On Bessel's Functions, and Relations connecting them with Hyper-Spherical and Spherical Harmonics. By E. W. HOBSON, Sc.D. Received and read December 14th, 1893.

The Bessel's functions $J_m(r)$ of positive integral order m make their appearance in the product $\begin{matrix}\cos\\\sin\end{matrix} m\theta \,.\, J_m(r)$, which is a particular integral of the differential equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+V=0,$$

where $$x=r\cos\theta,\quad y=r\sin\theta.$$

The Bessel's functions $J_{m+\frac{1}{2}}(r)$ of order half an odd integer (known sometimes as spherical functions) make their appearance in the product

$$\frac{1}{\sqrt{r}}J_{m+\frac{1}{2}}(r)\,.\,Y_m(\theta,\phi),$$

which satisfies the equation $\nabla^2V+V=0$,

Y_m denoting a surface harmonic of order m. There is, however, another mode in which both kinds of functions may be considered to arise; it appears that, if we consider the equation in p variables corresponding to

$$\nabla^2V+V=0,$$

the function $\dfrac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$ plays the same part in relation to this equation that $J_0(r)$ does in relation to the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+V=0,$$

and thus that $\dfrac{J_m(r)}{r^m}$ may be considered to be the Bessel's function of zero order when there are $2m+2$ variables, and also that $\dfrac{J_{m+\frac{1}{2}}(r)}{r^{m+\frac{1}{2}}}$ will be the Bessel's function of zero order when $2m+3$ is the number of variables. In the present paper, various properties of the functions are developed from this point of view, the method having the advantage of dealing with both classes of functions at once. A considerable number of relations connecting the functions of different orders, both amongst themselves and with the corresponding hyper-spherical harmonics, are obtained, many of which are believed to be new. Many of these theorems arise from a comparison of different ways of expressing the same solution of one of the equations

$$\nabla^2 V = 0, \quad \nabla^2 V + V = 0,$$

the number of variables being unrestricted. Expressions are obtained for the zonal and tesseral harmonics as definite integrals involving Bessel's functions.

A Theorem concerning a certain Differential Operator.

1. In a paper* "On a Theorem in Differentiation and its Application to Spherical Harmonics," I proved a theorem which may be stated thus:—If $f_n(x_1, x_2, \dots x_p)$ denote a rational integral function of degree n of the p variables $x_1, x_2, \dots x_p$, then

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_p}\right)\phi(r)$$

$$= \left\{2^n \frac{d^n\phi}{d(r^2)^n} + \frac{2^{n-1}}{2}\frac{d^{n-1}\phi}{d(r^2)^{n-1}}\nabla_p^2 + \frac{2^{n-2}}{2.4}\frac{d^{n-2}\phi}{d(r^2)^{n-2}}\nabla_p + \dots\right\} f_n(x_1, x_2, \dots x_p) \quad \dots\dots\dots\dots\dots(1),$$

where $$r^2 = x_1^2 + x_2^2 + \dots + x_p^2,$$

and $$\nabla_p^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2}.$$

Now suppose that f_n satisfies the differential equation

$$\nabla_p^2 f_n = 0,$$

so that f_n is a spherical or hyper-spherical harmonic; in the above theorem the series on the right-hand side reduces to its first term, and we have, denoting by $S_n(x_1, x_2, \dots x_p)$ such a value of f_n,

$$S_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_p}\right)\phi(r) = 2^n \frac{d^n\phi(r)}{d(r^2)^n} S_n(x_1, x_2, \dots x_p) \quad \dots(2).$$

In this paper, I shall make use of the theorem (2), and I give here two examples of its application to the differential equations of physics.

* See *Proc. Lond. Math. Soc.*, Vol. XXIV., p. 67.

(*a*) It is well known that the real part of $\frac{1}{r} e^{\iota\kappa(at-r)}$ represents the potential due to a simple source of vibrations in a gas, the expression

$$V = \frac{1}{r} e^{\iota\kappa(at \mp r)}$$

satisfying the differential equation

$$\frac{\partial^2 V}{\partial t^2} = a^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right);$$

it follows from the linear character of the equation that

$$S_n \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left[\frac{1}{r} e^{\iota\kappa(at \mp r)} \right],$$

where $S_n(x, y, z)$ is a solid harmonic of degree n, also satisfies the differential equation; applying the theorem (1), we see that the function

$$S_n(x, y, z) \frac{d^n}{d(r^2)^n} \left[\frac{1}{r} e^{\iota\kappa(at \mp r)} \right],$$

and therefore also $S_n(x, y, z)\, e^{\iota\kappa at} \dfrac{J_{n+\frac{1}{2}}(\kappa r)}{(\kappa r)^{n+\frac{1}{2}}}$,

satisfies the differential equation. It has been remarked by Lord Rayleigh* that the potential of a multiple source, which is of the form

$$\frac{\partial^n}{\partial h_1 \partial h_2 \dots \partial h_n} \frac{e^{\iota\kappa(at-r)}}{r},$$

does not in general contain a spherical harmonic $S_n(x, y, z)$ as a factor, as it does in the case $\kappa = 0$, of the gravitation potential; the reason of this is that $\dfrac{\partial^n}{\partial h_1 \partial h_2 \dots \partial h_n}$ differs from some operator of the form

$$S_n \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

by a multiple of

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

and thus

$$S_n(x, y, z)\, e^{\iota\kappa at} \frac{J_{n+\frac{1}{2}}(\kappa r)}{(\kappa r)^{n+\frac{1}{2}}}$$

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* *Theory of Sound*, Vol. II., p. 216.

is the potential due to a combination of sources of degrees n, $n-2$, &c., whereas, in the case $\kappa = 0$, we have, since

$$\nabla^2 \frac{1}{r} = 0,$$

$$\frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n} \frac{1}{r} = S_n \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{1}{r} = \frac{S_n (x, y, z)}{r^{2n+1}},$$

omitting numerical factors.

(*b*) It is well known that the equation

$$\frac{\partial V}{\partial t} = \kappa \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right)$$

is satisfied by $V = \int_0^t \frac{1}{\{2\sqrt{\pi\kappa (t-\lambda)}\}^3} f(\lambda)\, e^{-r^2/[4\kappa(t-\lambda)]}\, d\lambda;$

in fact this is the temperature at a point of an infinite solid of conductivity κ, due to a source of intensity $f(t)$, commencing at time $t = 0$. We see that

$$S_n \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) V$$

satisfies the differential equation; thus the expression

$$S_n (x, y, z) \int_0^t \frac{1}{(t-\lambda)^{n+\frac{3}{2}}} f(\lambda)\, e^{-r^2/[4\kappa(t-\lambda)]}\, d\lambda$$

satisfies the differential equation; putting

$$a^2 = \frac{r^2}{4\kappa (t-\lambda)},$$

we see that the function

$$\frac{S_n (x, y, z)}{r^{2n+1}} \int_{r/2\sqrt{\kappa t}}^{\infty} a^{2n} \,.\, e^{-a^2} \,.\, f\left(t - \frac{r^2}{4\kappa a^2} \right) da$$

satisfies the differential equation, S_n denoting any solid harmonic. The particular case $n = 1$ gives the class of solutions which I have applied, in my paper* on "Synthetic Solutions in the Conduction of Heat," to certain problems of conduction.

* See *Proc. Lond. Math. Soc.*, Vol. XIX., p. 289.

Bessel's Functions of rank p.

2. The equation $\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_p^2}\right)V=0$ ....(3),

or $$\nabla_p^2 V=0,$$

has $$\frac{p(p+1)\ldots(p+n-1)}{n!}-\frac{p(p+1)\ldots(p+n-3)}{(n-2)!},$$

or $$(2n+p-2)\frac{(p-1)p(p+1)\ldots(p+n-3)}{n!},$$

distinct solutions which are rational integral functions of $x_1, x_2, \ldots x_p$ of degree n; we shall denote such a solution by $S_n(x_1, x_2, \ldots x_p)$.

Consider the equation $\nabla_p^2 V+V=0$ ....(4).

