1.

De integralibus quibusdam definitis, seriebusque infinitis.

(Auct. *C. J. Malmstén*, prof. math. Upsaliens.)

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§. 1.

Si cognitam formulam

$$\int_0^\infty e^{-xz}\,\mathrm{Sin}\,uz\,.\,dz = \frac{u}{x^2+u^2}$$

per du multiplicemus atque ab $u=0$ integremus, habebimus

$$\tfrac{1}{2}.\qquad 2\int_0^\infty \frac{e^{-xz}-e^{-xz}\cos uz}{z}\,.\,dz = \log(x^2+u^2) - 2\log x,$$

unde, cum sit

$$\log x = \int_0^\infty \frac{e^{-z}-e^{-xz}}{z}\,dz,$$

obtinebitur

$$1.\qquad \log(x^2+u^2) = 2\int_0^\infty \frac{e^{-z}-e^{-xz}.\,\mathrm{Cos}\,uz}{z}\,dz.$$

Multiplicemus jam utrimque per

$$\frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\,.\,du, \qquad [a<\pi],$$

integratione inter $u=0$ et $u=\infty$ instituta, fit

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\log(x^2+u^2)\,du = 2\int_0^\infty \frac{dz}{z}\,.\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}(e^{-z}-e^{-xz}\,\mathrm{Cos}\,uz)\,du$$

unde beneficio notae formulae*) sequitur

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\log(x^2+u^2)\,du = \int_0^\infty \frac{e^{-z}\,dz}{z}\left[\mathrm{Tang}\,\tfrac{1}{2}a - \frac{2e^{-xz}\,\mathrm{Sin}\,a}{1+2e^{-z}\mathrm{Cos}\,a+e^{-2z}}\right]$$

sive, posito

$$e^{-z}=y, \qquad \text{unde} \qquad z=\log\frac{1}{y}, \qquad e^{-z} = -\,dy,$$

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*) Videas *Exerc. de Calc. Intégr. par Legendre.* Tom. 11. pag. 186.

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$$2.\qquad \int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\log(x^2+u^2)\,du = \int_0^1 \frac{dy}{\log\frac{1}{y}}\left(\operatorname{Tang}\tfrac{1}{2}a - \frac{2y^x \operatorname{Sin} a}{1+2y\operatorname{Cos} a+y^2}\right)$$
$$(a<\pi).$$

Facile vero demonstrari potest, si a in ratione qualibet commensurabili sit ad π, id est $a=\frac{m\pi}{n}$ (m et n numeri integri), integrale in dextero aequationis (2.) membro finite semper per functionem Γ exprimi posse. Nam sit brevitatis causa

$$T=\int_0^1 \frac{dy}{\log\frac{1}{y}}\left(\operatorname{Tang}\tfrac{1}{2}a - \frac{2y^x \operatorname{Sin} a}{1+2y\operatorname{Cos} a+y^2}\right),$$

differentiatione respectu ipsius x facta, prodit

$$3.\qquad \frac{dT}{dx} = 2\int_0^1 \frac{y^x \operatorname{Sin} a.\,dy}{1+2y\operatorname{Cos} a+y^2}.$$

In suppositione vero $a=\frac{m\pi}{n}$ cognitum est integrale*)

$$\int_0^1 \frac{y^x \operatorname{Sin} a.\,dy}{1+2y\operatorname{Cos} a+y^2} = \overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\sin ia.\left[\frac{d.\log\Gamma\left(\frac{x+n+i}{2n}\right)}{dx} - \frac{d.\log.\Gamma\left(\frac{x+i}{2n}\right)}{dx}\right]$$

ubi $m+n$ numerus impar est, et

$$\int_0^1 \frac{y^x \operatorname{Sin} a.\,dy}{1+2y\operatorname{Cos} a+y^2} = \overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\sin ia\left[\frac{d.\log\Gamma\left(\frac{x+n-i}{n}\right)}{dx} - \frac{d\log\Gamma\left(\frac{x+i}{n}\right)}{dx}\right]$$

ubi $m+n$ numerus par est; quod quidem in (3.) ductum, suppeditat

$$dT = 2\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia\,(d.\log\Gamma(\tfrac{x+n+i}{2n}) - d.\log\Gamma(\tfrac{x+i}{2n}))$$
$$(m+n=\text{num. imp.})$$

$$dT = 2\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia\,(d\log\Gamma(\tfrac{x+n-1}{n}) - d.\log\Gamma(\tfrac{x+i}{n}))$$
$$(m+n=\text{num. par.})$$

unde, integratione instituta,

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*) *Legendre Exerç.* Tom II. pag. 163—165. — Observamus errorem, qui loco cit. apud *Legendre* occurrit, ubi de formulis, a nobis in textu allatis, dicit, illam valere cum m numerus impar, hanc vero cum m numerus par est. At vero est statuendum, illam aut hanc valere, prout $m+n$ impar aut par est.

