On the Theory of Screws in Elliptic Space.

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Introduction.*

[As Grassmann's methods are not as well known as they should be, it may perhaps be convenient to the reader if I prefix a short account of them to my paper; I have not chosen the arrangement which would be best in itself, but that which connects itself most simply with familiar conceptions.

Let the coordinates of a point P in space be $(xyzw)$, referred to any tetrahedron $ABCD$, and let the coordinates be so chosen that

$$x+y+z+w=1;$$

then the fundamental idea of the *Ausdehnungslehre* is that this can be expressed in the form of an equation

$$P = xA+yB+zC+wD \quad (*).$$

If we choose, we can consider this equation as nothing more than a brief mode of expressing the statement immediately preceding it, but in Grassmann's view of the matter the equation (*) is fundamental, and we can, if we choose, express it in words by the statement immediately preceding it.

I now alter the notation so as to make it agree with Grassmann's: we call the vertices of the tetrahedron of reference e_1, e_2, e_3, e_4; the coordinates of a point are written x_1, x_2, x_3, x_4, and the point with these coordinates is called x; we therefore have

$$x = x_1e_1+x_2e_2+x_3e_3+x_4e_4.$$

In the same way, if y be any other point, we have

$$y = y_1e_1+y_2e_2+y_3e_3+y_4e_4.$$

Now an essential part of the *Ausdehnungslehre* is the *multiplication of points*. If we take the two equations just written down, and multiply the right-hand sides and the left-hand sides together in the most obvious way (remembering that multiplication is not to be taken as commutative until we have proved or explicitly assumed it to be so), we get a result which may be written

$$xy = \sum_{i=1}^{i=4}\sum_{k=1}^{k=4} x_i y_k e_i e_k;$$

* This introduction has been drawn up by request of one of the referees.

thus, for example, we have on the right-hand side a term $x_1 y_4 e_1 e_4$, and also a term $x_4 y_1 e_4 e_1$, and any reduction of the expression must depend on the law of multiplication assumed for the four *units* $e_1 e_2 e_3 e_4$. The law assumed by Grassmann is that known as *polar* multiplication; that is, we have

$$e_i e_k = -e_k e_i,$$
$$e_i^2 = 0.$$

Using these equations, we get at once

$$\begin{aligned} xy = \; & (x_1 y_4 - x_4 y_1)\, e_1 e_4 + (x_2 y_4 - x_4 y_2)\, e_2 e_4 + (x_3 y_4 - x_4 y_3)\, e_3 e_4 \\ & + (x_2 y_3 - x_3 y_2)\, e_2 e_3 + (x_3 y_1 - x_1 y_3)\, e_3 e_1 + (x_1 y_2 - x_2 y_1)\, e_1 e_2. \end{aligned}$$

But, if we consider the coefficients of $e_1 e_4$, &c., we see at once that they are the six coordinates of the line joining xy; and we may therefore say that *the product of two points is the line joining them.*

In precisely the same way, if we take three points x, y, z, and form their product, we find

$$xyz = \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix} e_2 e_3 e_4 + \begin{vmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{vmatrix} e_3 e_1 e_4 + \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix} e_1 e_2 e_4$$
$$+ \begin{vmatrix} x_3 & x_2 & x_1 \\ y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \end{vmatrix} e_3 e_2 e_1.$$

But, if we consider the coefficients of $e_2 e_3 e_4$, &c., we see at once that they are the four coordinates of the plane through xyz, and we may therefore say that *the product of three points is the plane through them.*

If we take four points, we get

$$xyzw = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} e_1 e_2 e_3 e_4.$$

If we take five points, we find that their product vanishes identically.

We have just seen that a plane X can be written in the form

$$X = X_1 e_2 e_3 e_4 + X_2 e_3 e_1 e_4 + X_3 e_1 e_2 e_4 + X_4 e_3 e_2 e_1 \ldots\ldots\ldots\ldots(\Theta),$$

where $e_2 e_3 e_4$, &c. are the faces of the tetrahedron of reference; or, denoting these by E_1, E_2, E_3, E_4, we have

$$X = X_1 E_1 + X_2 E_2 + X_3 E_3 + X_4 E_4.$$

But now there is a difficulty. If we take two planes given in the form (Θ), and try to multiply them together, we shall fail, for the product will vanish identically, since each term will necessarily contain at least one square factor; but the principle of duality obviously requires that the product of two planes shall be their line of intersection.

Grassmann obviates this difficulty by what he calls *regressive* multiplication. Before we can define this, we must introduce another conception of the greatest importance, viz., the *Ergänzung* (what I call the *conjugate*) of a quantity.

