NOTE ON A SOLUBLE DYNAMICAL PROBLEM

By L. J. ROGERS.

[Received March 10th, 1908.—Read March 12th, 1908.]

A DYNAMICAL system whose equation of energy in n independent co-ordinates $x_1, x_2, \ldots, x_n$ is

$$Y_1\dot{x}_1^2+Y_2\dot{x}_2^2+\ldots=\frac{P_1}{Y_1}+\frac{P_2}{Y_2}+\ldots, \tag{1}$$

where $Y_1, Y_2, \ldots, P_1, P_2$ are functions of $x_1, x_2, \ldots$, is satisfied by the equations

$$Y_1\dot{x}_1=\sqrt{P_1}, \quad Y_2\dot{x}_2=\sqrt{P_2}, \quad \ldots, \tag{2}$$

provided $P_1, P_2, \ldots$ are respectively functions of $x_1, x_2, \ldots$ only. For, in Lagrange's equation,

$$2\frac{d}{dt}Y_1\dot{x}_1-\dot{x}_1^2\frac{\partial Y_1}{\partial x_1}-\dot{x}_2^2\frac{\partial Y_2}{\partial x_1}-\ldots=\frac{\partial}{\partial x_1}\frac{P_1}{Y_1}+P_2\frac{\partial}{\partial x_1}\frac{1}{Y_2}+\ldots,$$

we have

$$2\frac{d}{dt}Y_1\dot{x}_1=2\frac{d}{dt}\sqrt{P_1}=\frac{\dot{P}_1}{\sqrt{P_1}}=\frac{\dot{x}_1}{\sqrt{P_1}}\frac{\partial P_1}{\partial x_1}=\frac{1}{Y_1}\frac{\partial P_1}{\partial x_1},$$

and

$$\dot{x}_r^2\frac{\partial Y_r}{\partial x_1}=-P_r\frac{\partial}{\partial x_1}\frac{1}{Y_r} \quad (r=1, 2, \ldots, n),$$

so that the equation is easily seen to be satisfied.

If $$Y_r=(X_r-X_1)(X_r-X_2)\ldots(X_r-X_{r-1})(X_r-X_{r+1})\ldots(X_r-X_n), \tag{3}$$

where $X_1, X_2, \ldots$ are respectively functions of $x_1, x_2, \ldots$ only, the solution of the dynamical problem is general and complete. For, if

$$P_r=F_r+a_0+a_1X_r+a_2X_r^2+\ldots+a_{n-2}X_r^{n-2}+CX_r^{n-1},$$

the equation of energy becomes

$$Y_1\dot{x}_1^2+\ldots=C+\frac{F_1}{Y_1}+\frac{F_2}{Y_2}+\ldots,$$

in which the constants $a_0, a_1, \ldots, a_{n-2}$ do not occur.

Moreover, equations (2) may be written

$$\left.\begin{aligned} &\frac{dx_1}{\sqrt{P_1}}+\frac{dx_2}{\sqrt{P_2}}+\ldots=0\\ &\frac{X_1\,dx_1}{\sqrt{P_1}}+\frac{X_2\,dx_2}{\sqrt{P_2}}+\ldots=0\\ &\ldots\quad\ldots\quad\ldots\quad\ldots\\ &\frac{X_1^{r-2}\,dx_1}{\sqrt{P_1}}+\frac{X_2^{r-2}\,dx_2}{\sqrt{P_2}}+\ldots=0\\ &\frac{X_1^{r-1}\,dx_1}{\sqrt{P_1}}+\frac{X_2^{r-1}\,dx_2}{\sqrt{P_2}}+\ldots=dt.\end{aligned}\right\} \tag{4}$$

The solution is therefore general, since we have $2n$ arbitrary constants, viz., a_0, a_1, ..., a_{n-2}, C, and the n constants obtained by integrating (4). It is also complete, as equations (4) are directly integrable.