Suppose $x_1, x_2, \ldots x_p$ to be expressed in terms of the usual hyper-polar system of variables $r, \theta_1, \theta_2, \ldots \theta_{p-1}$, and suppose V to be a function of r only; the equation (3) reduces in this case to

$$\frac{d^2V}{dr^2}+\frac{p-1}{r}\frac{dV}{dr}+V=0.$$

Now, in Bessel's equation of order m,

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\left(1-\frac{m^2}{r^2}\right)u=0,$$

put $$u=r^m v;$$

then we have $$\frac{d^2v}{dr^2}+\frac{2m+1}{r}\frac{dv}{dr}+v=0;$$

thus we see that (4) is satisfied by

$$V=\frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}\quad\text{or}\quad\frac{Y_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}},$$

where $J_{\frac{1}{2}p-1}$, $Y_{\frac{1}{2}p-1}$ are the two Bessel's functions of order $\frac{1}{2}p-1$.

For simplicity we shall for the most part consider the function J only; many of the theorems will apply equally to Y.

The solutions of (4) which contain r only I shall call the Bessel's functions of zero order and rank p; thus the ordinary Bessel's functions $J_0(r)$, $Y_0(r)$ are of rank 2. These solutions of (4) may be denoted by

$$J_0(p, r),\quad Y_0(p, r),$$

where $$J_0(p, r) = \frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}, \quad Y_0(p, r) = \frac{Y_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} \qquad \ldots\ldots\ldots\ldots(5);$$

we thus have

$$J_0(2, r) = J_0(r), \quad J_0(3, r) = \frac{J_{\frac{1}{2}}(r)}{r^{\frac{1}{2}}} = \sqrt{\frac{2}{\pi}}\,\frac{\sin r}{r},$$

$$J_0(2m, r) = \frac{J_{m-1}(r)}{r^{m-1}}, \quad J_0(2m+1, r) = \frac{J_{m-\frac{1}{2}}(r)}{r^{m-\frac{1}{2}}};$$

it thus appears that Bessel's functions of even rank are expressible in terms of the ordinary Bessel's functions of integral order, and that those of odd rank are expressible in terms of the ordinary functions of order half an odd integer.

3. In order to obtain unsymmetrical solutions of (3), the theorem (2) may be applied; thus

$$S_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots \frac{\partial}{\partial x}\right) J_0(p, r) = 2^n S_n(x_1, x_2, \ldots x_p)\,\frac{d^p}{d(r^2)^p} J_0(p, r);$$

now $$\frac{d^n}{d(r^2)^n} J_0(p, r) = \frac{d^n}{d(r^2)^n}\,\frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} = \frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}},$$

leaving out a numerical factor, as the form only of the result is required.

We see therefore that (4) is satisfied by

$$V = S_n(x_1, x_2, \ldots x_p)\,\frac{J_{n+\frac{1}{2}p-1}}{r^{n+\frac{1}{2}p-1}},$$

that is, a solution of (4) is obtained by multiplying a solution S_n of (3) by $\frac{J_{n-1+\frac{1}{2}p}(r)}{r^{n-1+\frac{1}{2}p}}$. The cases $p = 2$, $p = 3$ of this theorem are well-known; thus, when $p = 2$, we have

$$(x \pm \iota y)^n \frac{J_n(r)}{r^n}, \quad \text{or} \quad {\cos \atop \sin} n\phi \,.\, J_n(r),$$

as a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + V = 0,$$

where $$x = r\cos\phi, \quad y = \rho\sin\phi$$

again, when $p = 3$, we have

$$S_n(x, y, z)\frac{J_{n+\frac{1}{2}}(r)}{r^{n+\frac{1}{2}}},$$

as a solution of the equation

$$\nabla^2 V + V = 0.$$

4. We have considered only the case in which S_n is a rational algebraical solution of (3); it may, however, be shown that, if S_n is any solution of (3) of degree n in the p variables, $S_n.\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}}$ is a solution of (4).

In (4), put $V = S_n u$; the equation then becomes, on the assumption that u is a function of r only,

$$\frac{2}{r}\frac{\partial u}{\partial r}\left(x_1\frac{\partial S_n}{\partial x_1}+x_2\frac{\partial S_n}{\partial x_2}+\ldots+x_p\frac{\partial S_n}{\partial x_p}\right)+S_n\nabla_p^2 u+S_n u = 0;$$

or, using the theorem

$$x_1\frac{\partial S_n}{\partial x_1}+\ldots+x_p\frac{\partial S_n}{\partial x_p} = nS_n,$$

this becomes $$\frac{d^2u}{dr^2}+\frac{p+2n-1}{r}\frac{du}{dr}+u = 0,$$

of which the solution is

$$u = A\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}}+B\frac{Y_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}};$$

thus, whatever the nature of a function S_n of the n^{th} degree may be, which is a solution of (2), if it be multiplied by $\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}}$, we obtain a solution of (3). It will be observed that n may be negative, so that $S_{-n}\frac{J_{-n+\frac{1}{2}p-1}(r)}{r^{-n+\frac{1}{2}p-1}}$ satisfies (3), S_{-n} denoting a harmonic of negative degree.

5. In a paper* on "Systems of Spherical Harmonics," I have given a table of some spherical harmonics of degree zero; any one of these,

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* See *Proc.*, Vol. XXII., p. 435.

when multiplied by $\frac{\sin r}{\sqrt{r}}$, or $\frac{\cos r}{\sqrt{r}}$, is a solution of the equation

$$\nabla^2 V + V = 0;$$

we obtain, for example, as solutions of this equation

$$\frac{x}{r+z}\cdot\frac{\sin r}{\sqrt{r}},\quad \frac{x}{r+z}\,\frac{\cos r}{\sqrt{r}},\quad \frac{(r\pm z)^m}{(x^2+y^2)^{\frac{1}{2}m}}\,\frac{\cos}{\sin}\,m\phi\cdot\frac{1}{\sqrt{r}}\,\frac{\sin}{\cos}\,r,$$

$$\frac{1}{\sqrt{r}}\log\sqrt{\frac{r-z}{r+z}}\,\frac{\sin}{\cos}\,r,\quad \frac{1}{\sqrt{r}}\tan^{-1}\frac{y}{x}\,\frac{\sin}{\cos}\,r.$$

The most general harmonic of degree n is

$$r^{2n+1}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}f\left(\frac{x\pm \iota y}{r+z}\right)\right\};$$

we obtain therefore, as solutions of

$$\nabla^2 V + V = 0,$$

the expressions
$$r^{n+\frac{1}{2}}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}f\left(\frac{x\pm \iota y}{r+z}\right)\right\}J_{n+\frac{1}{2}}(r),$$

$$r^{n+\frac{1}{2}}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}f\left(\frac{x\pm \iota y}{r+z}\right)\right\}Y_{n+\frac{1}{2}}(r),$$

where f denotes any function.

Zonal and Tesseral Hyper-Spherical Harmonics.

6. The systems of zonal and tesseral harmonics for p variables have been discussed by various writers; as, however, some of their properties are needed below, I give what appears to me to be a simple method of investigating their forms.

The potential equation (3) is satisfied by

$$V = \frac{1}{\left\{(x_1-a_1)^2+(x_2-a_2)^2+\ldots+(x_p-a_p)^2\right\}^{\frac{1}{2}p-1}},$$

when $p > 2$; and by

$$V = -\tfrac{1}{2}\log_e\left\{(x_1-a_1)^2+(x_2-a_2)^2\right\},$$

when $p = 2$.

Denoting the zonal harmonic of rank p by $P_n(p, \cos\theta)$, we have, as in the case $p=3$,

$$\left.\begin{aligned}\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p-1}} &= \Sigma h^n P_n(p, \cos\theta)\\ -\tfrac{1}{2}\log_e(1-2h\cos\theta+h^2) &= \Sigma h^n P_n(2, \cos\theta)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6).$$

Also, as in the ordinary case $p=3$, we find, if $p \geqq 3$,

$$P_n(p, \cos\theta) = \frac{(-1)^n r^{2n+1}}{\Pi(n)} \frac{\partial^n}{\partial x_p^n} \frac{1}{(x_1^2+x_2^2+\ldots+x_p^2)^{\frac{1}{2}p-1}},$$

where $$x_p/r = \cos\theta = \mu.$$

Performing the differentiation by means of the theorem (1), we have at once

$$P_n(p, \mu) = 2^n \frac{\Pi(n+\frac{1}{2}p-2)}{\Pi(n)\,\Pi(\frac{1}{2}p-2)}$$

$$\times\left\{\mu^n - \frac{n(n-1)}{2.2n+p-4}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n+p-4)(2n+p-6)}\mu^{n-4} - \ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

In the case $p=2$, we have

$$P_n(2, \cos\theta) = \frac{1}{n}\cos n\theta;$$

thus $$P_n(2, \mu)$$

$$= 2^{n-1}\left\{\mu^n - \frac{n(n-1)}{2.2n-2}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.2n-2.2n-4}\mu^{n-4} - \ldots\right\};$$

thus the factor of $P_n(2, \mu)$ in the bracket agrees with the series factor in (7).