Grassmann supposes that e_1, e_2, e_3, e_4 are so chosen that

$$e_1 e_2 e_3 e_4 = 1,$$

or, if we please, we may say that this product is taken as the unit of such products.

Now this equation can be written

$$e_1 (e_2 e_3 e_4) = 1,$$

and $e_2 e_3 e_4$ is defined as the *conjugate* of e_1, and denoted by Ke_1,* and, in the same way, if E is any product of not more than four units, KE is defined by

$$E.KE = 1.$$

Thus $Ke_1e_2 = e_3e_4$, $Ke_1e_2e_3 = +e_4$, $Ke_4 = -e_1e_2e_3$, &c.: it will be seen hereafter that this term *conjugate* is appropriate. We have, if X is a plane,

$$KX = -(X_1 e_1 + X_2 e_2 + X_3 e_3 + X_4 e_4).$$

The definition of *regressive* multiplication is as follows:†—If E, E' are two different products of three units, EE' is defined by

$$K(EE') = KE.KE'.$$

Thus, if $E = e_1 e_2 e_3$, $E' = e_1 e_2 e_4$, we have $KE = e_4$, $KE' = -e_3$,

$$K(EE') = -e_4 e_3 = e_3 e_4,$$

$$EE' = e_1 e_2.$$

Using this definition of regressive multiplication, we can verify that *the product of two planes is their line of intersection*; that *the product of three planes is their point of intersection*; that *the product of four planes is the determinant of their coordinates*; and, lastly, that *the product of five planes vanishes identically.*

We can also verify that the product of a point and a plane vanishes if the plane contains the point; that the product of a point and a line is the plane through the point and line, and vanishes if the line contains the point; and that the product of a plane and a line is the point of intersection of the plane and line, and vanishes if the line is in the plane.

We must now consider the multiplication of lines. We saw, by

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* Grassmann denotes it by $|e_1$; my notation is, of course, borrowed from Hamilton.

† For the space of three dimensions we are considering.

multiplying two points, that the expression for a line was of the form

$$a = a_1 e_1 e_4 + a_2 e_2 e_4 + a_3 e_3 e_4 + a_4 e_2 e_3 + a_5 e_3 e_1 + a_6 e_1 e_2,$$

and it follows, from the expressions for a_1, &c., that we have

$$a_1 a_4 + a_2 a_5 + a_3 a_6 = 0.*$$

Now take two lines a, b, and multiply them together: we get, as is easily verified,

$$ab = a_1 b_4 + a_2 b_5 + a_3 b_6 + b_1 a_4 + b_2 a_5 + b_3 a_6, \dagger$$

and we thus see that the equation of condition satisfied by the coordinates of a line a may be written

$$a^2 = 0.$$

But it is known that

$$a_1 b_4 + a_2 b_5 + a_3 b_6 + a_4 b_1 + a_5 b_2 + a_6 b_3 = 0$$

is the condition that the two lines whose coordinates are

$$(a_1 a_2 a_3 a_4 a_5 a_6), \quad (b_1 b_2 b_3 b_4 b_5 b_6),$$

respectively, may intersect. We may therefore say (using Hamilton's indispensable word *scalar*) that *the product of two lines is a scalar, which vanishes if the lines intersect.*

It may be useful to give a few illustrations of these processes. Suppose we have three planes connected by a linear relation

$$a = \lambda b + \mu c;$$

to interpret this, let D be any point, then we have

$$aD = \lambda bD + \mu cD.$$

Therefore, if bD, cD both vanish, aD will also vanish; that is, any point in b and c is in a; that is, a, b, c pass through the same straight line.

In exactly the same way, we can show that, if a, b, c are points such that

$$a = \lambda b + \mu c,$$

the three points are collinear.

Now, suppose that a, b, c are three lines, connected by this linear relation; then, if P is any point, and p any plane, we have

$$aP = \lambda bP + \mu cP,$$

$$ap = \lambda bp + \mu cp.$$

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* In a more familiar notation, this is the condition $af + bg + ch = 0$.

† $af' + bg' + ch' + a'f + b'g + c'h$.

Therefore (1) the planes joining the lines to any point are collinear, therefore the three lines are concurrent; (2) the intersections of the lines with any plane are collinear, therefore the three lines are complanar.

Again, let a, b, c, d be four lines such that

$$a = \lambda b + \mu c + \nu d;$$

let α be any other line; then we have

$$a\alpha = \lambda b\alpha + \mu c\alpha + \nu d\alpha;$$

therefore $a\alpha$ vanishes if $b\alpha$, $c\alpha$, $d\alpha$ all vanish: that is, any line cutting b, c, d cuts a: that is, a, b, c, d are four generators of the same species of an hyperboloid of one sheet.

