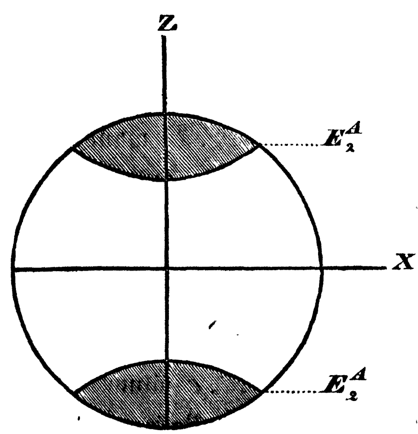


Fig. IV.



§. 6.

De figura orbitae.

Operis pretium videtur, varia, quae ad figuram orbitae cognoscendam adjuvant, colligere. — Quatuor ordinum, qui inter Constantes a, b, c, A, B intercedere possunt, quolibet designato per

$$m_1 < n_1 < m_2 < n_2 < r$$

invenimus supra, Variabilem λ_1 semper intervallo $m_1 \ldots n_1$ et Variabilem λ_2 semper intervallo $m_2 \ldots n_2$ contineri. Quod, designante k quemlibet numerorum 1, 2, brevius sic enuntiatur: Variabilis λ_k semper intervallo $m_k \ldots n_k$ continetur.

Quoniam Quantitates $(\lambda_1 - \lambda_2)$, $\lambda'_1 = \frac{d\lambda_1}{dt}$, $\lambda'_2 = \frac{d\lambda_2}{dt}$ discontinuos valores induere nequeunt, ex aequationibus supra (4.) inventis

$$\varepsilon_1 L_1 = (\lambda_1 - \lambda_2)\lambda'_1,$$
$$\varepsilon_2 L_2 = (\lambda_2 - \lambda_1)\lambda'_2$$

apparet, non nisi evanescente L_k, fieri posse, ut quantitas ambigua $\varepsilon_k = \pm 1$, omisso valore altero, alterum induat. Unde sequitur, non nisi $\lambda_k = m_k$ aut $\lambda_k = n_k$ habeatur, fieri posse, ut signum ambiguum ε_k e altero valore in alterum transeat.

Quoniam Quantitates $(\lambda_2 - \lambda_1)$, L_1, L_2, dt semper positivas esse constat, iisdem aequationibus hoc alterum apparet, signa $-\varepsilon_1$ et $+\varepsilon_2$ respective Variabilibus λ_1, λ_2 crescentibus positiva, decrescentibus negativa esse, binorumque signorum quodlibet ε_k semper vicem valoris obire, simulac λ_k vel a decrescendo ad crescendum vel a crescendo ad decrescendum transeat. Unde sequitur, signum ambiguum ε_k vicem valoris semper obire, simulac $\lambda_k = m_k$ vel $\lambda_k = n_k$ habeatur.

Si priori posterius jungitur hae emergunt propositiones:

(10.) *Simulac Variabilis λ_k, in intervallo $m_k \ldots n_k$ versans, in alterutrum istius intervalli finem adducitur, signum ambiguum ε_k vicem valoris obit; immutatum manet pro ceteris Variabilis λ_k valoribus.*

(11.) *Variabilis λ_k in intervallo $m_k \ldots n_k$ versatur ita, ut ab m_k ad n_k perpetuo crescens ascendat, tum ab n_k ad m_k perpetuo decrescens descendat, periodumque istam in aeternum continuet.*

Jam ad figuram orbitae cognoscendam aliae contemplationes in auxilium vocandae sunt. — Per locum, quem Mobile quolibet tempore t habet, ducantur binae Ellipses, altera generis E_1, altera generis E_2; per idemque punctum ducantur binae Rectae r_1, r_2, istas Ellipses tangentes, ita ut Recta r_1 normalis Ellipsi E_1, et Recta r_2 normalis Ellipsi E_2 fiat. Sint R_1, R_2 vires, Mobile sollicitantes, respective secundum directiones r_1, r_2 sumtae; porro sint V_1, V_2 Mobilis velocitates, respective secundum easdem directiones r_1, r_2 sumtae: fit

$$\begin{cases} R_1 = \frac{\partial U}{\partial x}\cos(xr_1)+\frac{\partial U}{\partial y}\cos(yr_1)+\frac{\partial U}{\partial z}\cos(zr_1), \\ R_2 = \frac{\partial U}{\partial x}\cos(xr_2)+\frac{\partial U}{\partial y}\cos(yr_2)+\frac{\partial U}{\partial z}\cos(zr_2), \end{cases}$$

$$\begin{cases} V_1 = x'\cos(xr_1)+y'\cos(yr_1)+z'\cos(zr_1), \\ V_2 = x'\cos(xr_2)+y'\cos(yr_2)+z'\cos(zr_2), \end{cases}$$

siquidem ponitur $\frac{dx}{dt}=x'$, $\frac{dy}{dt}=y'$, $\frac{dz}{dt}=z'$, atque (xr_1) angulum designat, quem Recta r_1 cum directione x-axis constituit, etc.