7. The potential equation (3) may be written

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \ldots + \frac{\partial^2 V}{\partial x_{p-1}^2} + \left(\frac{\partial}{\partial x_p}\right)^2 V = 0,$$

so that when $\dfrac{\partial}{\partial x_p}$ is treated as a quantity, this equation is of the form (4) with $p-1$ variables; we see therefore that

$$V = S_n(x_1, x_2 \ldots x_{p-1}) \frac{J_{n+\frac{1}{2}(p-3)}\left(\sqrt{x_1^2+x_2^2+\ldots+x_{p-1}^2}\,\dfrac{\partial}{\partial x_p}\right)}{\left(\sqrt{x_1^2+x_2^2+\ldots+x_{p-1}^2}\,\dfrac{\partial}{\partial x_p}\right)^{n+\frac{1}{2}(p-3)}} \chi(x_p),$$

where $\chi(x_p)$ is any function of x_p, and α is any integer, satisfies the above equation. Let $\chi(x_p) = x_p^{n-\alpha}$, and suppose $\alpha \leq n$; we thus have, as a solution of (3),

$$S_\alpha(x_1, x_2, \dots x_{p-1})\left\{x_p^{n-\alpha} - (x_1^2+x_2^2+\dots+x_{p-1}^2)\frac{(n-\alpha)(n-\alpha-1)}{2.2\alpha+p-1}x_p^{n-\alpha-2}+\dots\right\},$$

where S_α denotes any solid hyper-harmonic of rank $p-1$. S_α may itself be expressed as the product of a harmonic of rank $p-2$, and a series; thus, proceeding in this way, we obtain, as a solution of (3), an expression of the form of the product

$$L(p, n, \alpha, x_p)\, L(p-1, \alpha, \beta, x_{p-1}) \dots L(3, \kappa, \lambda, x_3)(x_1 \pm \iota x_2)^\lambda \dots\dots (8),$$

where $L(p-r, \eta, \theta, x_{p-r})$ denotes the series

$$x_{p-r}^{\eta-\theta} - \frac{(\eta-\theta)(\eta-\theta-1)}{2.2\theta+p-r-1}x_{p-r}^{\eta-\theta-2}(x_1^2+x_2^2+\dots+x_{p-r-1}^2)$$

$$+\frac{(\eta-\theta)(\eta-\theta-1)(\eta-\theta-2)(\eta-\theta-3)}{2.4.2\theta+p-r-1.2\theta+p-r+1}x_{p-r}^{\eta-\theta-4}(x_1^2+x_2^2+\dots+x_{p-r-1}^2)^2 - \dots$$

If different integral values are assigned to $\alpha, \beta, \dots \kappa, \lambda$, such that

$$n \geq \alpha \geq \beta \dots \geq \kappa \geq \lambda,$$

the form (8) expresses various hyper-spherical harmonics of degree n. It is easy to show that (8) gives a complete set of harmonics of degree n; the $p-1$ quantities $n-\alpha$, $\alpha-\beta$, ... $\kappa-\lambda$, λ, are capable of all positive integral values (including zero) which are such that their sum is n; corresponding to any choice of these numbers we have two harmonics given by (8), except when $\lambda = 0$, in which case only one harmonic is given; the number of solutions in which $\lambda = 0$ is the number of ways in which the $p-2$ numbers $n-\alpha$, $\alpha-\beta$ ... κ may be chosen so that their sum is n; it follows that the formula (8) represents

$$2\frac{(p-1)p(p+1)\dots(p+n-2)}{n!} - \frac{(p-2)(p-1)\dots(p+n-3)}{n!}$$

distinct harmonics; this number is equal to

$$(p+2n-2)\frac{(p-1)p(p+1)\dots(p+n-3)}{n!},$$

which, as we have seen, in Section 2, is the number of independent harmonics of degree n and rank p. It has thus been shown that all the hyper-spherical harmonics of degree n are included in (8); these correspond to the system of zonal, tesseral, and sectorial harmonics in the ordinary case $p = 3$.

Expansion of an Exponential Function.

9. The differential equation (4) is satisfied by

$$V = e^{\iota x_p} = e^{\iota r\cos\theta};$$

it follows that $e^{\iota r\cos\theta}$ can be expanded in the form

$$\sum_0^\infty a_n r^n P_n(p, \cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{n+\frac{1}{2}p-1}},$$

since Bessel's functions of the second kind are infinite when $r = 0$, and therefore cannot occur. We have therefore

$$e^{\iota r\cos\theta} = \sum_0^\infty a_n P_n(p, \cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}},$$

where a_n are constants to be determined; substituting the value for $P_n(p, \cos\theta)$ given by (7), and equating the coefficients of $r^n\cos^n\theta$ on both sides of the equation, we obtain

$$\frac{\iota^n}{\Pi(n)} = a_n\frac{1}{2^{n+\frac{1}{2}p-1}\Pi(n+\frac{1}{2}p-1)}2^n\frac{\Pi(n+\frac{1}{2}p-2)}{\Pi(n)\Pi(\frac{1}{2}p-2)};$$

thus $$a_n = \iota^n . 2^{\frac{1}{2}p-1}(n+\tfrac{1}{2}p-1)\Pi(\tfrac{1}{2}p-2),$$

whence we obtain the result

$$e^{\iota r\cos\theta} = 2^{\frac{1}{2}p-1}\Pi(\tfrac{1}{2}p-2)\sum_{n=0}^{n=\infty}\iota^n(n+\tfrac{1}{2}p-1)P_n(p, \cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} \quad \ldots(9).$$

Two cases of (9) are well known; putting $p = 3$, we have

$$e^{\iota r\cos\theta} = \sqrt{2\pi}\sum_0^\infty \iota^n(n+\tfrac{1}{2})P_n(\cos\theta)\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}};$$

again, putting $p = 2$, and taking account of the exceptional form of $P_n(2, \cos\theta)$, we have

$$e^{\iota r\cos\theta} = \Sigma\iota^n\cos n\theta . J_n(r).$$

Addition Theorems for Bessel's Functions.

10. Since $J_0(p, r)$ satisfies the equation (4), it follows that

$$J_0\left(p, \sqrt{r^2+r'^2-2rr'\cos\theta}\right)$$

satisfies the same equation, for

$$r^2+r'^2-2rr'\cos\theta = x_1^2+x_2^2+\ldots+(x_p-r')^2.$$

We see that if $\quad J_0\left(p, \sqrt{r^2+r'^2-2rr'\cos\theta}\right)$

is expanded in a series of the functions $P_n(p, \cos\theta)$, the coefficients must contain $\dfrac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$ and $\dfrac{J_{n+\frac{1}{2}p-1}(r')}{r'^{\frac{1}{2}p-1}}$ as factors; thus

$$J_0\left(p, \sqrt{r^2+r'^2-2rr'\cos\theta}\right)$$

$$= J_0(p, r)\, J_0(p, r') + \sum_{n=1} \beta_n \frac{J_{n+\frac{1}{2}p-1}(r)\, J_{n+\frac{1}{2}p-1}(r')}{(rr')^{\frac{1}{2}p-1}} P_n(p, \cos\theta),$$

where β_n is a constant to be determined. Write

$$R = \sqrt{r^2+r'^2-2rr'\cos\theta},$$

and differentiate both sides of the equation n times with respect to $\cos\theta$, remembering that

$$d(R^2) = -2rr'd(\cos\theta);$$

thus

$$(-2rr')^n \frac{d^n}{d(R^2)^n} \frac{J_{\frac{1}{2}p-1}(R)}{R^{\frac{1}{2}p-1}} = \beta_n \frac{J_{n+\frac{1}{2}p-1}(r)\, J_{n+\frac{1}{2}p-1}(r')}{(rr')^{\frac{1}{2}p-1}} \frac{d^n P_n(p, \cos\theta)}{d(\cos\theta)^n} + \ldots;$$

divide both sides of the equation by r'^n, and then put $r'=0$; we get

$$(-2r)^n \frac{d^n}{d(r^2)^n} \frac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$$

$$= \beta_n \frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} \frac{1}{2^{n+\frac{1}{2}p-1}\,\Pi(n+\frac{1}{2}p-1)} \frac{2^n\Pi(n+\frac{1}{2}p-2)}{\Pi(\frac{1}{2}p-2)}.$$