Lastly, suppose we have a homography,

$$\gamma = \lambda\alpha + \mu\beta,$$
$$\gamma' = \lambda\alpha' + \mu\beta',$$

on two lines $\alpha\beta$, $\alpha'\beta'$, then

$$\gamma\gamma' = \lambda^2\alpha\alpha' + \lambda\mu\,(\alpha\beta' - \alpha'\beta) + \mu^2\beta\beta';$$

therefore the connectors of corresponding points of two homographies generate an hyperboloid of one sheet; for, if we write down four of these equations we shall be able to eliminate $\alpha\alpha'$, $\alpha\beta' - \alpha'\beta, \beta\beta'$, so as to get a linear relation between four lines $\gamma\gamma'$; this is necessary because $\alpha\beta' - \alpha'\beta$ is not a line, as its square is $2\,(\alpha\beta\alpha'\beta')$.

We have now to extend our conceptions. An expression like $a_1 e_1 e_4 + \&c.$, in which every term contains a product of two units, will be called a *form of the second degree.* We have seen that such a form represents a straight line if its square vanishes; but, if its square does not vanish, it must obviously mean something. We can see what it means as follows: let a be any form of the second degree, and let x be a line,* then

$$ax = a_4 x_1 + a_5 x_2 + a_6 x_3 + a_1 x_4 + a_2 x_5 + a_3 x_6.$$

Therefore $ax = 0$ is a linear relation among the six coordinates of the line x; that is, it is the equation of a *linear complex,* and we may therefore say that *a form of the second degree whose square does not vanish represents a linear complex.* And it is worth while to notice that this is the only way in which we can get a definite notion of the meaning of a linear complex in three-dimensional space; the ordinary conceptions of it are either (1) as a locus of lines, which is too restricted, or (2) as a thing with six coordinates, that is, a point in five-

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* Coordinates $(x_1\, x_2\, x_3\, x_4\, x_5\, x_6)$.

dimensional space, which is too wide. The definition I have just given is perfectly definite and precise, and enables us to see the meaning, in our space, of all operations on the linear complex.*

There are two other interpretations of the form of the second degree, which are of equal importance.

Now, consider a system of forces acting on a rigid body. It is a known theorem in mechanics that they may be reduced in one way to the six forces acting along the six edges of a tetrahedron. Now we know that the edge $\overline{e_1 e_4}$ is represented by the product $e_1 e_4$, and we see that a force acting along this edge can be represented by $a_1 e_1 e_4$; where a_1 is a scalar proportional to the intensity of the force. Considering the forces along all the edges, we see that any system of forces can be represented in the form

$$a = a_1 e_1 e_4 + a_2 e_2 e_4 + a_3 e_3 e_4 + a_4 e_2 e_3 + a_5 e_3 e_1 + a_6 e_1 e_2.$$

But this is the general form of the second degree: therefore *the general form of the second degree represents a system of forces acting on a rigid body.*

Now we know that a system of forces can be reduced to what Prof. Ball calls a *wrench* about a certain screw. We may therefore say that the *general form of the second degree represents a wrench about a certain screw.*

There is also a kinematical interpretation. Any motion of a body can be reduced to rotations about the six edges of a given tetrahedron; now we habitually represent a rotation about a line by a length along the line, and we therefore see that a rotation about the edge $\overline{e_1 e_4}$ can be represented by $a_1 e_1 e_4$. Proceeding as above, we see that *the general form of the second degree represents a motion of a rigid body;* or, again, introducing Prof. Ball's terminology, we can say that *the general form of the second degree represents a twist about a certain screw.* Now, considering the three interpretations we have found, and attending only to what is common to the mechanical and kinematical interpretations, we can say that *the general form of the second degree represents a linear complex or a screw.* That is, the screw and the linear complex are the same thing regarded from different points of view. The *Ausdehnungslehre* furnishes a complete explanation of the theory of screws and the theory of the linear complex, and shows that they are identical and not merely analogous; this is the view on which the following paper is based.

* There is a paper by Prof. Reye in a recent number of *Crelle*, on systems of complexes, which is rendered much less clear than it need have been by the want of a precise definition of the linear complex as an element of space.

It remains to point out the connection with the non-Euclidean geometry.