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$$4.\quad \begin{cases} T = C + 2\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia \log \left\{ \dfrac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2}\right)} \right\} \\ (m+n = \text{num. imp.}) \\ T = C' + 2\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia . \log \left\{ \dfrac{\Gamma\left(\frac{x+n+i}{n}\right)}{\Gamma\left(\frac{x+i}{n}\right)} \right\}. \\ (m+n = \text{num. par}) \end{cases}$$

Hinc in primum $x = r$ et deinde $x = s$ ponamus, subtrahendo obtinebimus hoc integrale

$$5.\quad \begin{cases} \int_0^1 \dfrac{dy}{\log \frac{1}{y}} \cdot \dfrac{y^r(1-y^{s-r})}{1+2y\operatorname{Cos} a+y^2} \\ = \operatorname{Cosec}. a. \overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia \log \left\{ \dfrac{\Gamma\left(\frac{s+n+i}{2n}\right).\Gamma\left(\frac{r+i}{2n}\right)}{\Gamma\left(\frac{r+n+i}{2n}\right).\Gamma\left(\frac{s+i}{2n}\right)} \right\} \\ (m+n = \text{num. imp.}) \\ \int_0^1 \dfrac{dy}{\log \frac{1}{y}} \cdot \dfrac{y^r(1-y^{s-r})}{1+2y\operatorname{Cos} a+y^2} \\ = \operatorname{Cosec}. a \overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia \log \left\{ \dfrac{\Gamma\left(\frac{s+n-i}{n}\right).\Gamma\left(\frac{r+i}{n}\right)}{\Gamma\left(\frac{r+n-i}{n}\right).\Gamma\left(\frac{s+i}{n}\right)} \right\} \\ (m+n = \text{num. par}). \end{cases}$$

Harum vero formularum ope valores etiam C et C' jam determinari possunt. Nam si primum $r = 0$ et $s = 1$ et deinde $r = 1$, $s = 2$ ponamus, subtractione facta invenimus

$$\int_0^1 \frac{dy}{\log \frac{1}{y}} \cdot \frac{(1-2y+y^2)\operatorname{Sin} a}{1+2y\operatorname{Cos} a+y^2} = \overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia \log \left\{ \frac{\left(\Gamma\left(\frac{n+i+1}{2n}\right)\right)^2.\Gamma\left(\frac{i+2}{2n}\right).\Gamma\left(\frac{i}{2n}\right)}{\left(\Gamma\left(\frac{i+1}{2n}\right)\right)^2.\Gamma\left(\frac{n+i}{2n}\right).\Gamma\left(\frac{n+i+2}{2n}\right)} \right\}$$

$$(m+n = \text{num. imp.})$$

$$\int_0^1 \frac{dy}{\log \frac{1}{y}} \cdot \frac{(1-2y+y^2)\operatorname{Sin} a}{1+2y\operatorname{Cos} a+y^2} = \overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin} ia \log \left\{ \frac{\left(\Gamma\left(\frac{n+1-i}{n}\right)\right)^2.\Gamma\left(\frac{i+2}{n}\right).\Gamma\left(\frac{i}{2n}\right)}{\left(\Gamma\left(\frac{i+1}{n}\right)\right)^2.\Gamma\left(\frac{n-i}{n}\right).\Gamma\left(\frac{n+2-i}{n}\right)} \right\}$$

$$(m+n = \text{num. par})$$

atque etiam e formulis (4.), posito $x = 1$, prodit

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$$\int_0^1 \frac{dy}{\log\frac{1}{y}} \cdot \frac{(1-2y+y^2)\,\mathrm{Sin}\,a}{1+2y\,\mathrm{Cos}\,a+y^2}$$

$$= (1+\mathrm{Cos}.a)\left\{C' + 2\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\log\left\{\frac{\Gamma\left(\frac{n+i+1}{2n}\right)}{\Gamma\left(\frac{i+1}{2n}\right)}\right\}\right\}$$

