



I. On fallible measures of variable quantities, and on the treatment of meteorological observations

G.H. Darwin M.A.

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- I. *On Fallible Measures of Variable Quantities, and on the Treatment of Meteorological Observations.* By G. H. DARWIN, M.A., Fellow of Trinity College, Cambridge*.

IF we make any observation, for example the transit of a star, a definite numerical result is obtained. To say that that result is liable to errors of observation, is only correct from one point of view. It is true that the result does not really correspond with the time at which the star crossed the meridian, yet it is an accurate representation of a certain very complex event. Undoubtedly, the principal feature in that event is the time of crossing the meridian; but there is also involved in it various properties of the instruments, the atmosphere, and the observer himself, &c. The object of the observation is, of course, to get a result which shall represent that principal feature, after the elimination of the minor features. The comparative simplicity of this, and of many other observations, permits us to unravel the complex event into its constituent parts, and to estimate each numerically. One part consists of corrections of all sorts. But when all these have been made, there still remains a result representing a complex event, viz. the transit of the star, together with unknown properties of the circumstances of the observation. These unknown properties form the subject-matter of the theory of errors of observation; but it is only because there is a consistent theory of the principal phenomenon, that the most probable line of demarcation can be drawn between the two parts of the complex event. The final result is given as a definite value with a margin of uncertainty in either direction.

But in the case of astronomical observations there is com-

* Communicated by the Author.

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plete certainty that we cannot have affected the principal event in any way by the method of observing. In experiments, however, it is impossible to imitate exactly the proposed conditions ; so that, even when corrections have been applied, and when we can estimate the degree of uncertainty in the method of observing, there remains another sort of uncertainty, viz. as to the closeness with which the proposed conditions were imitated ; that is to say, the principal event is rendered uncertain and complex.

The case of experiments graduates into that of observations of natural phenomena, where we have no control over the disturbing causes, and have no opportunity of slightly altering the conditions.

The line of demarcation between the principal event, whose laws are to be determined, and the disturbances, here becomes still more undefined. And where we are still groping after a law of the phenomena (as in the case of meteorology) it is unknown what is to be classed as the principal event and what as disturbances. It is like looking at a series of irregular waves with ripples of various sizes on their surface ; until some law in the formation of the waves is discovered, it is unknown how large a ripple may be neglected in the discovery of that law. Nevertheless the only chance of discovery seems to be to neglect the ripples by some arbitrary rule, and to examine the main features of the series of waves.

The problem of how best to combine a number of discontinuous observations into a continuous law, so as to give a general representation of them after disturbances due to fallibility of measures, runs of chance (as in statistics), &c. have been set aside, is one that constantly presents itself for solution ; and a rational and methodical treatment can hardly fail to be of value.

The most frequent occasion for the solution of the problem arises from the necessity of drawing a curve passing close to the extremities of a number of ordinates ; and the usual way of solving it is to draw a curve without abrupt changes of curvature as close to the points as possible. If the changes of curvature are abrupt enough, the curve may be made to pass exactly through the points ; but then each observation is treated as exact, and we have exactly the case of the series of waves with ripples on them. But, by what precedes, it appears that we had better omit the ripples ; and the question remains as to how far we are justified in smoothing down the curve. This process of smoothing is often done by the free hand ; but it will probably be done better by a system ; and it will be an additional advantage if the system admits of arithmetical as well as graphical application.

In those cases in which an algebraic law can be assigned, to which the ordinates ought to conform, the best method of treating the problem is to determine the constants involved in the function by the method of least squares; but it might often be not worth while to carry out this process, as, for example, where the deviations of the various observations from the law are large, and where it would accordingly be pedantic to assign values to the constants with precision.

Where the law is unknown and the observations are equidistant, a method of treatment might, perhaps, be devised by the assumption of some form of function containing fewer constants than the number of given points, and consisting of a number of simple harmonic terms, none of which go through a large fraction of their period in passing from one ordinate to the next. The constants involved might then be determined by the method of least squares, so that the function should give the best representation of the observations. But the assumption of the form of function would be arbitrary, and the process very laborious.

On the whole it will be more convenient and equally satisfactory, as far the result is concerned, to proceed empirically from the first, remembering that the main object is to exclude ripples of short period.