Binas formulas posteriores etiam sic exhibere licet:

$$\begin{cases} V_1 = \left(\frac{\partial x}{\partial\lambda_1}\cos(xr_1)+\frac{\partial y}{\partial\lambda_1}\cos(yr_1)+\frac{\partial z}{\partial\lambda_1}\cos(zr_1)\right)\lambda_1' \\ \qquad +\left(\frac{\partial x}{\partial\lambda_2}\cos(xr_1)+\frac{\partial y}{\partial\lambda_2}\cos(yr_1)+\frac{\partial z}{\partial\lambda_2}\cos(zr_1)\right)\lambda_2' \\ V_2 = \left(\frac{\partial x}{\partial\lambda_1}\cos(xr_2)+\frac{\partial y}{\partial\lambda_1}\cos(yr_2)+\frac{\partial z}{\partial\lambda_1}\cos(zr_2)\right)\lambda_1' \\ \qquad +\left(\frac{\partial x}{\partial\lambda_2}\cos(xr_2)+\frac{\partial y}{\partial\lambda_2}\cos(yr_2)+\frac{\partial z}{\partial\lambda_2}\cos(zr_2)\right)\lambda_2'. \end{cases}$$

Jam adhibendo formulas notas:

$$\cos(xr_1):\cos(yr_1):\cos(zr_1) = \frac{\partial x}{\partial\lambda_1}:\frac{\partial y}{\partial\lambda_1}:\frac{\partial z}{\partial\lambda_1},$$

$$\cos(xr_2):\cos(yr_2):\cos(zr_2) = \frac{\partial x}{\partial\lambda_2}:\frac{\partial y}{\partial\lambda_2}:\frac{\partial z}{\partial\lambda_2},$$

adhibendoque formulas supra (3.) erutas:

$$\left(\frac{\partial x}{\partial\lambda_1}\right)^2+\left(\frac{\partial y}{\partial\lambda_1}\right)^2+\left(\frac{\partial z}{\partial\lambda_1}\right)^2 = \frac{\lambda_2-\lambda_1}{4\varDelta_1},$$

$$\left(\frac{\partial x}{\partial\lambda_2}\right)^2+\left(\frac{\partial y}{\partial\lambda_2}\right)^2+\left(\frac{\partial z}{\partial\lambda_2}\right)^2 = \frac{\lambda_1-\lambda_2}{4\varDelta_2},$$

$$\frac{\partial x}{\partial\lambda_1}\cdot\frac{\partial x}{\partial\lambda_2}+\frac{\partial y}{\partial\lambda_1}\cdot\frac{\partial y}{\partial\lambda_2}+\frac{\partial z}{\partial\lambda_1}\cdot\frac{\partial z}{\partial\lambda_2} = 0;$$

obtinentur haecce expressiones:

$$(12.)\quad \begin{cases} R_1^2 = 4\frac{(\lambda_1-a)(\lambda_1-b)(\lambda_1-c)}{\lambda_2-\lambda_1}, \\ R_2^2 = 4\frac{(\lambda_2-a)(\lambda_2-b)(\lambda_2-c)}{\lambda_1-\lambda_2}, \end{cases} \qquad \begin{cases} V_1^2 = 2\frac{(\lambda_1-A)(\lambda_1-B)}{\lambda_2-\lambda_1}, \\ V_2^2 = 2\frac{(\lambda_2-A)(\lambda_2-B)}{\lambda_1-\lambda_2}. \end{cases}$$

Si adnotamus, punctorum globi eorum, quae in yz-plano sita sint, Coordinatam $\lambda_1 = a$ esse: eorum, quae in zx-plano sita sint, aut Coordinatam $\lambda_1 = b$ aut Coordinatam $\lambda_2 = b$ esse; eorum denique, quae in xy-plano sita sint, Coordinatam $\lambda_2 = c$ esse: ac si brevitatis causa tres circulos eos, in quibus globus a planis Coordinatarum secatur, nominamus principales circulos: suppeditatur expressionibus Velocitatum V_1, V_2 haec propositio:

(13.) ***Si in quodlibet punctum cujuslibet circuli principalis Mobile adducitur, fieri nequit, ut ea Velocitatis Componens, quae plano circuli principalis normalis est, evanescat. Mobile igitur alicubi in circulum principalem adductum, transducitur per circulum.***

Expressionibus nostris Velocitatum V_1, V_2 aperitur methodus, qua figura orbitae investigari potest. Involvunt enim illae expressiones hancce legem:

Per unumquodque punctum orbitae Mobilis ducatur Ellipsis generis E_2, colligantur omnia puncta, in quibus directio orbitae cum directione Ellipsis transductae eundem angulum ω, ex arbitrio sumptum, constituit: sita sunt ista puncta omnia in curva, quae auxilio Coordinatarum ellipticarum sic exhibetur:

$$(\lambda_2 - A)(\lambda_2 - B) + \operatorname{tang}^2 \omega . (\lambda_1 - A)(\lambda_1 - B) = 0.$$

§. 7.