(m $+ n =$ num. imp.)

$$\int_0^1 \frac{dy}{\log\frac{1}{y}}\,\frac{(1-2y+y^2)\,\mathrm{Sin}\,a}{1+2y\,\mathrm{Cos}\,a+y^2}$$

$$= (1+\mathrm{Cos}\,a)\left\{C' + 2\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\log\left\{\frac{\Gamma\left(\frac{n+1-i}{n}\right)}{\Gamma\left(\frac{i+1}{n}\right)}\right\}\right\}$$

($m + n =$ num. par)

quae formulae, inter se comparatae, cum sit omnino

$$2\,\mathrm{Sin}\,ma\,\mathrm{Cos}\,a = \mathrm{Sin}\,(m+1)a + \mathrm{Sin}(m-1)a,$$

hos valores ipsorum C et C' suppeditant:

$$C = \mathrm{Tang}\,\tfrac{1}{2}a.\log 2n,$$

$$C' = \mathrm{Tang}\,\tfrac{1}{2}a\,\log.n.$$

Fit igitur, substitutione in (4.) facta,

$$T = \mathrm{Tang}\,\tfrac{1}{2}a.\log 2n + 2\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\log\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}$$

($m + n =$ num. imp.)

$$T = \mathrm{Tang}\,\tfrac{1}{2}a\,\log n + 2\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\log\left\{\frac{\Gamma\left(\frac{x+n+i}{n}\right)}{\Gamma\left(\frac{x+i}{n}\right)}\right\}$$

($m + n =$ num. par),

atque si brevitatis causa ponimus

$$6.\qquad L(a,x) = \int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}.\log(x^2+u^2)\,du,$$

ex formula (2.), posito $a = \frac{m\pi}{n}$ ubi $m < n$,

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$$7.\quad \left\{\begin{aligned} &L(a,x) = \text{Tang}\,\tfrac{1}{2}a\log 2n + 2\sum_{i=1}^{i=n-1}(-1)^{i-1}\,\text{Sin}\,ia\log\left\{\frac{\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\} \\ &\qquad (m+n = \text{num. par}) \\ &L(a,x) = \text{Tang}\,\tfrac{1}{2}a.\log n + 2\sum_{i=1}^{i=\frac{1}{2}(n-1)}(-1)^{i-1}\,\text{Sin}\,ia.\log\left\{\frac{\Gamma\left(\frac{x+n-i}{n}\right)}{\Gamma\left(\frac{x+i}{n}\right)}\right\} \\ &\qquad (m+n = \text{num. par}) \end{aligned}\right.$$

Pro $x = 1$ et $a = \frac{1}{2}\pi$, unde $m = 1$, $n = 2$, e priori accipies

$$8.\qquad \int_0^\infty \frac{\log(1+u^2)}{e^{\frac{1}{2}\pi u}+e^{-\frac{1}{2}\pi u}}\,du = \log\left(\frac{4}{\pi}\right).$$

§. 2.

Ut vero ex formulis (7.) nova quaedam integralia definita deducantur, demonstrare primum necesse est hasce formulas etiam pro $x = 0$ valere; quod quidem ut fiat, in promtu ponere sufficit, formulam (1.), unde tamquam e fonte illae derivant, justam etiam pro $x = 0$ manere.

Cognitae sunt formulae

$$\int_0^w e^{-xz}\,\text{Cos}\,uz\,dz = \frac{e^{-wx}(u\,\text{Sin}\,u.w - x\,\text{Cos}\,u.w)}{x^2+u^2} + \frac{x}{x^2+u^2},$$

$$\int_0^w e^{-uz}\,\text{Sin}\,xz\,dz = -\frac{e^{-w.u}(u\,\text{Sin}\,wx + x\,\text{Cos}\,wx)}{x^2+u^2} + \frac{x}{x^2+u^2},$$

unde subtractione facta habebimus

$$\int_0^\infty dz\,(e^{-xz}\,\text{Cos}\,uz - e^{-uz}\,\text{Sin}\,xz)$$
$$= \frac{e^{-wx}(u\,\text{Sin}\,u.w - x\,\text{Cos}\,u.w)}{x^2+u^2} + \frac{e^{-wu}(u\,\text{Sin}\,wx + x\,\text{Cos}\,wx)}{x^2+u^2}.$$