Now $\quad (-2r)^n \dfrac{d^n}{d(r^2)^n} \dfrac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}} = \dfrac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}};$

hence we find $\quad \beta_n = 2^{\frac{1}{2}p-1}(n+\frac{1}{2}p-1)\,\Pi(\frac{1}{2}p-2);$

we thus obtain the theorem

$$\frac{J_{\frac{1}{2}p-1}(R)}{R^{\frac{1}{2}p-1}} = \frac{1}{(rr')^{\frac{1}{2}p-1}} \{J_{\frac{1}{2}p-1}(r) J_{\frac{1}{2}p-1}(r')$$
$$+ 2^{\frac{1}{2}p-1}\Pi(\tfrac{1}{2}p-2) \sum_1^\infty (n+\tfrac{1}{2}p-1) J_{n+\frac{1}{2}p-1}(r) J_{n+\frac{1}{2}p-1}(r') P_n(p, \cos\theta)\} \quad \ldots\ldots\ldots\ldots\ldots(10).$$

This theorem has been proved by other methods by Gegenbauer and by Sonnine.

If we put, in (10), $p = 2$, we obtain C. Neumann's addition theorem

$$J_0(R) = J_0(r) J_0(r') + 2\sum_1^\infty J_n(r) J_n(r') \cos n\theta.$$

Putting $p = 3$, we obtain the addition theorem for the spherical functions,

$$\frac{J_{\frac{1}{2}}(R)}{R^{\frac{1}{2}}}$$
$$= \frac{1}{(rr')^{\frac{1}{2}}} \{J_{\frac{1}{2}}(r) J_{\frac{1}{2}}(r') + 2^{\frac{1}{2}}\Pi(-\tfrac{1}{2}) \Sigma (n+\tfrac{1}{2}) J_{n+\frac{1}{2}}(r) J_{n+\frac{1}{2}}(r') P_n(\cos\theta)\},$$

or

$$\frac{\sin R}{R}$$
$$= \frac{\sin r}{r}\frac{\sin r'}{r'} + \sum_1^\infty (2n+1)(4\rho\rho')^n \frac{d^n}{d(\rho^2)^n}\frac{\sin r}{r} \cdot \frac{d^n}{d(\rho'^2)^n}\frac{\sin r'}{r'} . P_n(\cos\theta).$$

It will be observed that the formula (10) is a general addition theorem for the functions $\frac{J_m(R)}{R^m}, \frac{J_{m+\frac{1}{2}}(R)}{R^{m+\frac{1}{2}}}$.

The Evaluation of a Surface Integral.

11. It can be shown by means of the differential equation (2), in the same way as in the case $p = 3$, that, if S_m, S_n are hyper-spherical harmonics of degrees m and n,

$$\int S_m S_n \, d\omega = 0,$$

provided m and n are unequal, the integration being taken over the surface of a sphere of unit radius, the centre being at the origin. We shall have occasion to use the value of

$$\int P_n(p, \cos\theta) S_n \, d\omega,$$

which I proceed to calculate. Using the system of hyper-polar coordinates, given by

$$\begin{aligned}
x_1 &= r\sin\theta\sin\phi_1\sin\phi_2\ldots\sin\phi_{p-2},\\
x_2 &= r\sin\theta\sin\phi_1\sin\phi_2\ldots\cos\phi_{p-2},\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
x_{p-1} &= r\sin\theta\cos\phi_1,\\
x_p &= r\cos\theta,
\end{aligned}$$

we find, for an elementary volume, the expression

$$r^{p-1}\sin^{p-2}\theta\sin^{p-3}\phi_1\ldots\sin\phi_{p-3}\,dr\,d\theta\,d\phi_1\,d\phi_2\ldots d\phi_{p-2},$$

and thus $d\omega = \sin^{p-2}\theta\sin^{p-3}\phi_1\ldots\sin\phi_{p-3}\,d\theta\,d\phi_1\,d\phi_2\ldots d\phi_{p-2}$.

Using the equation

$$\frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}(p-2)}} = \sum_{n=0} P_n(p,\cos\theta)\,h^n,$$

we find

$$(p-2)\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}} = \Sigma\,(2n+p-2)\,P_n(p,\cos\theta)\,h^n;$$

multiplying both sides of this equation by S_n, and integrating over the surface of the sphere of unit radius, we have

$$\int P_n(p,\cos\theta)\,S_n\,d\omega = \frac{p-2}{2n+p-2}\int\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}}\,S_n\,d\omega,$$

where, on the right-hand side, the expression has its limiting value when $h=1$; we therefore have

$$\int P_n(p,\cos\theta)\,S_n\,d\omega = \frac{p-2}{2n+p-2}\,S_n(1)\int\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}}\,d\omega,$$

where $S_n(1)$ is the value of S_n at the pole of P_n.

Further,

$$\begin{aligned}
\int\frac{1-h^2}{(1-2h\cos\theta+h^2)^{\frac{1}{2}p}}\,d\omega &= \frac{1}{p-2}\int\Sigma h^n\,(2n+p-2)\,P_n\,d\omega\\
&= \int d\omega = \frac{2\pi^{\frac{1}{2}p}}{\Pi\left(\frac{p-2}{2}\right)};
\end{aligned}$$

thus we obtain the theorem

$$\int P_n(p,\cos\theta)\,S_n\,d\omega = \frac{p-2}{2n+p-2}\,\frac{2\pi^{\frac{1}{2}p}}{\Pi\left(\frac{p-2}{2}\right)}\,S_n(1) \quad\ldots\ldots(11).$$

A particular case of this theorem is

$$\int \{P_n(p, \cos\theta)\}^2 (1-\mu^2)^{\frac{1}{2}(p-3)}\, d\mu$$

$$= \frac{p-2}{2n+p-2} \sqrt{\pi}\, \frac{\Pi(p+n-3)\,\Pi\left(\frac{p-3}{2}\right)}{\Pi(n)\,\Pi\left(\frac{p-2}{2}\right)\Pi(p-3)} \quad \text{.........(12).}$$

12. Next, let us evaluate $\int e^{a_1x_1+a_2x_2+\ldots+a_px_p} S_n\, d\omega$ over the sphere of unit radius; this integral is, by changing the variables, reducible to an integral $\int e^{\beta\cos\theta} S_n\, d\omega$, where

$$\beta^2 = a_1^2+a_2^2+\ldots+a_n^2.$$

Substitute for $e^{\beta\cos\theta}$ its value given by putting $-\iota\beta$ for r in the expansion (9); then, remembering that

$$\int S_n P_m(p, \cos\theta)\, d\omega = 0,$$

except when $m = n$, we have

$$\int e^{\beta\cos\theta} S_n\, d\omega = 2^{\frac{1}{2}p-1}\,\Pi(\tfrac{1}{2}p-2)\,\iota^n\,(n+\tfrac{1}{2}p-1)\,\frac{J_{n+\frac{1}{2}p-1}(-\iota\beta)}{(-\iota\beta)^{\frac{1}{2}p-1}} \int S_n P_n\, d\omega$$

$$= \frac{1}{2^n}\,\frac{\Pi(\frac{1}{2}p-2)(n+\frac{1}{2}p-1)}{\Pi(n+\frac{1}{2}p-1)} \int S_n P_n\, d\omega$$

$$\times \beta^n \left\{1+\frac{\beta^2}{2.2n+p}+\frac{\beta^4}{2.4.2n+p.2n+p+2}+\ldots\right\}.$$