De signis ambiguis ε_1, ε_2, ξ, η, ζ.

Legem, secundum quam ambi valores signi ε_1 et ambi valores signi ε_2 alternant, jam supra (10.) statuimus. Reliquum est, ut, quibus valoribus in statu initiali gaudeant, investigemus.

Aequatio quadratica

$$\frac{x^2}{\lambda - a} + \frac{y^2}{\lambda - b} + \frac{z^2}{\lambda - c} = 0$$

etiam sic exhiberi licet

$$\lambda^2 - \lambda \Sigma (b+c) x^2 + \Sigma bcx^2 = 0,$$

siquidem signum Σ denotat, formandos esse binos terminos, similes termino, qui sub signo Σ positus est, omnesque tres terminos summandos esse. Quoniam λ_1, λ_2 istius aequationis radices sunt ac $\lambda_1 < \lambda_2$ est, exprimuntur facile et sine ambiguitate λ_1, λ_2 per x, y, z, exprimunturque λ_1', λ_2' per $x, y, z; x', y', z'$. Quibus formulis erutis elucet, signa binorum expressionum:

$$(\Sigma(b+c)x^2 . \Sigma(b+c)xx' - 2\Sigma bcxx') - \Sigma(b+c)xx' \sqrt{(\Sigma(b+c)x^2)^2 - 4\Sigma bcx^2}$$

$$(\Sigma(b+c)x^2 . \Sigma(b+c)xx' - 2\Sigma bcxx') + \Sigma(b+c)xx' \sqrt{(\Sigma(b+c)x^2)^2 - 4\Sigma bcx^2}$$

aequales esse respective signis Quantitatum $-\lambda_1'$ et $+\lambda_2'$, vel aequales esse respective signis ε_1 et ε_2. Aequationibus (4.) scilicet apparet, signa ε_1 et ε_2 aequare respective signa Quantitatum $-\lambda_1'$ et $+\lambda_2'$.

Jam initialibus valoribus Coordinatarum x, y, z Velocitatumque x', y', z' datis, quomodo signorum ε_1, ε_2 initiales valores determinentur perspicuum est.

Coordinatis ellipticis λ_1, λ_2 per tempus t expressis, valores Coordinatarum orthogonalium x, y, z suppeditantur formulis:

$$\begin{cases} x = \xi \cdot \sqrt{\dfrac{(a-\lambda_1)(a-\lambda_2)}{(a-b)(a-c)}}, \\ y = \eta \cdot \sqrt{\dfrac{(b-\lambda_1)(b-\lambda_2)}{(b-c)(b-a)}}, \\ z = \zeta \cdot \sqrt{\dfrac{(c-\lambda_1)(c-\lambda_2)}{(c-a)(c-b)}}. \end{cases}$$

Dubitatio igitur, qui valores quoque tempore signis ξ, η, ζ tribuendi sint, tollenda est. Ac primum quidem patet, signum ξ non, nisi evanescente ipsius x expressione, vicem valoris obire posse. Sed, nisi $\lambda_1 = a$ habeatur, fieri nequit, ut ipsius x expressio evanescat, quoniam λ_1 extra intervallum $a \dots b$ et λ_2 extra intervallum $b \dots c$ egredi nequeunt. Similiter signa η, ζ examinare licet. Prodeunt haecce regulae:

Nequeunt alternare ambi valores signi ξ, nisi fiat $\lambda_1 = a$; ambi valores signi η, nisi fiat aut $\lambda_1 = b$ aut $\lambda_2 = b$; denique ambi valores signi ζ, nisi fiat $\lambda_2 = c$.

At Quantitates ambiguas ξ, η, ζ istis conditionibus ab altero valore in alterum transire non tantum posse, sed etiam cogi, perspicuum est ex iis, quae supra (13.) de Orbita Mobilis invenimus.

De initialibus valoribus signorum ξ, η, ζ nulla dubitatio est, quippe qui initialibus Coordinatarum x, y, z valoribus datis, dati ipsi sint.

§. 8.

Quomodo e dato statu initiali Valores Constantium arbitrarium l_1, l_2, A, B determinare liceat.