Jam si, multiplicatione per dx utrimque facta, integrale ab $x = 0$ ad $x = u$ sumitus, fit

$$\int_0^w \frac{dz}{z}\,(\text{Cos}\,uz - e^{-uz})$$
$$= e^{-wu}\int_0^u \frac{u\,\text{Sin}\,wx + x\,\text{Cos}\,wx}{x^2+u^2}\,dx + \int_0^u \frac{e^{-wx}(u\,\text{Sin}\,uw - x\,\text{Cos}\,uw)}{x^2+u^2}\,dx.$$

Facile vero patet pro valoribus ipsius w indefinite crescentibus integralia

$$e^{-wu}.\int_0^u \frac{u\,\text{Sin}\,wx + x\,\text{Cos}\,wx}{x^2+u^2}\,dx \quad \text{et} \quad \int_0^u \frac{e^{-wx}(u\,\text{Sin}\,uw - x\,\text{Cos}\,uw)}{x^2+u^2}\,dx$$

indefinite in nihilum convergere, unde fit ut

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$$\lim . \int_0^{w} \frac{dz}{z}(\operatorname{Cos} uz - e^{-uz}) = 0 \qquad (w = \infty)$$

id est

$$\int_0^\infty \frac{dz}{z}(\operatorname{Cos} uz - e^{-uz}) = 0.$$

Haec vero formula, si a cognita illa

$$\int_0^\infty \frac{e^{-z}-e^{-uz}}{z}\,dz = \log u$$

subtrahitur, obtinebitur

$$9. \qquad \int_0^\infty \frac{e^{-z}-\operatorname{Cos} uz}{z}\,dz = \log u$$

unde manifestum fit, formulam (1.), una cum omnibus ex ea derivantibus, etiam pro $x=0$ valere.

His ita demonstratis, sit jam in (7.) $x=0$, ponamusque

$$e^{\frac{\pi u}{n}} > y, \quad \text{unde} \quad u = \frac{n}{\pi}\log y,$$

cum sit $a = \frac{m\pi}{n}$, obtinebis

$$\int_0^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}}\left\{\log\left(\frac{n}{\pi}\right)\right\}dy$$

$$= \frac{\pi}{2n}.\operatorname{Tang}\frac{m\pi}{2n}.\log 2n + \frac{\pi}{n}.\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin}\frac{im\pi}{n}\log\left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\}$$

$$(m+n = \text{num. imp.})$$

$$\int_0^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y_{-n}}\left\{\log\left(\frac{n}{\pi}\right)+\log(\log y)\right\}dy$$

$$= \frac{\pi}{2n}\operatorname{Tang}\frac{m\pi}{2n}\log n + \frac{\pi}{n}.\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Sin}\frac{im\pi}{n}\log\left\{\frac{\Gamma\left(1-\frac{i}{n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\}$$

$$(m+n = \text{num. par}),$$

atque quia est

$$\int_0^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}}\,dy = -\int_0^1 \frac{z^{\frac{m}{2n}-\frac{1}{2}}-z^{-\frac{m}{2n}-\frac{1}{2}}}{1-z}\,dz = \frac{\pi}{2n}.\operatorname{Tang}\frac{m\pi}{2n},$$

etiam

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10. $$\begin{cases} \int_0^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}} \log(\log y)\,dy \\ = \frac{\pi}{2n}\,.\,\mathrm{Tang}\,\frac{m\pi}{2n} \log 2\pi + \frac{\pi}{n}\,.\,\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\mathrm{Sin}\,\frac{im\pi}{n} \log \left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\} \\ (m+n = \text{num. imp.}) \\ \int_0^\infty \frac{y^{m-1}-y^{-m-1}}{y^n-y^{-n}}\,.\log(\log y)\,dy \\ = \frac{\pi}{2n}\,\mathrm{Tang}\,\frac{m\pi}{2n} \log \pi + \frac{\pi}{n}\,.\,\underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}}(-1)^{i-1}\,.\,\mathrm{Sin}\,\frac{im\pi}{n} \log \left\{\frac{\Gamma\left(1-\frac{i}{n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\} \\ (m+n = \text{num. par}). \end{cases}$$

