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Note on the Centres of Similitude of a Triangle of Constant Form inscribed in a Given Triangle. By JOHN GRIFFITHS, M.A.

Received April 10th, 1893. Read April 13th, 1893.

I propose in the following note to discuss the following propositions, viz.:—

(1) A triangle DEF inscribed in a given triangle ABC so as to be similar to another given one $A'B'C'$ belongs to some one of twelve systems of similar in-triangles—each system having a centre of similitude of its own.

(2) The centres of similitude of the twelve systems in question can be formed into two groups of six points, which lie, respectively, on two circles, inverse to each other with respect to the circumcircle ABC. If we use isogonal coordinates, the equations of these circles are

$$x \operatorname{cosec} A + y \operatorname{cosec} B + z \operatorname{cosec} C = \cot \omega + \cot \omega',$$

and $$\Sigma x \operatorname{cosec} A = \cot \omega - \cot \omega',$$

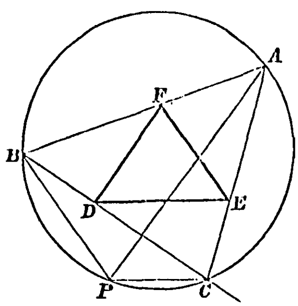
where ω and ω' are the Brocard angles of ABC and $A'B'C'$, respectively.

(3) The centre of similitude of any system of similar triangles inscribed in ABC, and having a common Brocard angle equal to that of $A'B'C'$, will lie on one or other of the above circles.

(4) As a particular case of the problem, I shall also notice the different systems formed by a triangle DEF inscribed in ABC so as to be either directly or inversely similar to it.

Section 1.

* When three points D, E, F are taken on the sides BC, CA, AB of a triangle ABC, then the circles CDE, AEF, BFD will intersect in one



or other of the pair of points whose isogonal coordinates are

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

and $$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad z = \frac{\sin(F-C)}{\sin F},$$

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* If $x, y, z, \alpha, \beta, \gamma$ denote respectively the isogonal and trilinear coordinates of a point, then

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{a\alpha + b\beta + c\gamma}{a\beta\gamma + b\gamma\alpha + c\alpha\beta},$$

where a, b, c are the sides of the triangle of reference ABC. Hence x, y, z satisfy the relation $\Sigma a(x - yz) = 0$, or $\Sigma (x - yz) \sin A = 0$,

it is seen that a circle is represented in this system of coordinates by a linear equation $$\lambda x + \mu y + \nu z = \delta.$$

See a note on "Secondary Tucker-Circles," recently communicated by me to the Society (*Proceedings*, Vol. xxiv., pp. 121 &c.).

where D, E, F denote the angles of the triangle DEF. These two points are inverse to each other with respect to the circumcircle ABC.

It follows from this theorem that a triangle DEF inscribed in ABC so as to be similar to $A'B'C'$, must belong to some one of the six pairs of similar in-triangles whose centres of similitude are the following points, viz.,

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

and $$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad z = \frac{\sin(F-C)}{\sin F},$$

where the angles D, E, F have the following values:—

(1) $D = A'$, $E = B'$, $F = C'$; (2) $D = A'$, $E = C'$, $F = B'$;

(3) $D = B'$, $E = C'$, $F = A'$; (4) $D = B'$, $E = A'$, $F = C'$;

(5) $D = C'$, $E = A'$, $F = B'$; (6) $D = C'$, $E = B'$, $F = A'$.

Section 2.

Since the equations $x = \dfrac{\sin(D+A)}{\sin D}$, &c.

give $$\Sigma x \operatorname{cosec} A = \Sigma \cot A + \Sigma \cot D,$$

it is easily seen that the centres of similitude of the six pairs of systems of similar in-triangles considered above must lie on the circles

$$\Sigma x \operatorname{cosec} A = \cot\omega + \cot\omega' \quad \text{and} \quad \Sigma x \operatorname{cosec} A = \cot\omega - \cot\omega',$$

where $\cot\omega = \Sigma \cot A$, and $\cot\omega' = \Sigma \cot D = \Sigma \cot A'$.

It is also evident that any point on either of these two circles can be considered to be the centre of similitude of a certain system of similar triangles inscribed in ABC which have a Brocard angle equal to that of $A'B'C'$.

This may be also stated as follows, viz.: The pedal triangle with respect to ABC of any point P on the above circles has the same Brocard angle as $A'B'C'$. The point P is, in fact, the centre of similitude of triangles inscribed in ABC so as to be directly similar to the pedal of P with respect to ABC.