The method here suggested is one which I believe is used in some form or other by meteorologists; but I am not aware that its merits have been discussed, or that it has been extended to the smoothing of surfaces and of functions of three or more independent variables, as I here propose to do*.

Empirical Rule.—The observations are supposed to be equidistant and to be functions of only one independent variable. The method may be most easily explained geometrically, and the transition afterwards made to its arithmetical equivalent. It will be convenient also to speak of the deviations of the several observations from the principal part of the complex event which those observations represent as *errors*.

It is proposed to substitute for each pair of consecutive points A, B, a point P which bisects the straight line AB. The points P then lie on a series of ordinates halfway between the original ones.

If the errors of A and B are of opposite sign, P is a better point than either of them; if they have errors of the

* Since this paper has been in the hands of the printer, I have learnt that M. Schiapparelli has written a work entitled *Sul modo di ricavare la vera espressione delle leggi della natura dalle curve empiriche*; (Milan, 1867), and that M. De Forest has written on the subject in the 'Annual Reports of the Smithsonian Institution' for 1871 and 1873, and in the 'Analyst (Iowa)' for May 1877.

same sign, P will be better or worse according to the direction of the curvature of the curve. But if the rate of change of curvature of the curve is small (as it must be assumed to be to justify the smoothing process), P is very little better or worse. Now, as on the average the series of points deviate as often to one side as to the other of the curve, there clearly will be on the average an improvement in accuracy from the substitution of P, whilst there will certainly be less abrupt changes of curvature in a curve passing through P than through A B. Where the points already lie on a fair curve with no contrary flexure, the chance is rather more than even that there will be a loss of accuracy of representation, because the substituted points all lie on the same side of the given ones, and the only case where there is improvement is where all the errors have one particular sign, and are not very small. The process of smoothing must then be applied cautiously, and especially at maximum- and minimum-points.

If the points P do not lie on a fair curve, the process may be applied again in part of the series or along the whole line; but when once our judgment leads us to think that the curve is smooth enough, every succeeding operation tends to spoil the representation.

Analytically the process may be stated as follows:—

If y_0, y_1, y_2 , &c. be the successive given ordinates, and if ϕ indicates a single smoothing operation, so that ϕy_x indicates the substituted ordinate corresponding to the abscissa x ; then clearly

$$\phi y_x = \frac{1}{2}(y_{x+\frac{1}{2}} + y_{x-\frac{1}{2}}),$$

and generally

$$\phi^n y_x = \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} \right)^n y_x.$$

It is clear that an odd number of operations will leave us with points on ordinates halfway between the original ones, whilst an even number will leave us on the original ones. There is a practical advantage in proceeding by two operations at a time, because it is not then necessary to draw the intermediate ordinates, and because a double operation has a very simple geometrical and analytical meaning.

From the above formula,

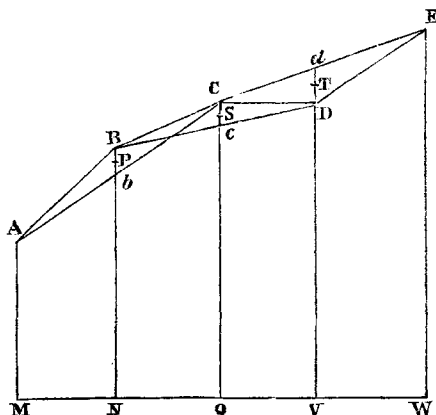
$$\phi^2 y_x = \frac{1}{2} \left\{ y_x + \frac{y_{x+1} + y_{x-1}}{2} \right\}.$$

If in the figure M A, N B, Q C are the three ordinates y_{x-1}, y_x, y_{x+1} , then $Nb = \frac{1}{2}(y_{x+1} + y_{x-1})$, and P, which bisects Bb, is the point to be substituted for B.

The practical rule of construction may be stated thus:—

Let A, B, C, &c. be the given points; join every point to

that next it and next but one to it. Then the points to be substituted bisect the intercepts Bb , Cc , &c. in P , S &c. If the curve which may be drawn through P , S &c. still seems too sinuous, repeat the operation.



The arithmetical application of this process is obviously very simple; for

$$\begin{aligned}\phi^2 y_x &= y_x + \frac{1}{4} \{y_{x-1} - 2y_x + y_{x+1}\} \\ &= y_x + \frac{1}{4} \Delta^2 y_{x-1};\end{aligned}$$

and therefore the correction to be applied to any ordinate y_x is $\frac{1}{4} \Delta^2 y_{x-1}$.