Hinc sequitur pro $m = 1$,

11. $$\begin{cases} \int_1^\infty \frac{y^{n-2}\log(\log y)\,dy}{1+y^2+y^4+\ldots+y^{2(n-1)}} \\ = \frac{\pi}{2n}\,.\,\mathrm{Tang}\,\frac{\pi}{2n} \log 2\pi + \frac{\pi}{n}\,.\,\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\mathrm{Sin}\,\frac{i\pi}{n}\,.\log \left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\} \\ (n = \text{num. par}) \\ \int_1^\infty \frac{y^{n-2}.\log(\log y)\,dy}{1+y^2+y^4+\ldots+y^{2(n-1)}} \\ = \frac{\pi}{2n}\,\mathrm{Tang}\,\frac{\pi}{2n} \log \pi + \frac{\pi}{n}\,.\,\underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}}(-1)^{i-1}\,\mathrm{Sin}\,\frac{i\pi}{n} \log \left\{\frac{\Gamma\left(1-\frac{i}{n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\} \\ (n = \text{num. imp.} \end{cases}$$

Ex. gr. posito in illa $n = 2$ et in hac $n = 3$, habebimus, si in hac y^2 in y mutatur,

12. $$\begin{cases} \int_0^\infty \frac{\log(\log y)\,dy}{1+y^2} = \frac{1}{2}\pi \log \left\{\frac{(2\pi)^{\frac{1}{2}}.\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right\} \\ \int_0^\infty \frac{\log(\log y)\,dy}{1+y+y^2} = \frac{\pi}{\sqrt{3}} \log \left\{\frac{(2\pi)^{\frac{1}{3}}.\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right\}. \end{cases}$$

§. 3.

Ex §. primo jam apparuit, transcendentem illam, quam per $L(a, x)$ designavimus, finite semper per Γ exprimi posse, cum a in ratione commensurabili ad π est. Nunc in hoc §. versabinur circa proprietates quasdam ejus maxime notandas. Sit igitur in priori formularum (7.) $x = \frac{1}{2}\pi$; tunc erit

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$$13. \qquad L(\tfrac{1}{2}\pi, x) = 2\log\left\{\frac{2\Gamma\left(\frac{x+3}{4}\right)}{\Gamma\left(\frac{x+1}{4}\right)}\right\},$$

unde, si $x+2$ loco ipsius x ponitur, addendo invenimus

$$14. \qquad L(\tfrac{1}{2}\pi, x+2) + L(\tfrac{1}{2}\pi, x) = 2\log(x+1).$$

Si vero in (13.) $2-x$ loco ipsius x substituimus, e relatione cognita

$$\Gamma(a).\Gamma(1-a) = \frac{\pi}{\operatorname{Sin} a\pi},$$

similiter addendo erit

$$15. \qquad L(\tfrac{1}{2}\pi, x) + L(\tfrac{1}{2}\pi, 2-x) = 2\log\left[(x-1).\operatorname{Cot}.\frac{(x-1)\pi}{4}\right],$$

id quod pro $x=1$ formulam (8.) reddit

$$16. \qquad L(\tfrac{1}{2}\pi, 1) = \log\left(\frac{4}{\pi}\right).$$

E formulis (14. et 15.) functionem $L(\frac{1}{2}\pi, x)$ pro quolibet ipsius x valore cognitam habemus, si modo per totam periodum ab $x=0$ ad $x=1$ cognita sit, Praeterea formulae (14. et 16.) docent, $L(\frac{1}{2}\pi, x)$ pro $x=$ quolibet numero integro impari finite per logarithmos et π exprimi posse.

Sit jam priori formularum (7.) $a=\frac{2}{3}\pi$; tunc erit

$$L(\tfrac{2}{3}\pi, x) = 2\operatorname{Sin}\tfrac{1}{3}\pi\log\left\{\frac{6.\Gamma\left(\frac{x+4}{6}\right).\Gamma\left(\frac{x+5}{6}\right)}{\Gamma\left(\frac{x+1}{6}\right).\Gamma\left(\frac{x+2}{6}\right)}\right\}$$

