are proportional to 1, $re^{\theta i}$, $re^{-\theta i}$, the required condition is that

$$r = \frac{2\,\mathrm{dn}\,\frac{4K}{n}}{1+\mathrm{cn}\,\frac{4K}{n}}, \quad \text{mod. } \cos\frac{1}{2}\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}.$$

It can be easily verified that for a triangle

$$r = 4\sin^2\frac{\theta}{2},$$

and for a quadrilateral $r+2\cos\theta = 0.$

On the Ideal Geometrical Form of Natural Cell-Structure.

By Mrs. Bryant, D.Sc.

[*Read March* 12*th*, 1885.]

Ideal natural cell-structure is not necessarily regular, in the strict geometrical sense of the word, as, by convention, it is used to denote solids with identically equal faces and solid angles. A cell-structure regular, in this conventional sense, would clearly consist of cells cubical in form; and such structure, as we shall see, is not natural.

The form of a natural structure is a logical result of its mode of genesis, and that form is ideal of which the mode of genesis is perfectly regular. Moreover, the original cell is spherical in form. Hence the solution of our problem turns upon the double question :—

1. If space is filled with equal spheres, and this space-ful of spheres is then crushed together symmetrically till the whole becomes a solid mass, what shape does each sphere ultimately assume?

2. If a homogeneous solid has equally efficient centres of excavation or absorption distributed uniformly in it, what is the ultimate form of the cells excavated? it being supposed that, when the excavating or absorbing agents cease their work, the walls of the cells are uniform in thickness, *i.e.*, the excavation is complete.

The second question is manifestly the counterpart of the first, and is answered in the answer to the first.

Our first step must be to determine the mode of arrangement of the spheres which, by the terms of the question, fill space in a natural, as well as a regular, manner. Now, there are three conceivable ways in which the spheres can be arranged regularly. Only one of these is, however, a natural, because a *mechanically stable*, arrangement, *i.e.*, the one in which the conditions of *maximum density* and *maximum stability* are fulfilled,—in which, therefore, the centres of all the spheres that touch any given sphere are at the minimum mutual distance of the common diameter.

The three geometrically possible arrangements may be conceived as follows, for convenience in deducing the corresponding forms of the crushed spheres:—

(1) The central sphere touches the *six* faces of a cube at their mid-points, and six surrounding spheres at the same points. No two of these six touch each other; the centres of any adjacent two are at a distance equal to $\sqrt{2}$ of the diameter, and they are not, therefore, situated similarly with respect to the central sphere and to one another. This may be called a cubical arrangement, and, if crushed, will yield cubes filling space without interstices.

(2) The central sphere passes through, and touches *eight* surrounding spheres at, the eight vertices of a cube. No two of these eight surrounding spheres touch each other, and the centres of any adjacent two are at a distance equal to $\frac{2}{\sqrt{3}}$ of the diameter. This arrangement is, therefore, denser than the first; but, like it, is deficient in the mutual support of its parts, and in the more perfect symmetry which belongs, as we shall see, to the third. If crushed, it will yield octohedra with tetrahedral interstices, these together filling space, as is well known; but, since there are interstices, the crushing cannot be complete. The proof of this need not detain us here.

(3) The central sphere touches the *twelve* edges of a cube, and twelve surrounding spheres, at the mid-points of the edges. The radius of the spheres is in this case equal to the semi-diagonal of the cube's face, and this is clearly equal to the distance between the points of contact of two adjacent spheres with the central sphere. The distance between their centres is, therefore, by similar triangles twice, the radius. Hence they touch; and thus the twelve surrounding spheres are in contact with each other three-and-three about the eight corners of the cube, while they are in contact four-and-four about the six faces of the cube.

This arrangement is of *maximum density*, since the surrounding spheres have their centres at the *minimum distance* of the common diameter. It is also the arrangement of *maximum stability*, because the *mutual support* of its parts is the greatest possible, as each of the surfaces bounding intervening spaces touches all the others. This is, therefore, the natural arrangement; and, as a matter of fact, if a quantity of shot be thrown into a box and shaken about freely, it will arrange itself in this way. Hence, too, the natural form of a pile of balls is a pyramid on a regular hexagonal, or, which is the same thing, an equilateral triangular, base. Piles on square bases are also common; but, as can easily be seen, the elementary arrangement is exactly the same; in the triangular pile, a face of the elementary tetrahedron of adjacent centres is horizontal, and in the square pyramidal pile, an edge.

While the mechanical instability of any but this dodecahedral arrangement of spheres determines it as the natural arrangement in a space-ful of spheres, its property of maximum density is a reason for considering it of fundamental importance in considering the natural mode of distribution of excavators or absorbents in a solid, since by it the maximum of excavation or absorption in a given space can be secured. Moreover, it is, as we have seen, more perfectly symmetrical than any other arrangement.