SECTION 3. *Particular Case of the Problem.*

When the triangle $A'B'C'$ is similar to ABC, the centres of similitude of the six pairs of systems of in-triangles formed by DEF, similar to ABC, are the points

(1) $(2\cos A,\ 2\cos B,\ 2\cos C)$; $(0, 0, 0)$.

(2) $\left(2\cos A,\ \frac{\sin A}{\sin C},\ \frac{\sin A}{\sin B}\right)$; $\left(0,\ \frac{\sin(C-B)}{\sin C},\ \frac{\sin(B-C)}{\sin B}\right)$.

(3) $\left(\frac{\sin C}{\sin B},\ \frac{\sin A}{\sin C},\ \frac{\sin B}{\sin A}\right)$; $\left(\frac{\sin(B-A)}{\sin B},\ \frac{\sin(C-B)}{\sin C},\ \frac{\sin(A-C)}{\sin A}\right)$.

(4) $\left(\frac{\sin C}{\sin B},\ \frac{\sin C}{\sin A},\ 2\cos C\right)$; $\left(\frac{\sin(B-A)}{\sin B},\ \frac{\sin(A-B)}{\sin A},\ 0\right)$.

(5) $\left(\frac{\sin B}{\sin C},\ \frac{\sin C}{\sin A},\ \frac{\sin A}{\sin B}\right)$; $\left(\frac{\sin(C-A)}{\sin C},\ \frac{\sin(A-B)}{\sin A},\ \frac{\sin(B-C)}{\sin B}\right)$.

(6) $\left(\frac{\sin B}{\sin C},\ 2\cos B,\ \frac{\sin B}{\sin A}\right)$; $\left(\frac{\sin(C-A)}{\sin C},\ 0,\ \frac{\sin(A-C)}{\sin A}\right)$.

Six of these points lie on the Brocard circle

$$\Sigma x \operatorname{cosec} A = 2\cot\omega,$$

and the remaining six on the Lemoine line

$$\Sigma x \operatorname{cosec} A = 0,$$

which is the inverse of the circle in question with respect to the circumcircle ABC.

The two systems whose centres of similitude are

$$\left(\frac{\sin(B-A)}{\sin B},\ \frac{\sin(C-B)}{\sin C},\ \frac{\sin(A-C)}{\sin A}\right)$$

and $$\left(\frac{\sin(C-A)}{\sin C},\ \frac{\sin(A-B)}{\sin A},\ \frac{\sin(B-C)}{\sin B}\right),$$

i.e., the inverses of the Brocard points $\left(\frac{c}{b},\ \frac{a}{c},\ \frac{b}{a}\right)$, $\left(\frac{b}{c},\ \frac{c}{a},\ \frac{a}{b}\right)$ with respect to the circumcircle ABC, have been already noticed in a paper recently communicated by me to the Society.

In either of these two systems we have a series of in-triangles inversely similar to ABC, and this is also the case with regard to each of the systems whose centres of similitude are the points

$$\left(2\cos A,\ \frac{a}{c},\ \frac{a}{b}\right);\ \left(\frac{b}{c},\ 2\cos B,\ \frac{b}{a}\right);\ \left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right);\ (0,\ 0,\ 0).$$

The point (0, 0, 0) must be regarded as an infinitely distant one on the Lemoine line

$$\Sigma x \operatorname{cosec} A = 0,$$

and, corresponding to each of the remaining six centres of similitude, viz. :—

$$(2\cos A,\ 2\cos B,\ 2\cos C),\quad \left(\frac{b}{c},\ \frac{c}{a},\ \frac{a}{b}\right),$$

$$\left(\frac{c}{b},\ \frac{a}{c},\ \frac{b}{a}\right),\quad \left(0,\ \frac{\sin(C-B)}{\sin C},\ \frac{\sin(B-C)}{\sin B}\right),$$

$$\left(\frac{\sin(C-A)}{\sin C},\ 0,\ \frac{\sin(A-C)}{\sin A}\right),\quad \left(\frac{\sin(B-A)}{\sin B},\ \frac{\sin(A-B)}{\sin A},\ 0\right),$$

the inscribed triangles are directly similar to ABC; that is to say, every one of them may be brought by rotation in its own plane into a position wherein its sides are respectively parallel to the corresponding sides of ABC.

A few theorems with regard to the systems of similar in-triangles whose centres of similitude are the points

$$\left(2\cos A,\ \frac{a}{c},\ \frac{a}{b}\right),\quad \left(\frac{b}{c},\ 2\cos B,\ \frac{b}{a}\right),\quad \left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right),$$

i.e., the vertices of the second Brocard triangle of ABC, may be noticed here.