So far, the tetrahedron of reference has been perfectly arbitrary; we now suppose it real and self-conjugate with respect to the absolute, then the point-equation to the absolute will be

$$x_1^2+x_2^2+x_3^2+x_4^2 = 0.$$

The plane-equation (tangential equation) will be

$$X_1^2+X_2^2+X_3^2+X_4^2 = 0.$$

The line equation (condition that a line may touch a surface)* will be

$$a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 = 0;$$

Now, if we use these forms of the equations of the absolute, the ordinary expressions for the distance between two points become

$$\cos xy = \frac{x_1y_1+x_2y_2+x_3y_3+x_4y_4}{\sqrt{(x_1^2+x_2^2+x_3^2+x_4^2)}\,.\,\sqrt{(y_1^2+y_2^2+y_3^2+y_4^2)}},$$

$$\sin xy = +\frac{\sqrt{\left\{\begin{array}{c}(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2+(x_1y_2-x_2y_1)^2\\ +(x_1y_4-x_4y_1)^2+(x_2y_4-x_4y_2)^2+(x_3y_4-x_4y_3)^2\end{array}\right\}}}{\sqrt{(x_1^2+x_2^2+x_3^2+x_4^2)}\,.\,\sqrt{(y_1^2+y_2^2+y_3^2+y_4^2)}}$$

Now, it is easy to verify that these expressions can be written

$$\cos xy = \frac{x\,.\,Ky}{\sqrt{(xKx)}\,.\,\sqrt{(yKy)}},$$

$$\sin xy = \frac{\sqrt{(xy\,.\,Kxy)}}{\sqrt{(xKx)}\,.\,\sqrt{(yKy)}}.$$

I now write Sxy for $x\,.\,Ky$, Tx for $+\sqrt{(xKx)}$.†

Then we see that the above expressions can be written

$$\cos xy = \frac{Sxy}{Tx\,Ty}, \qquad \sin xy = \frac{T\,(xy)}{Tx\,Ty}.$$

We have, of course, precisely similar expressions for the sine and cosine of the angle between two planes, and to these can be added the expressions for the angle between two lines or two complexes used in the paper.]

The Theory of Screws in Elliptic Space has been considered by Clifford and by Professor Ball,‡ but I venture to hope that the

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* Clifford's *rank-equation.*

† This is not Grassmann's notation.

‡ *Clifford*—"Preliminary Sketch of Biquaternions," *Mathematical Papers*, pp. 180 to 200; also Fragments numbered xli., xlii., xliv.

Ball—"Certain Problems in the Dynamics of a Rigid System moving in Elliptic Space" (Trans. R.I.A., t. xxviii., pp. 159 to 187).

methods employed in the following paper will be found to justify its existence.

My special object is to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in elliptic space. Clifford was apparently led to construct his theory of biquaternions by the want of such a calculus; but Grassmann's method seems to afford a simpler and more natural means of expression than biquaternions: thus an expression that Clifford writes

$$\frac{\alpha-\beta+\omega V\alpha\beta}{1-S\alpha\beta},$$

would be written

$$\frac{\alpha\beta}{S\alpha\beta};$$

with the modifications of Grassmann's notation used in this paper: and the meaning of the expression seems more obvious if we use the second form.

Another result of the application of the *Ausdehnungslehre* is, that we are able to make use of certain definitions given by Grassmann, and, by so doing, to discuss the metric properties of elliptic space without explicitly introducing the absolute.

I.

The four points of reference are denoted by e_1, e_2, e_3, e_4. The "Ergänzung" of any quantity a I call its conjugate and denote by Ka. I write Sqb for the product aKb, if a, b are of the same order, and I write Tx for the positive square root of Sxx.

The general expression for a form of the second order is

$$a = a_1e_1e_4+a_2e_2e_4+a_3e_3e_4+a_4e_2e_3+a_5e_3e_1+a_6e_1e_2,$$

A form of the second order represents a linear complex, or, what is the same thing, a screw or a motor, to use Clifford's expression.*

If a represents a line (*rotor*), it is the product of two forms of the first or third order, and we therefore have

$$0=\frac{a^2}{2}=a_1a_4+a_2a_5+a_3a_6.$$

If a, b are two complexes,

$$ab = a_1b_4+a_2b_5+a_3b_6+a_4b_1+a_5b_2+a_6b_3.$$

If ab vanishes, the complexes are said to be reciprocal: two lines are reciprocal if they cut.

* As a screw (motor) and a linear complex are exactly the same, I shall use the three words for the form of the second order according to the context.

Two quantities a, b determine two angles defined by the equations

$$\sin[ab] = \frac{Tab}{TaTb},$$

$$\cos(ab) = \frac{T(aKb)}{TaTb}.*$$

These angles are identical, unless a, b are both complexes (or lines) or one is a point (or plane) and the other a complex (not a line).

If a is a point on any line and b is a point on the conjugate line, Sab vanishes. Two lines are at right angles if either meets the conjugate of the other.

II.