Designantibus x_0, y_0, z_0 initiales valores Coordinatarum x, y, z, patet, l_1, l_2 esse radices aequationis quadraticae:

$$\frac{x_0^2}{\lambda-a} + \frac{y_0^2}{\lambda-b} + \frac{z_0^2}{\lambda-c} = 0,$$

atque esse l_1 minorem, l_2 majorem radicem. Aequationibus (4.), quae in hunc quoque modum exhibere licet:

$$\begin{cases}(\lambda_1 - A)(\lambda_1 - B) = \dfrac{(\lambda_1-\lambda_2)^2\lambda'_1\lambda'_1}{8\varDelta_1},\\[2mm] (\lambda_2 - A)(\lambda_2 - B) = \dfrac{(\lambda_2-\lambda_1)^2\lambda'_2\lambda'_2}{8\varDelta_2},\end{cases}$$

determinantur Constantium A, B valores, siquidem pro Quantitatibus λ_1, λ_2, λ'_1, λ'_2 initiales valores sumti sunt. Unde, in auxilium vocata aequatione identica:

$$\frac{(\varrho-A)(\varrho-B)}{(\varrho-\lambda_1)(\varrho-\lambda_2)} \equiv \frac{(\lambda_1-A)(\lambda_1-B)}{\lambda_1-\lambda_2}\frac{1}{\varrho-\lambda_1} + \frac{(\lambda_2-A)(\lambda_2-B)}{\lambda_2-\lambda_1}\frac{1}{\varrho-\lambda_2} + 1$$

prodit aequatio

$$0 = \frac{(\lambda_1-\lambda_2)\lambda'_1\lambda'_1}{8\varDelta_1}\frac{1}{\varrho-\lambda_1} + \frac{(\lambda_2-\lambda_1)\lambda'_2\lambda'_2}{8\varDelta_2}\frac{1}{\varrho-\lambda_2} + 1$$

respectu Quantitatis ϱ quadratica, conformataque ita, ut quaesitae Constantes A, B fiant radices. Quae aequatio etiam sic exhiberi potest:

$$0 = (\varrho-\lambda_1)(\varrho-\lambda_2) + \frac{(\lambda_1-\lambda_2)\lambda'_1\lambda'_1}{8\varDelta_1}(\varrho-\lambda_2) + \frac{(\lambda_2-\lambda_1)\lambda'_2\lambda'_2}{8\varDelta_2}(\varrho-\lambda_1)$$

vel etiam sic:

$$(14.)\qquad 0 = \varrho^2 - \varrho\left\{(\lambda_1+\lambda_2) + (\lambda_2-\lambda_1)\left(\frac{\lambda'_1\lambda'_1}{8\varDelta_1} - \frac{\lambda'_2\lambda'_2}{8\varDelta_2}\right)\right\}$$
$$+\left\{\lambda_1\lambda_2 + (\lambda_2-\lambda_1)\left(\frac{\lambda_2\lambda'_1\lambda'_1}{8\varDelta_1} - \frac{\lambda_1\lambda'_2\lambda'_2}{8\varDelta_2}\right)\right\}.$$

Ut hujus aequationis coefficientes, nunc per Coordinatas ellipticas expressae, exprimantur per Coordinatas orthogonales, adducimus formulas, jam plus semel adhibitas:

$$\Sigma\left(\frac{\partial x}{\partial\lambda_1}\right)^2 = \frac{\lambda_2-\lambda_1}{4\varDelta_1},$$
$$\Sigma\left(\frac{\partial x}{\partial\lambda_2}\right)^2 = \frac{\lambda_1-\lambda_2}{4\varDelta_2},$$
$$\Sigma\frac{\partial x}{\partial\lambda_1}\cdot\frac{\partial x}{\partial\lambda_2} = 0,$$

adducimusque has novas formulas:

$$\Sigma a\left(\frac{\partial x}{\partial\lambda_1}\right)^2 = \frac{\lambda_1(\lambda_2-\lambda_1)}{4\varDelta_1},$$
$$\Sigma a\left(\frac{\partial x}{\partial\lambda_2}\right)^2 = \frac{\lambda_2(\lambda_1-\lambda_2)}{4\varDelta_2},$$
$$\Sigma a\frac{\partial x}{\partial\lambda_1}\cdot\frac{\partial x}{\partial\lambda_2} = 0.$$

Jam ex aequationibus:

$$x' = \frac{\partial x}{\partial \lambda_1}\lambda_1' + \frac{\partial x}{\partial \lambda_2}\lambda_2',$$
$$y' = \frac{\partial y}{\partial \lambda_1}\lambda_1' + \frac{\partial y}{\partial \lambda_2}\lambda_2',$$
$$z' = \frac{\partial z}{\partial \lambda_1}\lambda_1' + \frac{\partial z}{\partial \lambda_2}\lambda_2',$$

adhibendo formulas priores, prodit:

$$\Sigma x' x' = (\lambda_2 - \lambda_1)\left(\frac{\lambda_1' \lambda_1'}{4\Delta_1} - \frac{\lambda_2' \lambda_2'}{4\Delta_2}\right)$$

et adhibendo formulas posteriores, prodit:

$$\Sigma a x' x' = (\lambda_2 - \lambda_1)\left(\frac{\lambda_2 \lambda_1' \lambda_1'}{4\Delta_1} - \frac{\lambda_1 \lambda_2' \lambda_2'}{4\Delta_4}\right).$$