If $f(x_1, x_2, \ldots x_p)$ be a function which is finite and continuous throughout the volume of the hyper-sphere, we have

$$f(x_1, x_2, \ldots, x_p) = e^{x_1.\partial/\partial\xi_1+x_2.\partial/\partial\xi_2+\ldots+x_p.\partial/\partial\xi_p} f(\xi_1, \xi_2, \ldots \xi_p),$$

where $\xi_1, \xi_2, \ldots \xi_p$ are put equal to zero after the operation is performed. Let

$$a_1 = \frac{\partial}{\partial\xi_1}, \quad a_2 = \frac{\partial}{\partial\xi_2}, \ldots\; a_p = \frac{\partial}{\partial\xi_p};$$

we then have

$$\int S_n \,.\, f(x_1, x_2, \ldots x_p)\, d\omega = \frac{1}{2^n}\,\frac{\Pi(\frac{1}{2}p-2)}{\Pi(n+\frac{1}{2}p-2)}\,\frac{p-2}{2n+p-2}\,\frac{2\pi^{\frac{1}{2}p}}{\Pi\left(\frac{p-2}{2}\right)}$$

$$\times S_n\left(\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, \ldots \frac{\partial}{\partial\xi_p}\right)\left(1+\frac{\nabla^2}{2.2n+p}+\frac{\nabla^4}{2.4.2n+p.2n+p+2}\right)$$

$$f(\xi_1, \xi_2, \ldots \xi_p) \quad \text{..(13),}$$

where $$\nabla_\xi^2 = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \ldots + \frac{\partial^2}{\partial \xi_p^2},$$

and where $\xi_1, \xi_2, \ldots \xi_p$ are put equal to zero after the operations on the right-hand side are performed.

13. A large number of integrals are included as particular cases of (13); an important one for our purpose is the case in which

$$S_n = P_n(p, \cos\theta), \quad f = (x_1^2 + x_2^2 + \ldots + x_{p-1}^2)^k;$$

we then have, supposing n even, keeping the only term which does not vanish,

$$\int P_n(p, \cos\theta) \sin^{2k}\theta \, d\omega$$

$$= \frac{2\pi^{\frac{1}{2}p}}{2^{2k}} (-1)^{\frac{1}{2}n} \frac{\Pi\left(\frac{p+n}{2}-2\right)}{\Pi\left(\frac{p+n}{2}+k-1\right)\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{p}{2}-2\right)\Pi\left(k-\frac{n}{2}\right)}$$

$$\times \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_{p-1}^2}\right)^k (x_1^2 + x_2^2 + \ldots + x_{p-1}^2)^k.$$

It can easily be shown that

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_{p-1}^2}\right)^k (x_1^2 + x_2^2 + \ldots + x_{p-1}^2)^k = \frac{4^k \Pi(k)\, \Pi\left(k + \frac{p-3}{2}\right)}{\Pi\left(\frac{p-3}{2}\right)};$$

hence the value of the surface integral is

$$(-1)^{\frac{1}{2}n} 2\pi^{\frac{1}{2}p} \frac{\Pi\left(\frac{p+n}{2}-2\right)\Pi(k)\,\Pi\left(k+\frac{p-3}{2}\right)}{\Pi\left(\frac{p+n}{2}+k-1\right)\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{p}{2}-2\right)\Pi\left(k-\frac{n}{2}\right)\Pi\left(\frac{p-3}{2}\right)};$$

or $$\int_{-1}^{1} P_n(p, \mu)(1-\mu^2)^{k+\frac{1}{2}(p-3)}\, d\mu$$

$$= (-1)^{\frac{1}{2}n} \pi^{\frac{1}{2}} \frac{\Pi\left(\frac{p+n}{2}-2\right)\Pi(k)\,\Pi\left(k+\frac{p-3}{2}\right)}{\Pi\left(\frac{p+n}{2}+k-1\right)\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{p}{2}-2\right)\Pi\left(k-\frac{n}{2}\right)} \quad \ldots (14),$$

where n is an even integer.

In the particular case $p = 3$, we have

$$\int_{-1}^{1} P_n(\mu)(1-\mu^2)^k\,d\mu$$

$$= (-1)^{\frac{1}{2}n}\pi^{\frac{1}{2}} \frac{\Pi\left(\frac{n-1}{2}\right)\Pi(k)\,\Pi(k)}{\Pi\left(\frac{n+1}{2}+k\right)\Pi\left(\frac{n}{2}\right)\sqrt{\pi}\,\Pi\left(k-\frac{n}{2}\right)}$$

$$= (-1)^{\frac{1}{2}n} \frac{\Pi(k)\,\Pi(k)}{\Pi\left(\frac{n}{2}\right)\Pi\left(k-\frac{n}{2}\right)}\,\frac{2^{2k+1}\,\Pi(n)\,\Pi\left(\frac{n}{2}+k\right)}{\Pi(n+2k+1)\,\Pi\left(\frac{n}{2}\right)}$$

$$= (-1)^{\frac{1}{2}n}\, 2^{2k+1}\, \frac{\Pi(n)\,\Pi\left(k+\frac{n}{2}\right)\Pi(k)\,\Pi(k)}{\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{n}{2}\right)\Pi\left(k-\frac{n}{2}\right)\Pi(n+2k+1)} \quad \ldots(15).$$

The particular case (15) agrees with the value obtained by Mr. W. D. Niven.*

Expansions in Zonal Hyper-Harmonics.

14. Since $\quad e^{\pm x_p} J_0\left(p-1,\ \sqrt{x_1^2+x_2^2+\ldots+x_{p-1}^2}\right),$

or $\quad e^{\pm r\cos\theta} J_0(p-1,\ r\sin\theta),$

satisfies the equation $\quad \nabla_p^2 V = 0,$

it is clear that this function must be capable of being exhibited in a series of zonal hyper-harmonics of rank p of positive integral degrees; thus

$$e^{r\cos\theta}\frac{J_{\frac{1}{2}(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}(p-3)}} = \sum_{n=0}^{n=\infty} a_n r^n P_n(p,\cos\theta);$$

putting $\theta = 0$, we have

$$e^r \frac{1}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)} = \sum_{n=0}^{n=\infty} a_n r^n P_n(p,1);$$

* See *Phil. Trans.* for 1879, p. 285.

thus
$$a_n = \frac{1}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)\Pi(n)\,P_n(p,1)}$$
$$= \frac{\Pi(p-3)}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)\Pi(p+n-3)};$$

we therefore obtain the theorem

$$e^{r\cos\theta}J_{\frac{1}{2}(p-3)}(r\sin\theta) = \frac{\Pi(p-3)(r\sin\theta)^{\frac{1}{2}(p-3)}}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)}\sum_{n=0}^{\infty}\frac{r^n}{\Pi(p+n-3)}P_n(p,\cos\theta) \quad \text{............(16)};$$

on changing r into $-r$, we have

$$e^{-r\cos\theta}J_{\frac{1}{2}(p-3)}(r\sin\theta) = \frac{\Pi(p-3)(r\sin\theta)^{\frac{1}{2}(p-3)}}{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)}\Sigma\frac{(-1)^n r^n}{\Pi(p+n-3)}P_n(p,\cos\theta) \quad \text{............(17)}.$$

In the particular case $p=3$, we have

$$e^{r\cos\theta}J_0(r\sin\theta) = \sum_{n=0}^{n=\infty}\frac{r^n}{n!}P_n(\cos\theta) \quad \text{...............(18)},$$

$$e^{-r\cos\theta}J_0(r\sin\theta) = \sum_{n=0}^{n=\infty}\frac{(-1)^n r^n}{n!}P_n(\cos\theta) \quad \text{............(19)}.$$

On multiplying the series (18), (19) together, we have

$$\{J_0(r\sin\theta)\}^2 = \left\{1+\frac{r^2}{2!}P_2(\cos\theta)+\frac{r^4}{4!}P_4(\cos\theta)+\dots\right\}^2$$
$$-\left\{rP_1(\cos\theta)+\frac{r^3}{3!}P_3(\cos\theta)+\dots\right\}^2.$$

Relations connecting Bessel's Functions of Different Orders.