We can see how it is that this process tends to improve the curve. The observed or given values of the function consist of two parts, the first representing the principal event or wave whose law of variation is to be found, and the second the errors or ripples which are to be eliminated. Now the observations are supposed to be so close as not to admit of very large differences between the successive values of the *principal event*; and therefore their second differences will be small. On the other hand, the errors will be some positive and some negative; and therefore their second differences will be very irregular, and probably much larger on the whole than the errors themselves. The second differences of the observed values are the sums formed by the addition of these two sets of second differences; and the justifiability of the process depends on the assumption that the increase of the latter will be sufficient to render the diminished values of the former insignificant by comparison. We thus obtain a series of quantities which depend principally on the errors, except in case the errors are small, or in case of a run of luck in the signs and magnitude of the errors, such as to make them apparently conform to law

and thus present small second differences. Now the proposed corrections to the various observed values are the quarters of this series of quantities; and thus in all probability our corrections depend principally on the errors. The process is therefore justifiable unless the points already lie in a smooth curve. The rough criterion of the applicability of the smoothing process is that the second differences of the observed values should not appear to conform to any law.

Every double operation causes the loss of one point at the beginning and one at the end; but perhaps the best course is to treat the first and last points as exact; and if the operation is repeated more than twice, the second and last but one as exact after one of these double operations, and so on.

Polar Coordinates.—The preceding method is applicable with equal justice to the case of polar coordinates, where the ordinates are replaced by radii vectores.

Irregular Observations.—With observations which are not equidistant, a strictly analogous process would be complex; but as it is empirical, a slight modification will be permissible. Thus we may omit the analogue of interpolation on intermediate ordinates, and only retain the double operation. The intercepts Bb , Cc , &c. may be bisected as before; for this gives less weight to observations which are more remote than to those which are near, as it clearly ought to do.

The corresponding numerical rule is to substitute for the ordinate y_r , the value

$$\frac{(x_{r+1} - x_r)(y_{r-1} + y_r) + (x_r - x_{r-1})(y_r + y_{r+1})}{2(x_{r+1} - x_{r-1})},$$

where

$$x_{r-1}, y_{r-1}; x_r, y_r; x_{r+1}, y_{r+1}$$

are the coordinates of the three successive points.

The merits of an empirical rule like this must of course depend on how it seems to work practically. I therefore devised the following scheme for testing it. A circular piece of card was graduated radially, so that a graduation marked x was $\frac{720}{\sqrt{\pi}} \int_0^x e^{-s^2} dx$ degrees distant from a fixed radius. The card was made to spin round its centre close to a fixed index. It was then spun a number of times, and on stopping it the number opposite the index was read off*. From the nature of the graduation the numbers thus obtained will occur in exactly the same way as errors of observation occur in practice; but

* It is better to stop the disk when it is spinning so fast that the graduations are invisible, rather than to let it run out its course.

they have no signs of addition or subtraction prefixed. Then by tossing up a coin over and over again and calling heads + and tails —, the signs + or — are assigned by chance to this series of errors. About a dozen equidistant values of some function (say sine or cosine) were next taken from a Table, and the errors added to or subtracted from them in order. The errors may be made either small or large by multiplying them by any constant. The falsified values may then be fairly taken to represent a series of observations; but we here know what are the true ones. The corrections were then applied, in some cases arithmetically and in others graphically, and the deviations of the corrected values from the true were observed.

In other cases a series of equidistant ordinates were taken, and a sweeping free-hand curve was drawn to represent the true curve, and the several ordinates of this curve were falsified by the roulette and then corrected by a graphical application of the rule. The general result of a good many trials was such as to justify the smoothing process. Where the errors were considerable the mean error was much reduced, although the actual error of some ordinates was increased; where the errors were very small the mean error was even slightly increased. Although the danger of over-smoothing was obvious, and the sharpness of the features of the curve was generally diminished, yet I think it was clear that the method might generally be employed with advantage, especially in such cases as the attempt to deduce some law from statistics or a series of barometric oscillations of considerable periods. The errors must be very large to justify a quadruple operation. This method of trial could not be so well applied to testing the case of an odd number of smoothing operations, where we are left finally at intermediate ordinates.