Quibus binis relationibus facile prodit haec nova relatio:

$$(\lambda_1 + \lambda_2)\Sigma a x' x' = (\lambda_2 - \lambda_1)\left(\frac{\lambda_2 \lambda_1' \lambda_1'}{4\Delta_1} - \frac{\lambda_1 \lambda_2' \lambda_2'}{4\Delta_2}\right).$$

Postremo adnotamus binas relationes

$$\Sigma(b+c)x^2 = \lambda_1 + \lambda_2,$$
$$\Sigma b c x^2 = \lambda_1 \lambda_2,$$

prodeuntes ex aequatione quadratica, cujus radices λ_1, λ_2 sunt. Quas tres postremas relationes in auxilium vocando nostra aequatio quadratica (14.) hanc induit formam:

$$(15.)\quad 0 = \varrho^2 - \varrho\{\Sigma(b+c)x^2 + \tfrac{1}{2}\Sigma x' x'\} + \{\Sigma b c x^2 + \tfrac{1}{2}\Sigma(b+c)x^2 . \Sigma x' x' - \tfrac{1}{2}\Sigma a x' x'\}.$$

En unde Coordinatarum x, y, z Velocitatumque x', y', z' initialibus valoribus datis, directe et sine ambiguitate derivantur Valores Constantium A, B, quippe quae istius aequationis radices sint, et relatione $A < B$ teneantur.

§. 9.

De quatuor casibus discernendis.

Designante $\varphi(\varrho)$ dexteram partem aequationis quadraticae (15.), formulisque

$$\varphi(-\infty) = \text{pos}, \quad \varphi(A) = 0, \quad \varphi(B) = 0, \quad \varphi(+\infty) = \text{pos}$$

adnotatis, patet

in primo casu $(a < A < b < B < c)$ fore: $\varphi(a) = \text{pos}$, $\varphi(b) = \text{neg}$, $\varphi(c) = \text{pos}$
in secundo casu $(a < A < b < c < B)$ fore: $\varphi(a) = \text{pos}$, $\varphi(b) = \text{neg}$, $\varphi(c) = \text{neg}$
in tertio casu $(a < b < A < B < c)$ fore: $\varphi(a) = \text{pos}$, $\varphi(b) = \text{pos}$, $\varphi(c) = \text{pos}$
in quarto casu $(a < b < A < c < B)$ fore: $\varphi(a) = \text{pos}$, $\varphi(b) = \text{pos}$, $\varphi(c) = \text{neg}$.

Itaque signa expressionum $\varphi(b)$ et $\varphi(c)$ discernendis quatuor casibus sufficiunt. Expressiones $\varphi(b)$, $\varphi(c)$ sic exhiberi possunt:

$$\varphi(b) \equiv (b-c)(b-a)y^2 + \tfrac{1}{2}(c+a-\Sigma ax^2)\Sigma x'x' - \tfrac{1}{2}\Sigma ax'x',$$
$$\varphi(c) \equiv (c-a)(c-b)z^2 + \tfrac{1}{2}(a+b-\Sigma ax^2)\Sigma x'x' - \tfrac{1}{2}\Sigma ax'x'.$$

Exempli causa inde perspicitur, semper primum casum locum habere, si initialis Velocitas $=0$ habeatur.

§. 10.

De periodo motus.

Designantibus

$$m_1 < n_1 < m_2 < n_2 < r$$

ordinem Constantium $a, b, c, A, B,$ secundum eorum magnitudinem exhibitum, fit:

$$L = \sqrt{8(\lambda-m_1)(\lambda-n_1)(\lambda-m_2)(\lambda-n_2)(\lambda-r)}.$$

Sint M, N numeri integri, sufficientes relationi:

$$0 = 2M\int_{m_1}^{n_1}\frac{d\lambda}{L} + 2N\int_{m_2}^{n_2}\frac{d\lambda}{L},$$

ponatur deinde

$$T = 2M\int_{m_1}^{n_1}\frac{\lambda d\lambda}{L} + 2N\int_{m_2}^{n_2}\frac{\lambda d\lambda}{L},$$

patet ex aequationibus nostris integralibus:

$$\begin{cases} 0 = \varepsilon_1\int_{l_1}^{\lambda_1}\frac{d\lambda}{L} + \varepsilon_2\int_{l_2}^{\lambda_2}\frac{d\lambda}{L} \\ t = \varepsilon_1\int_{l_1}^{\lambda_1}\frac{\lambda d\lambda}{L} + \varepsilon_2\int_{l_2}^{\lambda_2}\frac{\lambda d\lambda}{L} \end{cases}$$

valores Coordinatarum λ_1, λ_2 tempore $t+T$ eosdem ac tempore t fieri. Est igitur T id temporis spatium, quo elapso Mobile in eundem motum eandemque orbitam redit. Quum plerumque numeri integri $M, N,$ propositae relationi sufficientes, infinite magni evadant, ipsud T plerumque infinite magnum erit. Elucet, tempus T ea tantum conditione finitum esse, ut integralia

$$\int_{m_1}^{n_1}\frac{d\lambda}{L} \quad \text{et} \quad \int_{m_2}^{n_2}\frac{d\lambda}{L}$$

quantitates sint commensurabiles.