15. The equation $\nabla_p^2 V + V = 0$ .... (4)

is satisfied by $J_0(p-1, \sqrt{x_1^2+x_2^2+\dots+x_{p-1}^2})$,

which is independent of x_p; it is therefore clear that $J_0(p-1, r\sin\theta)$ can be exhibited in a series of the functions

$$P_n(p,\cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}};$$

thus $$\frac{J_{\frac{1}{2}(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}(p-3)}} = \sum_{n=0}^{\infty} \beta_n P_n(p, \cos\theta)\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}},$$

where β_n denotes constants which must be determined.

Multiply both sides of the equation by $P_n(p, \cos\theta)(1-\cos^2\theta)^{\frac{1}{2}(p-3)}$, and integrate with respect to $\cos\theta$ between the limits ± 1; we have then, in virtue of (12),

$$\beta_n \frac{p-2}{2n+p-2}\sqrt{\pi}\,\frac{\Pi(p+n-3)\,\Pi\left(\frac{p-3}{2}\right)}{\Pi(n)\,\Pi\left(\frac{p-2}{2}\right)\Pi(p-3)}\,\frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$$

$$= \int_{-1}^{1}\frac{J_{\frac{1}{2}(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}(p-3)}}P_n(p, \cos\theta)(1-\cos^2\theta)^{\frac{1}{2}(p-3)}\,d(\cos\theta);$$

equating the coefficients of r^n on both sides of this equation, we see that β_n is zero when n is odd, and that, when n is even,

$$\beta_n \frac{p-2}{2n+p-2}\,\frac{\sqrt{\pi}\,\Pi(p+n-3)\,\Pi\left(\frac{p-3}{2}\right)}{\Pi(n)\,\Pi\left(\frac{p-2}{2}\right)\Pi(p-3)}\,\frac{1}{2^{n+\frac{1}{2}p-1}\,\Pi(n+\frac{1}{2}p-1)}$$

$$= (-1)^{\frac{1}{2}n}\frac{1}{2^{n+\frac{1}{2}(p-3)}\,\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{p+n-3}{2}\right)}$$

$$\times\int_{-1}^{1}P_n(p, \cos\theta)(1-\cos^2\theta)^{\frac{1}{2}(p-3)+\frac{1}{2}n}\,d(\cos\theta)$$

$$= \frac{\sqrt{\pi}}{2^{n+\frac{1}{2}(p-3)}}\,\frac{\Pi\left(\frac{p+n}{2}-2\right)}{\Pi\left(n+\frac{p-2}{2}\right)\Pi\left(\frac{n}{2}\right)\Pi\left(\frac{p-4}{2}\right)}, \text{ by (14);}$$

hence $$\beta_n = \sqrt{2}\,\frac{\left(n+\frac{p-2}{2}\right)\Pi(n)\,\Pi(p-3)\,\Pi\left(\frac{p+n}{2}-2\right)}{\Pi\left(\frac{p-3}{2}\right)\Pi\left(\frac{n}{2}\right)\Pi(p+n-3)};$$

on changing n into $2n$, we have

$$\frac{J_{\frac{1}{2}(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}(p-3)}} = \sqrt{2}\,\frac{\Pi(p-3)}{r^{\frac{1}{2}p-1}}\sum_{n=0}^{\infty}\frac{\left(2n+\frac{p-2}{2}\right)\Pi(2n)\,\Pi\left(n+\frac{p-4}{2}\right)}{\Pi\left(\frac{p-3}{2}\right)\Pi(n)\,\Pi(2n+p-3)}$$

$$\times P_{2n}(p, \cos\theta)\,J_{2n+\frac{1}{2}p-1}(r) \quad\ldots\ldots\ldots\ldots(20).$$

In (20), put $p = 3$; we obtain

$$J_0(r\sin\theta) = \sqrt{\frac{2}{r}} \sum_0^\infty \frac{(2n+\frac{1}{2})\,\Pi\,(n-\frac{1}{2})}{\Pi\,(n)} P_{2n}(\cos\theta)\, J_{2n+\frac{1}{2}}(r)$$

$$= \sqrt{\frac{2\pi}{r}} \sum \frac{(2n+\frac{1}{2})(2n)!}{2^{2n+1}\,n!\,n!} P_{2n}(\cos\theta)\, J_{2n+\frac{1}{2}}(r) \quad \text{......(21).}$$

Again, put $p = 4$; we have then

$$\frac{J_{\frac{1}{2}}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}}} = \frac{2\sqrt{2}}{\pi}\,\frac{1}{r} \sum P_{2n}(4, \cos\theta)\, J_{2n+1}(r).$$

In the theorem (20), put $\theta = \frac{\pi}{2}$; we then have

$$J_{\frac{1}{2}(p-3)}(r) = \sqrt{\frac{2}{r}}\, \frac{\Pi\,(p-3)}{\Pi\left(\frac{p}{2}-2\right)\Pi\left(\frac{p-3}{2}\right)} \sum_{n=0}^{n=\infty} (-1)^n$$

$$\times \frac{\left(2n+\frac{p-2}{2}\right)\Pi\,(2n)\,\Pi\left(n+\frac{p}{2}-2\right)\Pi\left(n+\frac{p}{2}-2\right)}{\Pi\,(n)\,\Pi\,(n)\,\Pi\,(2n+p-3)} J_{2n+\frac{1}{2}p-1}(r) \text{...(22);}$$

this expresses a Bessel's function of integral order in a series of Bessel's functions of order half an odd integer, and conversely.

A particular case of (22) is when $p = 3$;

$$J_0(r) = \sqrt{\frac{2}{\pi r}} \sum_{n=0}^{n=\infty} (-1)^n \left\{\frac{\Pi\,(n-\frac{1}{2})}{\Pi\,(n)}\right\}^2 (2n+\tfrac{1}{2})\, J_{2n+\frac{1}{2}}(r) \text{......(23).}$$

Again, when $p = 4$, we have

$$\sin r = 2 \sum_{n=0}^{n=\infty} (-1)^n J_{2n+1}(r) \quad \text{..................(24).}$$

16. It is interesting to obtain an expansion corresponding to (23), by another method; on comparing the results obtained by the two methods, the evaluation of certain definite integrals is obtained.

We have

$$\int_0^\infty J_0(u)\,\frac{e^{\iota(u+r)}}{u+r}\,du = \int_0^\infty\int_0^\infty e^{(\iota-t)(u+r)} J_0(u)\,du\,dt;$$

on carrying out the integration with respect to u, the right hand becomes, by a known theorem, equal to

$$\int_0^\infty \frac{e^{(\iota-t)r}}{\sqrt{1+(t-\iota)^2}}\,dt, \quad \text{or} \quad e^{\iota r}\int_0^\infty \frac{e^{-tr}}{\sqrt{t^2-2t\iota}}\,dt;$$

putting $$\iota - t = \iota t',$$

we have $$\int_0^\infty J_0(u)\frac{e^{\iota(u+r)}}{u+r}\,du = -\int_1^{\iota\infty}\frac{e^{\iota t' r}}{\sqrt{t'^2-1}}\,dt';$$

the integral $\int_1^{\iota\infty}\frac{e^{\iota t' r}}{\sqrt{t'^2-1}}\,dt'$ is one of a class of integrals which represent Bessel's functions, and has been considered by Hankel,* and other writers. Using a result obtained by Hankel, we have

$$\int_0^\infty J_0(u)\frac{e^{\iota(u+r)}}{u+r}\,du = -\frac{1}{2}Y_0(r) + \frac{\iota\pi}{2}J_0(r);$$

on equating the real and imaginary parts on both sides of this equation, we obtain the formulæ

$$J_0(r) = \frac{2}{\pi}\int_0^\infty \frac{\sin(u+r)}{u+r}J_0(u)\,du \qquad (25),$$

$$Y_0(r) = -2\int_0^\infty \frac{\cos(u+r)}{u+r}J_0(u)\,du \qquad (26).$$

These formulæ were first obtained, by a different method, by Sonnine.