On the whole, I think the process is justifiable if applied with caution. Nevertheless it undoubtedly tends to spoil the results if applied to a series of points which are already in a sweeping curve; and therefore I have tried to find some other process which should not have this disadvantage. This can only be done by taking more than three points of the curve into consideration; and therefore the process must be more cumbrous.

The method pursued was as follows:—

Let $-2, y''$; $-1, y'$; $0, y$; $1, y_1$; $2, y_2$ be the coordinates of five consecutive points on the curve. Suppose them to be represented by a curve whose equation is $y = a + bx + cx^2 + dx^3$; and make the following expression a minimum, viz.

$$\Sigma(a + bx + cx^2 + dx^3 - y)^2,$$

where the summation is made for the values of $x = -2, -1, 0, 1, 2$, and the corresponding values of y . In other words, the values of a, b, c, d are to be determined by the method of least squares, so that this curve shall give the best representation of the five points.

The equations for finding a, b, c, d are therefore

$$\begin{aligned} 5a + b\Sigma x + c\Sigma x^2 + d\Sigma x^3 &= \Sigma y, \\ a\Sigma x + b\Sigma x^2 + c\Sigma x^3 + d\Sigma x^4 &= \Sigma xy, \\ a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 + d\Sigma x^5 &= \Sigma x^2y, \\ a\Sigma x^3 + b\Sigma x^4 + c\Sigma x^5 + d\Sigma x^6 &= \Sigma x^3y. \end{aligned}$$

From the manner in which the origin has been chosen the sums of the odd powers of x are all zero, and $\Sigma x^2 = 10$, $\Sigma x^4 = 34$, $\Sigma x^6 = 130$.

Thus the first and third equations are

$$\begin{aligned} 5a + 10c &= y'' + y' + y + y_1 + y_2, \\ 10a + 34c &= 4y'' + y' + y_1 + 4y_2; \end{aligned}$$

and the second and fourth may be easily written down. It will be noticed that the first and third equations would be exactly the same if we assumed as the form of the equation $y = a + bx + cx^2$.

Now the proposed method is to substitute for every point of the series of given points the intersection with the ordinate of that point of the curve of the form $y = a + bx + cx^2$ or $y = a + bx + cx^2 + dx^3$ which best represents that point and the two preceding and two succeeding points. In the case we have been considering we are, therefore, to substitute for the point 0, y the intersection of this curve with the axis of y ; that is to say, we are to substitute the point 0, a , because when $x = 0$, $y = a$.

Now $35a = -3y'' + 12y' + 17y + 12y_1 - 3y_2$;
or

$$\begin{aligned} a &= y + \frac{3}{35} \{-y'' + 4y' - 6y + 4y_1 - y_2\} \\ &= y - \frac{3}{35} \Delta^4 y''. \end{aligned}$$

Thus the correction δy , to be applied to y , is $-\frac{3}{35} \Delta^4 y''$.

Hence generally, since the process is supposed to be applied all along the series,

$$\delta y_x = -\frac{3}{35} \Delta^4 y_{x-2}.$$

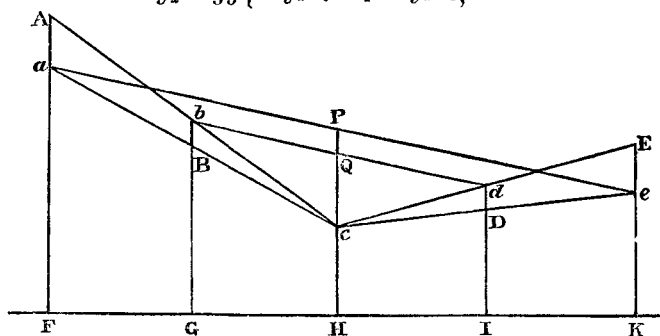
To give a geometrical meaning to the rule, it may be observed that

$$-y'' + 4y' - 6y + 4y_1 - y_2 = -(y'' - 2y + y_2) + 4(y' - 2y + y_1);$$

and therefore if Δ' be a symbol denoting the operation of dif-

ferencing with the omission of alternate ordinates.

$$\delta y_x = \frac{1}{3} \frac{\Delta^2}{\Delta x} \left\{ \Delta^2 y_{x-1} - \frac{1}{4} \Delta^2 y_{x-2} \right\}^*.$$