If $x_1, x_2, \ldots$ are the generalized elliptic coordinates of a particle moving in n-fold space, the *vis viva* of the particle can be reduced to a form corresponding to that in (1), subject to condition (3). If the functions F are all zero, we shall get the conditions for a straight line.

Hence the equations of a straight line in n-fold elliptic coordinates are the algebraic solutions of the system (4) of Abelian differential equations.

We may, moreover, extend the generality of the solution by supposing that the *vis viva* contains terms such as $L\dot{\phi}^2$, in addition to those assumed above, L being a function of $x_1, x_2, \ldots, x_n$ only, of a form such that

$$\frac{1}{L}=\frac{1}{\phi_1(X_1)\,Y_1}+\frac{1}{\phi_2(X_2)\,Y_2}+\ldots+\frac{1}{\phi_n(X_n)\,Y_n}.$$

In such a case ϕ is an ignorable coordinate, leading to an integral of the type $L\dot{\phi}=h$, and as such is subject to the ordinary laws which modify the form of the energy-equation in which ignorable coordinates exist. When, however, as in this case, the ignorable coordinate occurs in one term only, through the square of its time-flux, in the kinetic energy, it is worth noticing that we may eliminate the variables entirely before applying the other Lagrangian equation, provided the corresponding term be placed in the force-function.

For instance, if $T+\frac{1}{2}L\dot{\phi}^2=U$ be the equation of energy where T, L, U are independent of ϕ, and θ any coordinate of the system, the true Lagrangian equation for θ is

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}}-\frac{\partial T}{\partial\theta}-\tfrac{1}{2}\dot{\phi}^2\,\frac{\partial L}{\partial\theta}=\frac{\partial U}{\partial\theta},$$

i.e.,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}}-\frac{\partial T}{\partial \theta}+\tfrac{1}{2}h^2\frac{\partial}{\partial \theta}\frac{1}{L}=\frac{\partial U}{\partial \theta},$$

which is the same as if the equation had been derived from an energy-equation

$$T = U-\tfrac{1}{2}\frac{h^2}{L}.$$

If, then, L has the form assumed above, the dynamical system has now a complete solution for all variables except ϕ.

The last, however, is given by

$$\frac{\dot{\phi}}{h}=\frac{1}{L}=\frac{1}{\phi_1(X_1)\,Y_1}+\frac{1}{\phi_2(X_2)\,Y_2}+\dots$$

$$=\frac{\dot{x}_1}{\phi_1(X_1)\sqrt{P_1}}+\frac{\dot{x}_2}{\phi_2(X_2)\sqrt{P_2}}+\dots,$$

so that

$$\frac{d\phi}{h}=\frac{dx_1}{\phi_1(X_1)\sqrt{P_1}}+\frac{dx_2}{\phi_2(X_2)\sqrt{P_2}}+\dots,$$

where now

$$P_r = F_r+a_0+a_1X_r+\dots+CX_r^{n-1}-\tfrac{1}{2}\frac{h^2}{\phi(X_r)}.$$

As an instance of the introduction of extra terms such as $\tfrac{1}{2}L\dot{\phi}^2$ into the kinetic energy, we may shew how Euler's problem of the motion of a particle attracted by Newtonian forces towards two fixed centres of force may be *fully* solved, *i.e.*, when the path of the particle is not confined to a plane.

If the force-centres lie at the foci of the ellipse

$$\frac{x^2}{a}+\frac{y^2}{b}=1 \quad (a>b),$$

and

$$x^2=\frac{(a+\lambda)(a+\mu)}{a-b},\qquad y^2=\frac{(b+\lambda)(b+\mu)}{b-a},$$

then the focal distances are $\sqrt{a+\lambda}\pm\sqrt{a+\mu}$.