Let a, b be any two lines : if a line c is to meet them both at right angles, it must satisfy the four equations

$$0 = ac = bc = Ka \, . \, c = Kb \, . \, c.$$

But obviously, if c satisfies these equations, Kc also satisfies them; but we know that in general there are only two lines satisfying the conditions, therefore we have the theorem: there are in general two lines meeting two given lines at right angles, and these two lines are at right angles.

If there are more than two lines meeting a, b at right angles, there is an infinite number, and a, b, Ka, Kb must be connected by a linear relation with scalar coefficients: in this case the lines a, b are said to be parallel.

If we have $$a = \lambda b + \mu Kb + \nu Ka,$$

we have also $$Ka = \lambda Kb + \mu b + \nu a.$$

Substituting,

$$a = b(\lambda + \mu\nu) + Kb(\mu + \lambda\nu) + a\nu^2 = 0;$$

but in general we shall not have a, b, Kb connected by a linear relation: for, if this were the case, they would be complanar and concurrent.

Therefore $$\nu^2 = 1, \quad \lambda + \mu\nu = 0.$$

That is, either $\nu = 1$, $\lambda = -\mu$, or $\nu = -1$, $\lambda = \mu$. The first pair gives

$$a - Ka = \lambda(b - Kb).$$

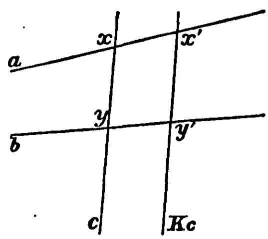
The second pair gives $$a + Ka = \lambda(b + Kb).$$

We thus see that there are two species of parallelism. Clifford distinguishes them as left and right parallelism respectively.

* It must be borne in mind that the tensor of a scalar is the scalar itself, whether it be positive or negative.

We shall now treat the subject of parallelism in a different way leading, however, to the same results.

Let xx', yy' be the points in which a, b are cut by c, Kc.



Then we have

$$0 = Sxx' = Sxy' = Syx' = Syy'.$$

Let $xy = \phi$, $x'y' = \phi'$.

Take any points $\xi = \alpha x + \alpha' x'$, $\eta = \beta y + \beta' y'$ on a, b respectively.

Then $$\xi = \alpha\beta xy + \alpha\beta' xy' + \alpha'\beta x'y + \alpha'\beta' x'y',$$

$$S(xx'.\xi\eta) = \alpha\beta' T^2 x Sx'y' - \alpha'\beta T^2 x' Sxy.$$

Now we can suppose that the tensors of all points are unity: then $(\xi\eta)$ is at right angles to a if

$$\frac{\beta}{\beta'} = \frac{\alpha \cos\phi'}{\alpha' \cos\phi}.$$

But all our expressions are homogeneous, and we may therefore use this equation in the form

$$\beta = \alpha \cos\phi', \quad \beta' = \alpha' \cos\phi.$$

Now $$\cos(\xi\eta) = \frac{\alpha\beta\cos\phi + \alpha'\beta'\cos\phi'}{(\alpha^2+\alpha'^2)^{\frac{1}{2}}(\beta^2+\beta'^2)}.$$

Substituting the values of β, β', we get

$$\cos(\xi\eta) = \cos\phi\cos\phi' . \frac{(\alpha^2+\alpha'^2)^{\frac{1}{2}}}{(\alpha^2\cos^2\phi' + \alpha'^2\cos^2\phi)^{\frac{1}{2}}}.$$

This gives the perpendicular distance from b of any point in a. It follows (1) that ϕ, ϕ' are one the greatest and the other the least values of $\xi\eta$, and (2) that, if $\cos\phi = \pm\cos\phi'$, $\cos\xi\eta = \pm\cos\phi$.

If $\phi = \phi'$, a, b are said to be right parallel.

If $\phi + \phi' = \pi$, a, b are said to be left parallel.

We have now to express these conditions in terms of the coordinates of the lines.

We have $$Sab = S(xx'.yy') = \begin{vmatrix} Sxy & Sxy' \\ Sx'y & Sx'y' \end{vmatrix}$$
$$= \cos\phi\cos\phi',$$

$$T^2 ab = S(xx'yy'.xx'yy').$$

$$= \begin{vmatrix} T^2x & 0 & Sxy & 0 \\ 0 & T^2x' & 0 & Sx'y' \\ Sxy & 0 & T^2y & 0 \\ 0 & Sx'y' & 0 & T^2y' \end{vmatrix}.$$

Therefore $\sin^2 [ab] = \sin^2\phi \sin^2\phi'$.

But all angles are supposed less than π; we therefore have

$$\sin [ab] = \sin\phi \sin\phi'.$$

And we found $\cos (ab) = \cos\phi \cos\phi'$.