Now in the figure let A F, B G, C H, D I, E K be any five consecutive ordinates, and suppose that it is proposed to correct the ordinate C H. Then, if the construction shown in the figure be carried out, it is clear that

$$C P = \Delta^2(B G), \text{ and } C Q = \frac{1}{4} \Delta^2(A F),$$

and therefore

$$P Q = \Delta^2(B G) - \frac{1}{4} \Delta^2(A F).$$

Thus the correction to be applied to the ordinate H C is $\frac{1}{3} \frac{\Delta^2}{\Delta x}$ (or very nearly $\frac{1}{3}$) of P Q. The same process must be applied all along the series for each set of five points.

Four points are lost out of the series, two at each end. For example, if A, B are the first two points, the rule gives no substituted points on those ordinates. The results obtained from the use of this rule do not seem markedly superior to those given by the empirical method, except where the points lie in a fair curve; and as the rule is more cumbrous to apply, it does not seem likely to be of much practical value.

The construction of a fair surface near a number of points.

The preceding process may be extended to the case where the function involves two independent variables. The observed values may, for the sake of clearness, be considered as consisting of a number of ordinates standing on the intersections of the lines of a chess-board, of which two intersecting edges are the axes of x and y .

Let $[x, y]$ indicate the given ordinate which stands on the

* If the points lie in a fair curve, $\Delta^2 y_{x-1} - \frac{1}{4} \Delta^2 y_{x-2}$ is very small, a property which I have used to give a rule of graphical interpolation on intermediate ordinates (see the 'Messenger of Mathematics,' January 1877). Thus in this case the correction applied is very small.

point x, y ; let E, Δ be the operations of writing $x+1$ for x , and of differencing with respect to x ; and let E', D be the like operations with respect to y . Let ϕ represent a single smoothing operation on any series of ordinates which are in a plane parallel to x ; and ψ the same with respect to y .

Now apply a smoothing operation ϕ to all the points parallel to x , and then apply the operation ψ to all these new points. The order in which these operations are performed is immaterial; and the result is

$$\phi\psi[x, y] = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} \cdot \frac{E'^{\frac{1}{2}} + E'^{-\frac{1}{2}}}{2} [x, y] = \frac{1}{4} \{ [x + \frac{1}{2}, y + \frac{1}{2}] + [x + \frac{1}{2}, y - \frac{1}{2}] + [x - \frac{1}{2}, y + \frac{1}{2}] + [x - \frac{1}{2}, y - \frac{1}{2}] \}.$$

This, interpreted geometrically, means that we are to erect in the middle of each square an ordinate which is the mean of the four surrounding ordinates. Again,

$$\phi^2\psi^2[x, y] = \frac{1}{16} \{ [x+1, y+1] + [x+1, y-1] + [x-1, y+1] + [x-1, y-1] + 2([x+1, y] + [x-1, y] + [x, y+1] + [x, y-1]) + 4[x, y] \}.$$

If the figure represents any four squares of the chess-board (in which observe that nine ordinates stand on the intersections), the rule given by a double operation is to substitute for the ordinate at K,

$\frac{1}{16}$ of sum of ordinates at A, B, C, D
 $+$ $\frac{1}{8}$ of sum of ordinates at E, F, G, H
 $+$ $\frac{1}{4}$ of ordinate at K.

	A	E	B
		K	F
H			
D		G	C

It is clear that the operations of smoothing parallel to the two axes are quite independent, and that there is no necessity to smooth the same number of times in each direction. The symbolical way of writing the operation makes it perfectly easy to construct any desired modification of this formula, where ϕ and ψ are each performed any number of times.

The process may also be extended with equal justice to the case where there are three independent variables, although that case no longer admits of geometrical interpretation.

Let t be a third variable; then, if the former notation be extended, and if only a type of each form of term be written down preceded by the sign of summation, it will be found that

$$\phi^2\psi^2\chi^2[x, y, t] = \frac{1}{64} \{ \Sigma [x \pm 1, y \pm 1, t \pm 1] + 2 \Sigma [x, y \pm 1, t \pm 1] + 4 \Sigma [x, y, t \pm 1] + 8 [x, y, t] \}.$$

There are eight terms of the first kind, such as $[x+1, y-1, t+1]$; twelve of the second, such as $[x+1, y, t-1]$; six of the third, such as $[x+1, y, t]$; and only one of the last, viz. $[x, y, t]$.