Taking the coordinates of the particle as x, y in the plane containing the particle and the force-centres, and ϕ the angle this plane makes with a fixed plane with which it is coincident at some epoch, the energy-equation is

$$\tfrac{1}{2}(\dot{x}^2+\dot{y}^2+y^2\dot{\phi}^2)=C+\frac{\kappa_1}{r_1}+\frac{\kappa_2}{r_2},$$

where r_1, r_2 are the distances from the centres of force.

In the elliptic coordinates this becomes

$$\tfrac{1}{8}(\lambda-\mu)\frac{\dot{\lambda}^2}{(a+\lambda)(b+\lambda)}+\tfrac{1}{8}(\mu-\lambda)\frac{\dot{\mu}^2}{(a+\mu)(b+\mu)}-\tfrac{1}{2}\frac{(b+\lambda)(b+\mu)}{a-b}\dot{\phi}^2$$

$$=C+\frac{(\kappa_1+\kappa_2)\sqrt{a+\lambda}}{\lambda-\mu}+\frac{(\kappa_2-\kappa_1)\sqrt{a+\mu}}{\mu-\lambda}.$$

Here L, the coefficient of $\dot{\phi}^2$ satisfies the relation

$$\frac{1}{L}=-(a-b)\left\{\frac{1}{(b+\lambda)(\lambda-\mu)}+\frac{1}{(b+\mu)(\mu-\lambda)}\right\}.$$

so that a complete solution of the problem is obtained according to the method above indicated.*

When $n=3$, and X, Y, Z are elliptic coordinates, the *vis viva* takes the form

$$\tfrac{1}{4}(X-Y)(X-Z)\,\dot{x}^2+\tfrac{1}{4}(Y-Z)(Y-X)\,\dot{y}^2+\tfrac{1}{4}(Z-X)(Z-Y)\,\dot{z}^2,$$

where $\left(\frac{dX}{dx}\right)^2=X^3+pX^2+qX+r,\quad \left(\frac{dY}{dy}\right)^2=Y^3+pY^2+qY+r,$

and $$\left(\frac{dZ}{dz}\right)^2=Z^3+pZ^2+qZ+r.$$

Taking the equation of energy to be

$$(X-Y)(X-Z)\,\dot{x}^2+(Y-Z)(Y-X)\,\dot{y}^2+(Z-X)(Z-Y)\,\dot{z}^2$$

$$=C+\frac{P}{(X-Y)(X-Z)}+\frac{Q}{(Y-Z)(Y-X)}+\frac{R}{(Z-X)(Z-Y)},$$

we have seen that the dynamical system is completely soluble if P, Q, R are respectively functions of X, Y, Z.

These functions can be put into a simple general form when subjected to the condition that the forces are due to gravitation only.

* When the attractions are zero the path is a straight line whose equations are algebraic in λ, μ and circular functions of ϕ. There are three equations for the solution, involving elliptic integrals. The equation involving λ and μ, leads to the addition equation of elliptic integrals of the first kind; while the equations giving t and ϕ lead to the fundamental properties of those of the second and third kind respectively. There is probably no analytical process which yields these results so concisely.

For it is known that if the square of the velocity of a particle be denoted by

$$\frac{\dot{x}^2}{h_1^2}+\frac{\dot{y}^2}{h_2^2}+\frac{\dot{z}^2}{h_3^2},$$

where x, y, z are orthogonal coordinates, then Laplace's operation is equivalent to

$$h_1 h_2 h_3 \left(\frac{\partial}{\partial x}\frac{h_1}{h_2 h_3}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{h_2}{h_3 h_1}\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\frac{h_3}{h_1 h_2}\frac{\partial}{\partial z}\right),$$

i.e.,
$$(Y-Z)\frac{\partial^2}{\partial x^2}+(Z-X)\frac{\partial^2}{\partial y^2}+(X-Y)\frac{\partial^2}{\partial z^2}, \tag{5}$$

so that this operation must annihilate the function

$$C+\frac{P}{(X-Y)(X-Z)}+\frac{Q}{(Y-Z)(Y-X)}+\frac{R}{(Z-X)(Z-Y)}.$$

It will be found convenient to write X_1 for $\frac{dX}{dx}$, and X_2 for

$$\frac{d^2X}{dx^2}, = \tfrac{1}{2}(3X^2+2pX+q),$$

with corresponding meanings for Y_1, Z_1, Y_2, Z_2; and to use Ω for the operation (5), which, of course, only differs from Laplace's ∇^2 by a factor here unnecessary.