On substituting in (25), the value of $\frac{\sin(u+r)}{u+r}$, given by the addition formula of Section 10, we have

$$J_0(r) = \frac{\sin r}{r}\,\frac{2}{\pi}\int_0^\infty J_0(u)\frac{\sin u}{u}\,du$$
$$+\sum_{n=1}^{n=\infty}(-1)^n(2n+1)\frac{2}{\pi}(-2r)^n\frac{d^n}{d(r^2)^n}\frac{\sin r}{r}$$
$$\times\int_0^\infty J_0(u)(-2u)^n\frac{d^n}{d(u^2)^n}\frac{\sin u}{u}\,du,$$

which may be written

$$J_0(r) = \frac{1}{\sqrt{r}}J_{\frac{1}{2}}(r)\int_0^\infty J_0(u)\frac{J_{\frac{1}{2}}(u)}{\sqrt{u}}\,du$$
$$+\sum_{n=1}^{n=\infty}(-1)^n(2n+1)\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}\int_0^\infty J_0(u)\frac{J_{n+\frac{1}{2}}(u)}{\sqrt{u}}\,du.$$

* See *Mathematische Annalen*, Vol. I.

On comparing this expansion with (23), we see that

$$\int_0^\infty \frac{J_0(u)\,J_{n+\frac{1}{2}}(u)}{\sqrt{u}}\,du = 0,$$

when n is odd; and when n is even, writing $2n$ for n, we have

$$\int_0^\infty \frac{J_0(u)\,J_{2n+\frac{1}{2}}(u)}{\sqrt{u}}\,du \equiv \sqrt{\frac{2}{\pi}}\,\frac{2n+\frac{1}{2}}{2n+1}\left\{\frac{\Pi\left(n-\frac{1}{2}\right)}{\Pi(n)}\right\}^2 \quad\text{.........(27).}$$

It is clear that, by using the addition theorem for $\dfrac{\cos(u+r)}{u+r}$, the equation (26) could be applied to obtain a development of $Y_0(r)$ in Bessel's functions of the second kind, and of orders equal to half an odd integer.

Definite Integral Relations between Bessel's Functions.

17. Since $r\sin\theta = (x_1^2 + x_2^2 + \ldots + x_{p-1}^2)^{\frac{1}{2}}$,

we see that $J_0(p-1,\ r\sin\theta)$ satisfies the equation

$$\nabla_p^2 V + V = 0,$$

being a solution which is independent of x_p; it follows that the mean value of $J_0(p-1,\ r\sin\theta)$ taken over the sphere of radius r is a solution of

$$\nabla_p^2 V + V = 0,$$

which is such that it depends only on r, and is therefore, except for a constant factor, equal to $J_0(p, r)$, as it is clear that the Bessel's function of the second kind cannot be involved.

We have $\displaystyle\int_0^{\frac{1}{2}\pi} J_0(p-1,\ r\sin\theta)\sin^{p-2}\theta\,d\theta$

$$= \int_0^{\frac{1}{2}\pi} \frac{J_{\frac{1}{2}(p-3)}(r\sin\theta)}{(r\sin\theta)^{\frac{1}{2}(p-3)}}\sin^{p-2}\theta\,d\theta$$

$$= \frac{1}{2^{\frac{1}{2}(p-3)}\,\Pi\left(\frac{p-3}{2}\right)}\int_0^{\frac{1}{2}\pi}\left\{1 - \frac{r^2\sin^2\theta}{2.p-1} + \frac{r^4\sin^4\theta}{2.4.p-1.p+1} - \ldots\right\}\sin^{p-2}\theta\,d\theta$$

$$= \frac{\sqrt{\pi}}{2^{\frac{1}{2}(p-1)}\,\Pi\left(\frac{p-2}{2}\right)}\left\{1 - \frac{r^2}{2.p} + \frac{r^4}{2.4.p(p+2)} - \ldots\right\}$$

$$= \sqrt{\frac{\pi}{2}}\,\frac{J_{\frac{1}{2}(p-2)}(r)}{r^{\frac{1}{2}(p-2)}} = \sqrt{\frac{\pi}{2}}\,J_0(p, r);$$

it follows that

$$\sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}\pi} J_{\frac{1}{2}(p-3)}(r\sin\theta)\sin^{\frac{1}{2}(p-1)}\theta\,d\theta = \frac{J_{\frac{1}{2}p-1}(r)}{\sqrt{r}} \quad\text{.........(28).}$$

Putting $p = 2n+3$, we have

$$\sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}\pi} J_n(r\sin\theta)\sin^{n+1}\theta\,d\theta = \frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}} \quad\text{...........(29).}$$

Again, putting $p = 2n+2$, we have

$$\sqrt{\frac{2}{\pi}}\int_0^{\frac{1}{2}\pi} J_{n-\frac{1}{2}}(r\sin\theta)\sin^{n+\frac{1}{2}}\theta = \frac{J_n(r)}{\sqrt{r}} \quad\text{..............(30).}$$

A particular case of (29) is

$$\int_0^{\frac{1}{2}\pi} J_0(r\sin\theta)\sin\theta\,d\theta = \frac{\sin r}{r} \quad\text{..................(31);}$$

thus we have relations connecting Bessel's functions of orders differing by $\frac{1}{2}$.

The mode of verification of (28) shows that the relation holds even when p is not restricted to be a positive integer.

Expressions for Zonal and Tesseral Harmonics as Definite Integrals involving Bessel's Functions.

18. Let
$$\rho^2 = x_1^2 + x_2^2 + \ldots + x_{p-1}^2;$$

then we have the well known theorem

$$\frac{1}{r} = \frac{1}{(\rho^2+x_p^2)^{\frac{1}{2}}} = \int_0^\infty e^{-\lambda x_p} J_0(\lambda\rho)\,d\lambda.$$

Differentiating both sides of this equation m times with respect to ρ^2, we have

$$\frac{1}{r^{2m+1}} = \frac{1}{(\rho^2+x_p^2)^{m+\frac{1}{2}}} = \frac{2^m\Pi(m)}{\Pi(2m)}\int_0^\infty e^{-\lambda x_p}\lambda^m \frac{J_m(\lambda\rho)}{\rho^m}\,d\lambda \quad\text{......(32).}$$

In order to find a corresponding expression for the even powers of $\frac{1}{r}$, we have

$$\frac{1}{r^2} = \frac{1}{x_p^2+\rho^2} = \int_0^\infty e^{-\lambda x_p}\frac{\sin\lambda\rho}{\rho}\,d\lambda$$
$$= \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-\lambda x_p}\lambda\frac{J_{\frac{1}{2}}(\lambda\rho)}{(\lambda\rho)^{\frac{1}{2}}}\,d\lambda;$$

on differentiation m times with respect to ρ^2, we have

$$\frac{1}{r^{2m+2}} = \frac{1}{(x_p^2+\rho^2)^{m+1}} = \frac{2^{m+\frac{1}{2}}\Pi\left(m+\frac{1}{2}\right)}{\Pi(2m+1)}\int_0^\infty e^{-\lambda x_p}\lambda^{2m+1}\frac{J_{m+\frac{1}{2}}(\lambda\rho)}{(\lambda\rho)^{m+\frac{1}{2}}}d\lambda \ldots (33);$$

both the equations (32), (33) are included in the formula

$$\frac{1}{r^n} = \frac{1}{(x_p^2+\rho^2)^{\frac{1}{2}n}} = \frac{2^{\frac{1}{2}(n-1)}\Pi\left(\frac{n-1}{2}\right)}{\Pi(n-1)}\int_0^\infty e^{-\lambda x_p}\lambda^{\frac{1}{2}(n-1)}\frac{J_{\frac{1}{2}(n-1)}(\lambda\rho)}{\rho^{\frac{1}{2}(n-1)}}d\lambda \ldots (34);$$

the most important case of this theorem is obtained by putting

$$n = p-2,$$

in which case we have

$$\frac{1}{r^{p-2}} = \frac{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)}{\Pi(p-3)}\int_0^\infty e^{-\lambda x_p}\lambda^{p-3}J_0(p-1, \lambda\rho)\,d\lambda \ldots\ldots\ldots (35).$$