Therefore, if the lines are right parallel,

$$\sin [ab] + \cos (ab) = 1.$$

If they are left parallel,

$$\sin [ab] - \cos (ab) = 1.$$

Taking the first condition, we have

$$\frac{ab + aKb}{TaTb} = 1.$$

But $(a+Ka)(b+Kb) = 2(ab+aKb)$,

and $T(a+Ka)\,T(b+Kb) = 2Ta\,Tab$.

Therefore $\dfrac{(a+Ka)(b+Kb)}{T(a+Ka)\,T(b+Kb)} = 1$.

Now it is easy to see that the coordinates of $(a+Ka)$ are of the form $(a_1 a_2 a_3 a_1 a_2 a_3)$, and that those of $b+Kb$ are of the form $(b_1 b_2 b_3 b_1 b_2 b_3)$.

Therefore $\dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{(a_1^2+a_2^2+a_3^2)^{\frac{1}{2}}(b_1^2+b_2^2+b_3^2)^{\frac{1}{2}}} = 1$.

Therefore $0 = (b_2 a_3 - a_3 b_2)^2 + (b_3 a_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$.

That is, if a, b are real,

$$a_1 : a_2 : a_3 = b_1 : b_2 : b_3.$$

That is, $a + Ka = \lambda (b+Kb)$,

where λ is a scalar.

In precisely the same way, we get as the condition for left parallelism* $a - Ka = \lambda (b - Kb)$.

III.

If a is a motor, (aa) vanishes and $[aa]$ is called the pitch of the motor. If a, b are two motors, $[ab]$ is called their moment. If a is a motor, and $a - Ka = 0$, a is called a right vector; if $a + Ka = 0$, a is called a left vector.

Any motor can be expressed in one way only, as the sum of a right and a left vector.

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* If we use the absolute, there is no difficulty in obtaining these conditions geometrically.

For, let a be the given motor, α, β the required vectors: we have

$$\alpha+\beta = a,$$

and therefore $$K\alpha+K\beta = Ka,$$

that is, $$\alpha-\beta = Ka.$$

Therefore $$\alpha = \tfrac{1}{2}(a+Ka), \quad \beta = \tfrac{1}{2}(a-Ka).$$

If a motor is reduced to the form $\lambda b+\mu Kb$, where b is a line and λ, μ scalars, b is called the axis of the motor.

To find the axis of a given motor.

We are to have $$a = \lambda b+\mu Kb,$$

and therefore $$Ka = \lambda Kb+\mu b.$$

Therefore $$b = \frac{\lambda a-\mu Ka}{\lambda^2-\mu^2}.$$

Since b can be multiplied by any scalar, we can take

$$b = \lambda a-\mu Ka,$$

and then $$a^2 = 2\lambda\mu T^2 b,$$

$$T^2 a = (\lambda^2+\mu^2)\, T^2 b.$$

Therefore $$\frac{\lambda^2+\mu^2}{\lambda\mu} = \frac{2T^2 a}{a^2},$$

that is, $$\frac{\lambda}{\mu}+\frac{\mu}{\lambda} = \frac{2}{\sin\phi},$$

if ϕ is the pitch of a.

A solution of this is $$\frac{\lambda}{\mu} = \cot\tfrac{1}{2}\phi;$$

and then we can take $\lambda = \cos\tfrac{1}{2}\phi, \quad \mu = \sin\tfrac{1}{2}\phi.$

Therefore $$b = a\cos\tfrac{1}{2}\phi-Ka\sin\tfrac{1}{2}\phi.$$

The other solution $\lambda/\mu = \tan\tfrac{1}{2}\phi$ would simply change b and Kb into $-Kb$ and $-b$.

Now let a, a' be two motors: let β, β' be their axes and ϕ, ϕ' their pitches. Then

$$\beta = a\cos\tfrac{1}{2}\phi-Ka\sin\tfrac{1}{2}\phi,$$
$$\beta' = a'\cos\tfrac{1}{2}\phi'-Ka'\sin\tfrac{1}{2}\phi'.$$

Using these expressions to calculate $\beta\beta'$ and $S\beta\beta'$, and then solving for aa', and remembering that (as is easily proved)

$$T\beta = Ta\cos\phi, \quad T\beta' = Ta'\cos\phi',$$

we get $$\sin[aa'] = \sin[\beta\beta']\cos\tfrac{1}{2}(\phi-\phi')+\cos(\beta\beta')\sin\tfrac{1}{2}(\phi+\phi').$$

This equation corresponds to the expression

$$\Delta \sin \phi + (k+k') \cos \phi,$$

for the moment of two complexes in parabolic space.