1. Considering, first, the case of the spheres to be symmetrically crushed together, let us limit our attention to the central sphere, which touches the twelve adjacent spheres at the mid-points of the twelve edges of a cube, its intermediate portions bulging out through the six faces of the cube. When the spheres are crushed together, these twelve points of contact move inwards along the radii, and the six intermediate portions are squeezed out into the over-arching spaces which lie between the points of contact of the surrounding spheres. Since there are four spheres round every face, these portions will be squeezed into four-sided pyramids, the faces of each being evidently conterminous with those of the adjacent pyramid, both being the ultimate position of the original plane of contact. Each, therefore, makes half a right angle with the face of the cube, the sum of the two being supplementary to the angle between the cube's faces. Hence, in the final position, we have the twelve points of contact represented by the mid-points of the edges of a smaller cube, and the intermediate portions heaped up into six square pyramids on the faces of the cube, whose faces make half a right angle with those of the cube. The form thus generated is a solid with twelve rhombic faces, the well-known rhombic dodecahedron.

There will be no intervening spaces in the mass of solids when the spheres are completely crushed together; because dodecahedra of this kind can be packed so as to fill space without interstices. To prove this, it is convenient to consider the dodecahedron as built up by dividing a cube into the six equal pyramids which have their vertices at the intersection of its diagonals, and placing these on the faces of an equal cube. The twenty-four faces of the figure reduce, as in the above, to twelve; since the diagonal plane of a cube makes half a right-angle with the cube's face, and hence two adjacent faces of any two pyramids lie in one plane and form a rhombus.

Six of the solid angles are enclosed by four planes, perpendicular to each other, two and two, since they were originally the diagonal planes of a cube. Therefore, four of the solids superposed on these faces at any vertex will just leave room for the vertex of another solid in the remaining space.

The other eight solid angles are situated at the vertices of the original cube, being enclosed by three plane angles, which are the obtuse angles of the rhombi, and are easily seen to be equal to those between the opposite diagonals of the cube. Now, by parallels, the diagonal of the dodecahedron through one of these vertices makes, with an edge of the dodecahedron, an angle equal to that which it makes with the opposite diagonal. Hence, the diagonal makes an exterior angle with each of the edges equal to the obtuse angle of the faces. When, therefore, three solids are superposed on the faces at such a vertex, their edges, meeting in that vertex, coincide along the diagonal of the central solid.

Space can, therefore, be filled with rhombic dodecahedra; and the crushed spheres, consequently, form a mass without intervening spaces.

We should expect to find this dodecahedral form in nature wherever originally spherical cells, packed together in the most natural or in the closest manner, have been subjected to uniform and complete pressure. The two conditions, (1) of initial symmetrical arrangement, and (2) of complete symmetrical pressure, are probably seldom fulfilled simultaneously, as a matter of fact; and so, nature transgresses her own ideal of naturalness in this as in other respects. In the centre of a mass of soap-bubbles their chance of fulfilment is perhaps at its best: but, in the fact that the bubbles tend to stick to one another, there is a disturbing element, even in the centre of the mass; and the difficulty of seeing the form within the mass is great.

2. Reverting now to our second question, it is, as before remarked, evident that the structure produced by complete and symmetrical pressure of spherical cells, symmetrically distributed in a space, is of the same form as that produced by the complete and symmetrical activity of equally efficient centres of excavation or absorption,

symmetrically distributed in a solid. The dodecahedral form is, therefore, the ideal natural form of cell-structure which is caused by absorption, as organic structures often are, or by excavation as in the case of the honey-comb cell.

In organic structure it is not likely that the ideal conditions are ever completely fulfilled, and frequently the actual conditions are quite different. But, as regards the honey-comb, we might reasonably expect that the bees, who are the cell-excavators, should by natural instinct distribute themselves *as densely as possible*, and with a considerable degree of regularity, and that their activities should be equal and symmetrical about the working parts of their bodies. The facts confirm this reasonable expectation. The bees distribute themselves, with apparent uniformity, at the two sides of a homogeneous cake of wax which has been previously deposited. In it they excavate cells, at doubtless uniform rates of work, and continue excavating till their work is as complete as possible, and the walls of the cells therefore of uniform thickness. Meanwhile, the excavated wax is used to build up higher the open cell walls. Hence, the cells ought to be elongated rhombic semi-dodecahedra; and this is just what they are, the axis of the cell corresponding to a diagonal of the primary cube, and the apex being one of the trihedral vertices of the dodecahedron. Each face at the apex fits exactly against one face of a cell in the opposite system. Each cell, therefore, is in contact with three cells of the opposite system.

It follows, from this last mentioned fact, that the bees must distribute themselves with maximum density, not only on each side separately, but on the two considered jointly. This, as a case of instinct, is certainly remarkable, but the possibilities of trial and error are sufficient to account for it. It is not unreasonable to expect that the bees should learn how to employ the largest possible number of themselves on a piece of wax to be excavated, this being a thing which they would *naturally try to do;* though it would be strange, in comparison, if they tried to effect those other two ends, of maximum economy in wax, and maximum strength of structure, which, as a matter of fact, they do effect. Whatever it is natural that they should try to do, it is natural that they should succeed in doing. And so, it is no less intelligible than remarkable, that our one clear example of nature fulfilling her own ideal of a natural cell-structure, should be the work of simple animal instinct in the construction of the honey-comb.

The following presents were received in the Recess:—

Carte-de-visite likeness of the Rev. J. J. Milne, M.A.

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