§. 11.

Seriebus, quas Cl. Rosenhain invenit, in auxilium vocatis, Coordinatas orthogonales x, y, z per tempus t exprimi posse.

Cl. **Rosenhain** in commentatione „Sur les fonctions de deux variables et à quatre periodes" inscripta, docuit, quod accommodatum nostris aequationibus:

$$\varepsilon_1\int_{l_1}^{\lambda_1}\frac{d\lambda}{L}+\varepsilon_2\int_{l_2}^{\lambda_2}\frac{d\lambda}{L}=0,$$

$$\varepsilon_1\int_{l_1}^{\lambda_1}\frac{\lambda d\lambda}{L}+\varepsilon_2\int_{l_2}^{\lambda_2}\frac{\lambda d\lambda}{L}=t,$$

ubi

$$L=\sqrt{8(\lambda-a)(\lambda-b)(\lambda-c)(\lambda-A)(\lambda-B)},$$

sic enuntiare licet.

Designetur Quantitatum a, b, c, A, B ordo secundum eorum magnitudinem exhibitus per

$$0_1<0_2<0_3<0_4<0_5;$$

introducantur v, w, a, p, q per formulas:

$$\begin{pmatrix}0_2 & 0_4\\ 0_1 & 0_3\end{pmatrix}\frac{2v}{\pi} = \varepsilon_1\begin{pmatrix}l_1 & 0_4\\ 0_1 & 0_3\end{pmatrix}+\varepsilon_2\begin{pmatrix}l_2 & 0_4\\ 0_3 & 0_3\end{pmatrix}-t\int_{0_3}^{0_4}\frac{d\lambda}{L},$$

$$\begin{pmatrix}0_2 & 0_4\\ 0_1 & 0_3\end{pmatrix}\frac{2w}{\pi} = -\varepsilon_1\begin{pmatrix}l_1 & 0_2\\ 0_1 & 0_1\end{pmatrix}-\varepsilon_1\begin{pmatrix}l_2 & 0_2\\ 0_3 & 0_1\end{pmatrix}+t\int_{0_1}^{0_2}\frac{d\lambda}{L},$$

$$\begin{pmatrix}0_2 & 0_4\\ 0_1 & 0_3\end{pmatrix}\frac{2a}{i\pi} = \begin{pmatrix}0_5 & 0_4\\ 0_4 & 0_3\end{pmatrix},$$

$$\begin{pmatrix}0_2 & 0_4\\ 0_1 & 0_3\end{pmatrix}\frac{\log p}{i\pi} = \begin{pmatrix}0_1 & 0_4\\ -\infty & 0_3\end{pmatrix},$$

$$\begin{pmatrix}0_2 & 0_4\\ 0_1 & 0_3\end{pmatrix}\frac{\log q}{i\pi} = -\begin{pmatrix}0_5 & 0_2\\ 0_4 & 0_1\end{pmatrix},$$

ubi brevitatis causa posita sunt:

$$\pi=3{,}14159\ldots \qquad i=\sqrt{-1}$$

$$\begin{pmatrix}\beta & \delta\\ \alpha & \gamma\end{pmatrix}=\int_\alpha^\beta\frac{d\lambda}{L}\int_\gamma^\delta\frac{\lambda d\lambda}{L}-\int_\alpha^\beta\frac{\lambda d\lambda}{L}\int_\gamma^\delta\frac{d\lambda}{L};$$

definiantur, literis k, h quoslibet binos numerorum 0, 1, 2, 3 designantibus, functiones φ_{kh} per formulam:

$$\varphi_{kh}(iv, iw, p, q, \alpha)=\sum_{-\infty}^{+\infty}{}_m\sum_{-\infty}^{+\infty}{}_n(-1)^{ms_k+ns_h}p^{\frac14 m_k^2}q^{\frac14 n_h^2}e^{m_k iv+n_h iw+m_k n_h\alpha},$$

ubi est:

$$m_0=2m, \quad m_1=2m+1, \quad m_2=2m+1, \quad m_3=2m,$$
$$n_0=2n, \quad n_1=2n+1, \quad n_2=2n+1, \quad n_3=2n,$$
$$s_0=1, \quad s_1=1, \quad s_2=0, \quad s_3=0,$$

solvuntur propositae nostrae aequationes respectu iporum λ_1, λ_2 per quaslibet binas harum quinque aequationum:

$$(16.)\begin{cases}(0_1-\lambda_1)(0_1-\lambda_2)=-\sqrt{(0_2-0_1)(0_3-0_1)(0_4-0_1)(0_5-0_1)}\cdot\dfrac{\varphi^2_{10}(iv,iw,p,q,\alpha)}{\varphi^2_{00}(iv,iw,p,q,\alpha)},\\ (0_2-\lambda_1)(0_2-\lambda_2)=-\sqrt{(0_2-0_1)(0_3-0_2)(0_4-0_2)(0_5-0_2)}\cdot\dfrac{\varphi^2_{20}(iv,iw,p,q,\alpha)}{\varphi^2_{00}(iv,iw,p,q,\alpha)},\\ (0_3-\lambda_1)(0_3-\lambda_2)=\sqrt{(0_3-0_1)(0_3-0_2)(0_4-0_3)(0_5-0_3)}\cdot\dfrac{\varphi^2_{31}(iv,iw,p,q,\alpha)}{\varphi^2_{00}(iv,iw,p,q,\alpha)},\\ (0_4-\lambda_1)(0_4-\lambda_2)=\sqrt{(0_4-0_1)(0_4-0_2)(0_4-0_3)(0_5-0_4)}\cdot\dfrac{\varphi^2_{32}(iv,iw,p,q,\alpha)}{\varphi^2_{00}(iv,iw,p,q,\alpha)},\\ (0_5-\lambda_1)(0_5-\lambda_2)=\sqrt{(0_5-0_1)(0_5-0_2)(0_5-0_3)(0_5-0_4)}\cdot\dfrac{\varphi^2_{33}(iv,iw,p,q,\alpha)}{\varphi^2_{00}(iv,iw,p,q,\alpha)}.\end{cases}$$

Supra demonstravimus, l_1 semper intervallo $0_1 \dots 0_2$ et l_2 semper intervallo $0_3 \dots 0_4$ contineri. Simulac λ in quolibet trium intervallorum $0_1 \dots l_1 \dots 0_2$, $0_3 \dots l_2 \dots 0_4$, $0_5 \cdots +\infty$ situm est, Quantitas L realis est; simulac λ in quolibet trium intervallorum $-\infty \dots 0_1$, $0_2 \dots 0_3$, $0_4 \dots 0_5$ situm est, Quantitas L imaginaria est.

Itaque Quantitates v, w, α, $\log p$, $\log q$, denuo introductae, omnes inveniuntur reales esse. Atque adnotando, expressionem

$$\binom{\beta\delta}{\alpha\gamma}=\int_\alpha^\beta\frac{d\lambda}{L}\int_\gamma^\delta\frac{\lambda d\lambda}{L}-\int_\alpha^\beta\frac{\lambda d\lambda}{L}\int_\gamma^\delta\frac{d\lambda}{L}$$

etiam sic exhiberi posse

$$\binom{\beta\delta}{\alpha\gamma}=\int_\alpha^\beta\int_\gamma^\delta\frac{(\lambda_1-\lambda)\,d\lambda\,d\lambda_1}{L L_1},$$

Quantitates $\log p$, $\log q$ semper negativae esse, ipsaque p, q semper intervallo $0 \cdots +1$ contineri inveniuntur.

Postremo adnotemus, functiones φ_{kh}, Quantitatibus exponentialibus e^{iv}, e^{iw} per trigonometricas Quantitates expressis, hanc induere formam:

$$\varphi_{kh}(iv, iw, p, q, \alpha)=\sum_{-\infty}^{+\infty}{}_m\sum_{-\infty}^{+\infty}{}_n R^{mn}_{kh}$$

vel

$$=R^{00}_{kh}+2.\sum_0^{+\infty}{}_m R^{m0}_{kh}+2.\sum_0^{+\infty}{}_n R^{0n}_{kh}+4.\sum_1^{+\infty}{}_m\sum_1^{+\infty}{}_n R^{mn}_{kh}$$

ubi, posito:

$$\eta_0 = 1, \quad \eta_1 = -1, \quad \eta_2 = 1, \quad \eta_3 = 1,$$

R_{kh}^{mn} hanc designat expressionem:

$$R_{kh}^{mn} = \tfrac{1}{2}\left\{(-1)^{ms_k + ns_h} p^{\frac{1}{4}m_k^2} q^{\frac{1}{4}n_h^2}\right\}.$$

$$.\left\{\left[\left(\frac{\eta_h+\eta_k}{2}\right)^2 e^{m_k n_h \alpha}\cos(m_k v + n_h w) + \left(\frac{\eta_h+\eta_k}{2}\right) e^{-m_k n_h \alpha}\cos(m_k v - n_h w)\right] + \right.$$

$$\left. + i\left[\left(\frac{\eta_h-\eta_k}{2}\right)^2 e^{m_k n_h \alpha}\sin(m_k v + n_h w) + \left(\frac{\eta_h-\eta_k}{2}\right) e^{-m_k n_h \alpha}\sin(m_k v - n_h w)\right]\right\}.$$

Unde, litera r quemlibet trium numerorum 0, 2, 3 designante, elucet, Quantitates φ_{11} et φ_{rr} reales, Quantitates φ_{1r} et φ_{r1} pure imaginarias esse.