Now it can readily be shown that

$$\Omega\frac{X_1}{(X-Y)(X-Z)}$$

$$=\frac{\partial^2}{\partial x^2}\left(\frac{X_1}{X-Y}-\frac{X_1}{X-Z}\right)-X_1\frac{\partial^2}{\partial y^2}\frac{1}{X-Y}+X_1\frac{\partial^2}{\partial z^2}\frac{1}{X-Z}$$

$$=\frac{\partial}{\partial x}\left\{\frac{X_2}{X-Y}-\frac{X_2}{X-Z}-\frac{X_1^2}{(X-Y)^2}+\frac{X_1^2}{(X-Z)^2}\right\}$$
$$-X_1\frac{\partial}{\partial y}\frac{Y_1}{(X-Y)^2}+X_1\frac{\partial}{\partial z}\frac{Z_1}{(X-Z)^2}$$

$$=\frac{\partial}{\partial x}\left\{\frac{Y_2}{X-Y}-\frac{Z_2}{X-Z}-\frac{2Y_2}{X-Y}-\frac{Y_1^2}{(X-Y)^2}+\frac{2Z_2}{X-Z}+\frac{Z_1^2}{(X-Z)^2}\right\}$$
$$-X_1\frac{\partial}{\partial y}\frac{Y_1}{(X-Y)^2}+X_1\frac{\partial}{\partial z}\frac{Z_1}{(X-Z)^2}$$

$$=0, \tag{6}$$

so that it will be better to substitute UX_1 for P, since the coefficient of U in ΩUX_1 will then be zero.

Thus
$$\Omega \frac{UX_1}{(X-Y)(X-Z)}$$
$$= \frac{\partial^2}{\partial x^2} UX_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) + \dots$$
$$= X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) \frac{\partial^2 U}{\partial x^2} + 2 \frac{\partial U}{\partial x} \frac{\partial}{\partial x} X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right),$$

which, if we put $\frac{\partial U}{\partial x} = \frac{L}{X_1^2}$, becomes

$$X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) \frac{\partial}{\partial x} \frac{L}{X_1^2} + 2 \frac{L}{X_1^2} \frac{\partial}{\partial x} X_1 \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right)$$
$$= \left(\frac{1}{X-Y} - \frac{1}{X-Z}\right) \frac{\partial L}{\partial X} - 2L \left\{ \frac{1}{(X-Y)^2} - \frac{1}{(X-Z)^2} \right\}. \quad (7)$$

Hence, if M and N be functions respectively of y and z derived from Q and R, just as L is derived from P, we see that the final condition required is that the sum of the three expressions of which (7) is the type must be equal to zero.

The form of the relation being purely algebraic in X, Y, Z, we are naturally led to test for what value of n the assumptions $L = X^n$, $M = Y^n$, $N = Z^n$ lead to the satisfying of the identity. Such values are easily seen to be $n = 0, 1, 2, 3, 4$ and no other.

Finally, we see that the necessary value of P, *i.e.*, $X_1 \int \frac{L}{X_1^3} dX$, is

$$(X^3+pX^2+qX+r)^{\frac{1}{2}} \int \frac{AX^4+BX^3+CX^2+DX+E}{(X^3+pX^2+qX+r)^{\frac{3}{2}}} dX,$$

while Q and R are the same functions of Y and Z respectively. In consequence of (6) it is evident that the lower limit of these integrals may be taken as arbitrary.