Application to Ocean Meteorology.—This last process appears to me to be applicable here. Meteorologists have divided the ocean up into squares of 5° of latitude and 5° of longitude. The logs of ships sailing over those squares are consulted for meteorological observations; and the results are classified by months and squares; and the mean result for any one element, such as the height of the barometer, is taken to be the average for the middle of that square and the middle of that month*. Now it seems as though this were a case where smoothing is justifiable, and that it is allowable to make the result for each month depend in some degree on its neighbours both in space and time. There are three independent variables, viz. latitude, longitude, and time; and in the previous formula we may take these to be represented by x, y , and t respectively.

Suppose, for example, we want to modify the mean height of the barometer for any square e for, say, February, both with reference to surrounding squares and to the heights for January and March. Then the rule for finding the amended height is:—Take the sum of the heights for a, c, g, k for January and for March + twice the sum of the heights for b, d, h, f for January and March + twice the sum of the heights for a, g, c, k for February + four times the sum of the heights for b, d, h, f for February + four times the sum of the heights for e for January and March + eight times the height for e for February, and divide the result by 64.

	a	b	c
	d	e	f
	g	h	k

It must be observed that it is not necessary that the smoothing should be carried to the same extent for x, y , and t . If, for example, we wish to smooth only once for time, the formula will be different, and the result will be applicable to the beginnings of the months instead of the middles. A knowledge of the particular requirements of the case is the only guide to the amount of smoothing which is expedient; but the formulæ are so easy to construct that it does not seem worth while to give any other forms.

In concluding this part of the subject, I may mention that the proposed processes may be extended so as to allow various weights to the various observations.

Terrestrial Meteorology.—There are a number of observa-

* I owe this explanation to Mr. Francis Galton.

tories on the land at which observations are taken at the same hours of the day all over the country; from these results, maps are drawn showing the form of the "isobars" for each day. After the observations have been reduced to the sea-level and corrected in other ways, they may be considered as correct, and the isobars give a graphical illustration of the successive deformations of the barometric surface. Land meteorology serves, then, to give quite a different kind of result from those of ocean meteorology. In the latter the result is the mean heights of the barometer at stated places and times. The oceanic barometric surface, as far as we know it, is ideal, and does not correspond with its real form at any one time. In oceanic meteorology the smoothing process seems justifiable; for we only seek to study the main features of the changes. In land meteorology this is not the case; for we seek to discover the details of the changes. To return to the former metaphor—in one case the law of the waves is sought, in the other the law of the ripples.

The observatories are scattered irregularly over the country; and it seems probable that the results would be more useful and more easily interpreted if they could be distributed at regular intervals of space. They are already regularly distributed as regards time. My present object is, then, to give a formula (which is, as far as I am aware, new) for the reduction of observations scattered irregularly, to regular stations equidistant in latitude and longitude. It is a problem in interpolation of the ordinary kind where the ordinates are not fallible.

The problem is to find a continuous surface passing through the tops of a number of irregularly spaced ordinates; and it may be solved by an extension of Lagrange's well-known formula for interpolation in two dimensions.

Let x_0, y_0 ; x_1, y_1 ; &c.; x_n, y_n be the coordinates (latitude and longitude) of a number of points, and let z_0, z_1 , &c., z_n be ordinates (barometric heights) corresponding to these points. Lagrange's formula suggests the following as the equation to a surface passing through the tops of z_0, z_1 , &c., z_n .

$$\begin{aligned}
 z = & z_0 \frac{(x-x_1)(y-y_1)(x-x_2)(y-y_2) \dots (x-x_n)(y-y_n)}{(x_0-x_1)(y_0-y_1)(x_0-x_2)(y_0-y_2) \dots (x_0-x_n)(y_0-y_n)} \\
 & + z_1 \frac{(x-x_0)(y-y_0)(x-x_2)(y-y_2) \dots (x-x_n)(y-y_n)}{(x_1-x_0)(y_1-y_0)(x_1-x_2)(y_1-y_2) \dots (x_1-x_n)(y_1-y_n)} \cdot \\
 & + \&c. \\
 & + z_n \frac{(x-x_0)(y-y_0)(x-x_1)(y-y_1) \dots (x-x_{n-1})(y-y_{n-1})}{(x_n-x_0)(y_n-y_0)(x_n-x_1)(y_n-y_1) \dots (x_n-x_{n-1})(y_n-y_{n-1})}.
 \end{aligned}$$

Then this formula will give the height z of the barometer at any station whose latitude and longitude are x, y , as deduced from the heights at the several observing-stations. The applicability of this interpolation depends, of course, on the assumption that the surface is not contorted between the given ordinates; and if the observatories are numerous enough, this assumption is probably justifiable.