Jam Coordinatarum ellipticarum valores aequationibus (16.) exhibiti directe suppeditant Coordinatarum orthogonalium x, y, z valores, quippe quae illis conjunctae sint formulis

$$x^2 = \frac{(a-\lambda_1)(a-\lambda_2)}{(a-b)(a-c)},$$

$$y^2 = \frac{(b-\lambda_1)(b-\lambda_2)}{(b-c)(b-a)},$$

$$z^2 = \frac{(c-\lambda_1)(c-\lambda_2)}{(c-a)(c-b)}.$$

Exempli causa in primo casu, quo pro Quantitatibus $0_1, 0_2, 0_3, 0_4, 0_5$ sumenda sunt a, A, b, B, c, Quantitates auxiliares v, w, α, p, q hos induunt valores

$$\begin{pmatrix} A & B \\ a & b \end{pmatrix}\frac{2v}{\pi} = \varepsilon_1\begin{pmatrix} l_1 & B \\ a & b \end{pmatrix} + \varepsilon_2\begin{pmatrix} l_2 & B \\ b & b \end{pmatrix} - t\int_b^B \frac{d\lambda}{L},$$

$$\begin{pmatrix} A & B \\ a & b \end{pmatrix}\frac{2w}{\pi} = -\varepsilon_1\begin{pmatrix} l_1 & A \\ a & a \end{pmatrix} - \varepsilon_1\begin{pmatrix} l_2 & A \\ b & a \end{pmatrix} + t\int_a^A \frac{d\lambda}{L},$$

$$\begin{pmatrix} A & B \\ a & b \end{pmatrix}\frac{2\alpha}{i\pi} = \begin{pmatrix} c & B \\ B & b \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ a & b \end{pmatrix}\frac{\log p}{i\pi} = \begin{pmatrix} a & B \\ -\infty & b \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ a & b \end{pmatrix}\frac{\log q}{i\pi} = -\begin{pmatrix} c & A \\ B & a \end{pmatrix},$$

ipsique Coordinatarum x, y, z valores fiunt:

$$x = \pm\sqrt[4]{\frac{(A-a)(B-a)}{(b-a)(c-a)}}\cdot\frac{A}{\varDelta},$$

$$y = \pm\sqrt[4]{\frac{(b-A)(B-b)}{(b-a)(c-b)}}\cdot\frac{B}{\varDelta},$$

$$z = \pm\sqrt[4]{\frac{(c-A)(c-B)}{(c-a)(c-b)}}\cdot\frac{\Gamma}{\varDelta},$$

ubi $\mathsf{A}, \mathsf{B}, \Gamma, \varDelta$ quinta superioribusque dimensionibus Quantitatum q, q neglectis, hasce significant expressiones:

$$\begin{aligned}\mathsf{A} = p^{\frac{1}{4}}[&\sin v - q\,(e^{2\alpha}\sin(v+2w) + e^{-2\alpha}\sin(v-2w))\\ &+ q^4(e^{4\alpha}\sin(v+4w) + e^{-4\alpha}\sin(v-4w))\\ &- 2p^2\,2\sin 3v + 2p^2q\,(e^{6\alpha}\sin(3v+w) + e^{-6\alpha}\sin(3v-w))],\end{aligned}$$

$$\begin{aligned}\mathsf{B} = q^{\frac{1}{4}}[&\sin w + p\,(e^{2\alpha}\sin(w+2v) + e^{-2\alpha}\sin(w-2v))\\ &+ p^4(e^{4\alpha}\sin(w+4v) + e^{-4\alpha}\sin(w-4v))\\ &- 2q^2\,2\sin 3w - 2q^2p\,(e^{6\alpha}\sin(3w+v) + e^{-6\alpha}\sin(3w-v))],\end{aligned}$$

$$\begin{aligned}\Gamma = 1 &+ 2q\cos 2w + 2q^4\cos 4w + 2p\cos 2v + 2p^4\cos 4v\\ &+ 2pq\,(e^{4\alpha}\cos 2(v+w) - e^{-4\alpha}\cos 2(v-w)),\end{aligned}$$

$$\begin{aligned}\varDelta = 1 &- 2q\cos 2w + 2q^4\cos 4w - 2p\cos 2v + 2p^4\cos 2w\\ &+ 2pq\,(e^{4\alpha}\cos 2(v+w) - e^{-4\alpha}\cos 2(v-w)).\end{aligned}$$

Expressiones Coordinatarum x, y, z, quae in ceteris casibus obtinentur, conscribendo supersedeo.