IV.

The axes of two complexes α, β intersect at right angles if $S\alpha\beta=0$, $\alpha\beta = 0$. Every congruence contains two complexes satisfying these conditions. Let the congruence be referred to these complexes, so that, if γ is any complex of the congruence, we have

$$\gamma = x\alpha + y\beta.$$

Let the pitches of α, β, γ be A, B, Γ; let $(\alpha\gamma) = a$, $(\beta\gamma)=b$. Then, remembering that $S\alpha\beta = \alpha\beta = 0$, we get

$$\gamma^2 = x^2\alpha^2 + y^2\beta^2, \quad T^2\gamma = x^2 T^2\alpha + y^2 T^2\beta\,;$$
$$S\alpha\gamma = xT^2\alpha, \quad S\beta\gamma = yT^2\beta.$$

The last two equations give

$$x = \frac{T\gamma \cos a}{T\alpha}, \quad y = \frac{T\gamma \cos b}{T\beta},$$

$$\sin \Gamma = \frac{\gamma^2}{T^2\gamma} = \frac{x^2\alpha^2 + y^2\beta^2}{x^2T^2\alpha + y^2T^2\beta}$$
$$= \frac{\cos^2 a \sin A + \cos^2 b \sin B}{\cos^2 a + \cos^2 b}.$$

But
$$\cos^2 a + \cos^2 b = \frac{x^2T^2\alpha + y^2T^2\beta}{T^2\gamma} = 1.$$

Therefore
$$\sin \Gamma = \cos^2 a \sin A + \cos^2 b \sin B.$$

This answers to the expression

$$P = P_\alpha \cos^2 l + P_\beta \sin^2 l,$$

for the parameter of any complex of a congruence.*

V.

Since the condition that two screws may be reciprocal is linear in the coefficients of both, and since any screw can be expressed in terms of any six, the whole theory of reciprocal screws and of the screw set† remains true for elliptic space. It is also possible to extend the theory of the principal screws of inertia, if we make use of the following definitions and of the kinetic postulate involved in them:—

"Whatever may be the nature of the constraints acting on a rigid body, there is always a matrix α of the sixth order (called the funda-

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* Or for the pitch of any screw on a cylindroid. (*Theory of Screws*, p. 15.)

† I venture to use this word for the screw *complex*, as the word *complex* is used in Plücker's sense in this paper.

mental matrix) such that, if an impulsive wrench be applied to the body about a screw A, the body will begin to turn about the screw aA, which is called the instantaneous screw corresponding to the impulsive screw A: aA means that the coordinates of A are transformed by the matrix a.

"The body is said to have freedom of the n^{th} order if it is possible to determine a screw set of order $(6-n)$, such that, if B be any screw of the set, $aB=0$. The set of the n^{th} order reciprocal to this set is called the set corresponding to the freedom of the n^{th} order possessed by the rigid body."

This being so, it follows at once that, if the body is free, there are six screws satisfying the equation $(a-\lambda)A=0$, and that, if referred to these six screws, the coordinates of an impulsive screw are $(x_1, x_2, x_3, x_4, x_5, x_6)$, the coordinates of the corresponding instantaneous screw are $(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \lambda_4 x_4, \lambda_5 x_5, \lambda_6 x_6)$, where λ_1, &c. are the roots of $\text{Det}\,(a-\lambda)=0$.

And then the whole theory in §§ 52 to 58 of the *Theory of Screws* can be at once extended to elliptic space.

VI.

It is worth while to consider the theory of the set of the third order.

Let $\xi = \lambda\alpha+\mu\beta+\nu\gamma$ be any complex of the set: then ξ is a line if

$$(\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta \between \lambda\mu\nu)^2 = 0.$$

Now we may take $(\lambda\mu\nu)$ as point coordinates in a plane; and then we may say that the points of the plane answer to complexes of a threefold set, and that the points of a certain conic answer to the lines of the set: two points, conjugate with respect to the conic, answer to reciprocal complexes. We can, in precisely the same way, represent the complexes of a fourfold set as points in space.

VII.

Clifford has applied biquaternions to the free motion of a solid in elliptic space.* We can also employ Grassmann's methods. If the velocity system of the body is given by a motor a, the rates of change of the coordinates of a point x are given by the equation

$$\dot{x} = K(ax);\dagger$$

and then, if T is the kinetic energy,

$$\begin{aligned} 2T &= \Sigma\, T^2(ax)\, dx \\ &= A(a^2), \end{aligned}$$

* *Mathematical Papers*, pp. 378 to 384.