The application of the formula would in general entail a great detail of arithmetic; but in the case of the reduction from irregular to regular stations, the great mass of the work might be done once for all. In this case the coordinates of the observing stations $x_0, y_0; x_1, y_1; \&c.$ are the same day after day, and the coordinates of the fixed stations x, y are constant for each of them. Hence the coefficients of $z_0, z_1, \&c.$ in the formula may be calculated once for all.

It would be very laborious and unnecessary to make the heights of the barometer at the equidistant stations depend on all the observatories in the country; and it would be probably quite sufficient to make each one depend on the five or six nearest observatories. The practical rule would then run somewhat in this fashion (the numbers being purely hypothetical):—

Height of bar at lat. $\} = \cdot 705$ Oxford $+ \cdot 20$ Kew
 51° long. 1° W. $\} + \cdot 092$ Southampton $+ \cdot 002$ Cambridge.

Every separate point to which the reductions were to be made would require a different set of coefficients, which would depend on the four, five, or six nearest actual observing-stations.

If the heights of the barometer were taken as the excess above 28 inches, the various heights need not be given to more than three figures; and as the coefficients would probably have also three figures, the multiplications might be very easily made by means of Crelle's Rechentafeln. By these means the daily observations might be very quickly reduced, and the results of each day's observations would be given by a series of numbers on a map spaced out at regular intervals of latitude and longitude. This would, I think, facilitate the drawing of the "isobars," and it would also be more intelligible than are the results as given at irregularly dispersed stations.

It may be noticed that the same set of coefficients would also be proper for the reduction of any other meteorological element which could be fairly represented by a surface. The calculation of the coefficients would be rather laborious; but if there is any real advantage in thus classifying the observa-

tions, this would be of slight consequence, as the work would be performed once for all.

In conclusion, I will add one other rule—namely, for interpolation between the oceanic meteorological observations when smoothed, as before suggested. This is a formula for interpolation in the case of a function of three independent variables, the values of which are given at equal intervals, as is the case in the mean barometer-heights in latitude, longitude, and time.

Let Δ , D be the differences between successive barometer-heights in latitude and longitude respectively, and δ the difference in time (that is to say, between the values for successive months). Then, following the former notation,

$$\begin{aligned} [x + \xi, y + \eta, t + \tau] &= [x, y, t] + \xi\Delta + \eta D + \tau\delta \\ &+ \frac{1}{1 \cdot 2} \{ \xi(\xi-1)\Delta^2 + \eta(\eta-1)D^2 + \tau(\tau-1)\delta^2 \\ &+ 2\xi\eta\Delta D + 2\eta\tau D\delta + 2\tau\xi\delta\Delta \} \\ &+ \&c. \end{aligned}$$

The proof of this will be obvious to those acquainted with the Calculus of Finite Differences. No doubt it has been given before, although I do not happen to have met with it. This formula enables us to pass from the regular equidistant values for the middles of squares and months to those for any other neighbouring time and place.

II. *On the Production of Heat by Dynamical Action in the Compression of Gas.* By the Rev. J. M. HEATH*.

WHEN the equilibrium between the compressive and expansive forces in a given mass of gas has been disturbed by suddenly establishing an arbitrary, but finite, inequality (f) between them, the dynamical effect of this unbalanced force will be, a gradual and continuous alteration in the volume and temperature of the gas, which continues until the expansive force, which depends upon these two elements alone, again becomes equal to that of the compression, and the energy of the inequality is exhausted. At this moment a new condition of equilibrium obtains, in which the elements p' , v' , t' , differ from their former values by determinate finite quantities, δp , δv , and δt .

These quantities will, of course, satisfy the equation

$$v \cdot \delta p - p' \delta v = pv \cdot \alpha \delta t,$$

which expresses nothing but that they are the differences of

* Communicated by the Author.