As I have already stated, the in-triangles for each of these points are directly similar to each other and inversely similar to ABC.

If we consider the system of in-triangles corresponding to the centre of similitude $\left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right)$, we have the following results, viz. :—

(1) A reference to the figure will show that if $D = B$, $E = A$, and $F = C$, the sides DF, FE, ED are respectively parallel to lines AP, BP, CP which meet in a point P on the circumcircle ABC.

(2) There is no difficulty in finding a geometrical construction for the centre of similitude in question. In fact, if O denote the centre of the circumcircle ABC, the point $\left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right)$, or say U, will be the intersection of the circle BAO and that which can be drawn

through B and C to touch AC. If the angle C be acute, the co-ordinates of U can be written in the forms

$$x = \frac{\sin\theta}{\sin(\theta+A)},\quad y = \frac{\sin\phi}{\sin(\phi+B)},\quad z = \frac{\sin\psi}{\sin(\psi+C)},$$

where $$\theta = C,\quad \phi = C,\quad \text{and}\quad \psi = \pi - 2C.$$

In this case it follows that the angles subtended at U by the sides BC, CA, AB are $\pi - C$, $\pi - C$, and $2C$.

(3) If $D = B$, $E = A$, and $F = C$, the centre of similitude U of the in-triangle DEF (see Fig.) has the following property with regard to DEF, viz., the isogonal point of U with reference to the triangle DEF has the same relation to DEF as U has to ABC.

(4) The centre of similitude of the triangles ABC, DEF, where $D = B$, $E = A$, $F = C$, is a variable point whose locus is the conic represented by the trilinear equation

$$\gamma^2 \sin 2C - \beta\gamma \sin A - \gamma\alpha \sin B + \alpha\beta \sin C = 0.$$

This curve passes through the vertices A, B, and the centroid and orthocentre of ABC. It also touches the symmedian lines AK, BK of the triangle ABC.

(5) If $p\alpha + q\beta + r\gamma = 0$ represent the axis of similitude of ABC, DEF, the envelope of this line seems to be a curve of the third class whose equation is

$$\{a^2p + b^2q + (a^2+b^2-c^2)\,r\}\,\{qr + pr - 2pq\}$$
$$+ \{(a^2+b^2-c^2)(p+q) + c^2r\}\,\{pq - r^2\} = 0.$$

(6) If $D = B$, $E = A$, $F = C$, the circumcircle DEF has double contact with the conic

$$\Sigma \sqrt{\sin D \,.\, \sin \overline{D+A}\; x} = 0,$$

or $$\sqrt{\alpha \sin B} + \sqrt{\beta \sin A} + \sqrt{\gamma \sin 2C} = 0.$$

(7) If we take $D = B$, $E = A$, $F = C$, and the other centre of similitude—say, U'—of the in-triangle DEF to be the intersection of the Lemoine line

$$\Sigma x \operatorname{cosec} A = 0$$

with the side AB of the triangle ABC, it will be easily seen that the circle DEF passes through two fixed points, viz., the vertex C and U'. The system whose centre of similitude is the centre of the circumcircle ABC, does not seem to call for special notice. The foci of its double contact inscribed conic are the centre of the circumcircle and orthocentre of ABC.

On a Problem of Conformal Representation.

By Prof. W. Burnside. Read April 13th, 1893.

1. *Introductory.*

The conformal representation of a plane finite polygon, no two of whose sides cross each other, on the half of an infinite plane or on a circle, is due originally to Schwartz. If $a_1\pi$, $a_2\pi$, ... $a_n\pi$ are the internal angles of the polygon in order, and if w and z are complex variables moving respectively in the plane of the polygon and in the infinite half-plane by the functional relation between which the conformal representation is given, it is shown first that

$$\frac{d}{dz}\left(\log\frac{dw}{dz}\right)$$

is everywhere finite, continuous, and uniform in the positive half of the z-plane, and that it takes real values along the axis of real quantities. Hence, by a general theorem, also due to Schwartz, it follows that this function can be continued across the real axis, so that it takes conjugate imaginary values at conjugate points; and therefore that its value at every point of the z-plane is determined by its discontinuities, which all lie on the real axis, and are the points of that axis corresponding to the angular points of the original polygon. If a_1, a_2, ... a_n are these points, the value of the function thus obtained is given by the equation

$$\frac{d}{dz}\left(\log\frac{dw}{dz}\right) = \sum_1^n \frac{a_r-1}{z-a_r}.$$

That the quantities a_1, a_2, &c. are actually determinable in terms of