† There seem to be some mistakes in sign on p. 382, but they do not affect the final result.

where A is a symmetrical matrix: and Aa^2 denotes

$$(A \between a_1, a_2, a_3, a_4, a_5, a_6)^2.$$

The momentum of the body is given by

$$M = Aa.$$

And the equation of motion is $\dot{M} = 0$.

Clifford obtains the integral $T =$ constant; but two others can easily be found. It can be verified without difficulty that we have

$$\frac{1-a^2}{1-b^2-c^2}u^2 + \frac{1-b^2}{1-c^2-a^2}v^2 + \frac{1-c^2}{1-a^2-b^2} = \text{constant},$$

$$(1-a^2)\,a^2ux + (1-b^2)\,b^2vy + (1-c^2)\,c^2wz = \text{constant}.$$

VIII.

The whole theory admits of immediate extension to space of any odd number of dimensions. Let $2n-1$ be the number of dimensions: there is a form of order n which may be called a motor: the product of two motors is a scalar: if the square of a motor vanishes, it is called a rotor: the product of n forms of the first or $(2n-1)^{\text{th}}$ order is a rotor:* a motor has $\mu = \dfrac{2n!}{(n!)^2}$ coordinates: if the product of two motors vanishes, they are reciprocal: there are in general two rotors reciprocal to $(\mu-2)$ given motors: in particular there are two rotors reciprocal to $(\mu-2)/2$ given motors and their conjugates; if there is an infinite number, we have either

$$\Sigma\lambda\,(a+Ka) = 0 \quad \text{or} \quad \Sigma\lambda\,(a-Ka) = 0,$$

where the λ are scalars and the a are the given motors. The axis of a motor a is a rotor b, such that

$$a = \lambda b + \mu Kb.$$

It is obvious that the theory of screws can now be extended to space of $2n-1$ dimensions without any change in the notation or results.

IX.

It remains to explain the connection between the methods used in this paper and Clifford's Biquaternions: it is not possible to work out the correspondence between the two methods, any more than it is possible to work out in detail the correspondence between the *Ausdehnungslehre* and quaternions. The important point is, that Clifford's operator ω is the same as what is here denoted by K: if $\alpha = \alpha_1 i + \alpha_2 j + \alpha_3 k$, $\beta = \beta_1 i + \beta_2 j + \beta_3 k$, the motor that Clifford writes

* The converse is not true.

$\alpha+\omega\beta$ would be written $\alpha_1 e_1 e_4+\alpha_2 e_2 e_4+\alpha_3 e_3 e_4+\beta_1 e_2 e_3+\beta_2 e_3 e_1+\beta_3 e_1 e_2$ and denoted by a single symbol.

The difficulty in the application of biquaternions to parabolic space seems to arise from the definition of ω. Clifford's definition (p. 186) gives $\omega^2=0$: but there is nothing to prevent us from keeping to the definition $\omega=K$; that is to say, the symbol ω changes rotation about any axis into translation parallel to that axis, and translation parallel to any vector into rotation about a rotor equal to the vector and passing through a fixed point. The addition to Clifford's definition is contained in the words "and rotation .... a fixed point." With this definition we have $\omega^2=1$; we can introduce Clifford's operators

$$\xi=\frac{1+\omega}{2},\quad \eta=\frac{1-\omega}{2},$$

and then we have $\xi^2=\xi$, $\eta^2=\eta$, $\xi\eta=0$; we can write any motor in the form $\xi\alpha+\eta\beta$, and we have

$$\frac{\xi\alpha+\eta\beta}{\xi\gamma+\eta\delta}=\xi q+\eta r;$$

where $q=\alpha/\gamma$, $r=\beta/\delta$.

Using the notations of my paper "On the Application of Quaternions to the Theory of the Linear Complex and the Linear Congruence,"* we can write the complex $(\alpha\beta)$ in the form $\alpha+\omega\beta$, and then $\omega(\alpha,\beta)=(\beta,\alpha)$.

[I think it right to state that, when the above paper was written, I had not seen Mr. Cox's paper in the *Camb. Phil. Trans.*, and only knew of it through a reference in Prof. Cayley's British Association address.—*June*, 1884.]

On Contact and Isolation, a Problem in Permutations.

By Mr. H. Fortey, M.A.

[*Read Jan. 10th*, 1884.]

1. Let there be n letters, and suppose a of them to be α, b of them to be β, c of them to be γ, and so on, where $n=a+b+c+\&c.$; it is required to find the number of permutations of these n letters, taken all together, in which there are exactly r contacts of the α's, s contacts of the β's, t contacts of the γ's, and so on.

Def.—If a number of letters are ranged in a row, any two adjacent letters are *in contact*.

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* *Messenger of Mathematics*, t. xii., p. 129.