From the equation (35), we find, by differentiating n times with respect to x_p,

$$\frac{\partial^n}{\partial x_p^n}\frac{1}{r^{p-2}} = \frac{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)}{\Pi(p-3)}\int_0^\infty (-\lambda)^n e^{-\lambda x_p}\lambda^{p-3}J_0(p-1, \lambda\rho)\,d\lambda;$$

hence

$$\frac{P_n(p,\mu)}{r^{n+p-2}} = \frac{2^{\frac{1}{2}(p-3)}\Pi\left(\frac{p-3}{2}\right)}{\Pi(p-3)\,\Pi(n)}\int_0^\infty \lambda^{n+p-3}e^{-\lambda x_p}J_0(p-1, \lambda\rho)\,d\lambda \ldots (36);$$

this is the expression for the zonal hyper-harmonic of degree $-(n+p-2)$, in terms of Bessel's functions. In particular we have

$$\frac{P_n(\mu)}{r^{n+1}} = \frac{1}{\Pi(n)}\int_0^\infty \lambda^n e^{-\lambda z}J_0(\lambda\rho)\,d\lambda \ \ldots\ldots\ldots\ldots\ldots (37).$$

19. For the ordinary system of zonal and tesseral harmonics of rank 3, we have from the equation

$$\frac{1}{r} = \frac{1}{(z^2+\xi\eta)^{\frac{1}{2}}} = \int_0^\infty e^{-\lambda z}J_0(\lambda\sqrt{\xi\eta})\,d\lambda,$$

where $\xi = x + \iota y, \quad \eta = x - \iota y,$

$$\frac{\partial^m}{\partial \xi^m}\frac{\partial^{n-m}}{\partial z^{n-m}}\frac{1}{r} = (-1)^{n-m}\int_0^\infty \lambda^{n-m} e^{-\lambda z}\frac{\partial^m}{\partial \xi^m} J_0(\lambda\sqrt{\xi\eta})\, d\lambda$$

$$= (-1)^n \frac{1}{2^m} e^{-m\iota\phi}\int_0^\infty \lambda^n e^{-\lambda z} J_m(\lambda\rho)\, d\lambda.$$

Now $$\frac{\partial^m}{\partial \xi^m}\frac{\partial^{n-m}}{\partial z^{n-m}}\frac{1}{r} = (-1)^n \frac{e^{-m\iota\phi}}{r^{n+1}}\frac{(n-m)!}{2^m} P_n^m(\mu);$$

hence we have

$$\frac{P_n^m(\mu)}{r^{n+1}} = \frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda z} J_m(\lambda\rho)\, d\lambda \quad \text{..............(38)},$$

which gives an expression for the tesseral harmonic $\frac{P_n^m(\mu)}{r^{n+1}}\cos m\phi$ as a definite integral.

We have, putting $r = 1$,

$$P_n^m(\mu) = \frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda\cos\theta} J_m(\lambda\sin\theta)\, d\lambda \quad \text{.........(39)},$$

$$P_n(\mu) = \frac{1}{n!}\int_0^\infty \lambda^n e^{-\lambda\cos\theta} J_0(\lambda\sin\theta)\, d\lambda \quad \text{..............(40)}.$$

Some potential problems may be solved either in terms of the system of zonal and tesseral harmonics, or in terms of Bessel's functions; the formulæ of the present section afford the means of passing from one form of solution to the other.

20. It might be expected that, corresponding to (36) and (37), there should exist expressions for the positive harmonics $r^n P_n(p, \mu)$, $r^n P_n(\mu)$, as definite integrals involving Bessel's functions. If, in (16), we write λr for r, we see, by Cauchy's theorem, that

$$r^n P_n(p, \cos\theta)\frac{\Pi(p-3)}{2^{\frac{1}{2}(p-3)}\,\Pi\left(\frac{p-3}{2}\right)\Pi(n+p-3)}$$

$$= \frac{1}{2\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}}\frac{J_{\frac{1}{2}(p-3)}(\lambda r\sin\theta)}{(\lambda r\sin\theta)^{\frac{1}{2}(p-3)}}\, d\lambda,$$

or $$r^n P_n(p, \cos\theta)$$

$$= \frac{2^{\frac{1}{2}(p-5)}\,\Pi\left(\frac{p-3}{2}\right)\Pi(n+p-3)}{\Pi(p-3)\,\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}}\frac{J_{\frac{1}{2}(p-3)}(\lambda r\sin\theta)}{(\lambda r\sin\theta)^{\frac{1}{2}(p-3)}}\, d\lambda \quad \text{...(41)},$$

where the integral is taken along a complex path represented by a closed curve round the origin $\lambda = 0$.

In particular, we have

$$r^n P_n(\cos\theta) = \frac{n!}{2\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}} J_0(\lambda r\sin\theta)\, d\lambda \quad \text{............(42).}$$

It may be shown that

$$r^n P_n^m(\cos\theta) = \frac{(n-m)!}{2\pi\iota}\int \frac{e^{\lambda r\cos\theta}}{\lambda^{n+1}} J_m(\lambda r\sin\theta)\, d\lambda \quad \text{.........(43).}$$

The expressions (42), (43) correspond exactly to (37) and (38), the only difference being that in the latter the integrals are taken along a real path, and in the former along a complex path.

Expressions for the Zonal and Tesseral Harmonics of the Second Kind in Terms of Bessel's Functions.

21. Let us evaluate the definite integral

$$\int_0^\infty e^{-\lambda z} Y_0(\lambda\rho)\, d\lambda.$$

Substituting for $Y_0(\lambda\rho)$, the value

$$\int_0^\infty \cos(\lambda\rho\cosh u)\, du,$$

we have

$$\begin{aligned}\int_0^\infty e^{-\lambda z} Y_0(\lambda\rho)\, d\lambda &= \int_0^\infty\int_0^\infty e^{-\lambda z}\cos(\lambda\rho\cosh u)\, du\, dr \\ &= \int_0^\infty \frac{z d(2u)}{2z^2+\rho^2+\rho^2\cosh 2u} \\ &= \frac{1}{2\sqrt{z^2+\rho^2}}\left[\cosh^{-1}\frac{(2z^2+\rho^2)\cosh 2u+\rho^2}{(2z^2+\rho^2)+\rho^2\cosh 2u}\right]_0^\infty \\ &= \frac{1}{2\sqrt{z^2+\rho^2}}\left[\cosh^{-1}\frac{2z^2+\rho^2}{\rho^2}-\cosh^{-1}1\right] \\ &= \frac{1}{2\sqrt{z^2+\rho^2}}\log_e\left(\frac{2z^2+\rho^2}{\rho^2}+\frac{2z\sqrt{z^2+\rho^2}}{\rho^2}\right) \\ &= \frac{1}{\sqrt{z^2+\rho^2}}\log\frac{z+\sqrt{z^2+\rho^2}}{\rho};\end{aligned}$$

thus we have the theorem

$$\frac{1}{r}\log_e\sqrt{\frac{1+\mu}{1-\mu}}=\int_0^\infty e^{-\lambda z}\,Y_0(\lambda\rho)\,d\lambda \quad\ldots\ldots\ldots\ldots\ldots\ldots(44),$$

which corresponds to the known theorem

$$\frac{1}{r}=\int_0^\infty e^{-\lambda z}\,J_0(\lambda\rho)\,d\lambda.$$

From (44), we obtain, by differentiation n times with respect to r, the formula

$$\frac{Q_n(\mu)}{r^{n+1}}=\frac{1}{n!}\int_0^\infty \lambda^n e^{-\lambda z}\,Y_0(\lambda\rho)\,d\lambda \quad\ldots\ldots\ldots\ldots\ldots\ldots(45),$$

where $Q_n(\mu)$ is the zonal harmonic of the second kind. As in the case of the harmonics of the first kind, we find

$$\frac{Q_n^m(\mu)}{r^{n+1}}=\frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda z}\,Y_m(\lambda\rho)\,d\lambda \quad\ldots\ldots\ldots\ldots(46);$$

thus the tesseral harmonic $\frac{Q_n^m(\mu)}{r^{n+1}}\cos m\phi$ is expressed as a definite integral involving the elements $e^{-\lambda z}\,Y_m(\lambda\rho)\cos m\phi$.

Note on a Variable Seven-points Circle, analogous to the Brocard Circle of a Plane Triangle. By JOHN GRIFFITHS, M.A.

Received December 13th, 1893. Read December 14th, 1893.

The object of this note is to show that a seven-points circle can be constructed from a variable point U taken on one of three given circles connected with a triangle ABC.

1. On the side BC of a triangle ABC describe a circular arc BUC touching AC in C, and let U be any point on this arc. This con-