unde simili fere modo uc supra obtinebimus

$$17. \qquad \begin{cases} L(\tfrac{2}{3}\pi, x+3) + L(\tfrac{2}{3}\pi, x) = 2\operatorname{Sin}\tfrac{1}{3}\pi\log[(x+2)(x+1)] \\ \qquad L(\tfrac{2}{3}\pi, x) + L(\tfrac{2}{3}\pi, 3-x) \\ = 2\operatorname{Sin}\tfrac{1}{3}\pi\log\left[(x-2)(x-1).\operatorname{Tang}\frac{(x+2)\pi}{6}.\operatorname{Tang}\frac{(x+1)\pi}{6}\right], \end{cases}$$

atque e posteriore pro $x=\frac{3}{2}$:

$$18. \qquad L(\tfrac{2}{3}\pi, \tfrac{3}{2}) = 2\operatorname{Sin}\tfrac{1}{3}\pi\log\left(\tfrac{1}{2}\operatorname{Cotang}\frac{\pi}{12}\right).$$

E formulis (17.) sequitur, ut functio $L(\frac{2}{3}\pi, x)$ pro omnibus ipsius x valoribus cognita sit, si modo pro quolibet valore inter $x=0$ et $x=\frac{3}{2}$ cognitam habeamus; praeterea prior harum formularum una cum (18.) docet, $L(\frac{2}{3}\pi, x)$ finite per logarithmos et functiones trigonometricas exprimi posse pro $x=\frac{1}{2}((2i+1).3)$, designante i numerum quemlibet integrum.

At vero relationes longe generaliores invenire possumus, e quibus

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praecedentes tamquam casus speciales derivari possunt. Nam ubi $a=\frac{m\pi}{n}$ et $m+n=$ numerus impar est, habemus etiam

$$19.\qquad \text{Tang}\,\tfrac{1}{2}a = \underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia,$$

quare prior formularum (7.) in hanc formam transformari potest:

$$20.\qquad L(a,x) = 2\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia\,\log\left\{\frac{(2n)^{\frac{1}{2}}.\Gamma\left(\frac{x+n+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}.$$

$$(m+n=\text{num. imp.})$$

Substituamus hic $x+n$ pro x; tunc addendo habebimus

$$21.\qquad L(a,x+n)+L(a,x) = 2\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia\,\log(x+i)$$

$$m+n=\text{num. imp.}$$

Si porro in (20.) $n-x$ loco x substituitur, prodit etiam, addendo,

$$L(a,x)+L(a,n-x) = 2.\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia\,\log\left\{\frac{(2n)^{\frac{1}{2}}.\Gamma\left(\frac{1}{2}+\frac{x+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right)}\right\}$$

$$+2\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia\,\log\left\{\frac{(2n)^{\frac{1}{2}}.\Gamma\left(1+\frac{i-x}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{x+i}{2n}\right)}\right\},$$

sive, cum sit

$$\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia\,\log\left\{\frac{(2n)^{\frac{1}{2}}.\Gamma\left(1+\frac{i-x}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{i-x}{2n}\right)}\right\}$$

$$=\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia\,\log\left\{\frac{(2n)^{\frac{1}{2}}.\Gamma\left(\frac{3}{2}-\frac{x+i}{2n}\right)}{\Gamma\left(1-\frac{x+i}{2n}\right)}\right\},$$

etiam

$$L(a,x)+L(a,n-x)$$

$$=2\underset{i=1}{\overset{i=n-1}{S}}(-1)^{i-1}\,\text{Sin}\, ia\,\log\left\{\frac{2n.\Gamma\left(\frac{3}{2}-\frac{x+i}{2n}\right).\Gamma\left(\frac{1}{1}+\frac{x+i}{2n}\right)}{\Gamma\left(\frac{x+i}{2n}\right).\Gamma\left(1-\frac{x+i}{2n}\right)}\right\}.$$

Est autem

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$$\Gamma(\tfrac{3}{2}-\tfrac{x+i}{2n}).\Gamma(\tfrac{1}{2}+\tfrac{x+i}{2n}) = \frac{n-x-i}{2n}\cdot\frac{\pi}{\text{Sin}\,\frac{(x+i)\pi}{2n}},$$

$$\Gamma(\tfrac{x+i}{2n}).\Gamma(1-\tfrac{x+i}{2n}) = \frac{\pi}{\text{Sin}\,\frac{(x+i)\pi}{2n}},$$

unde sequitur

$$L(a,x)+L(a,n-x) = 2\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\,\text{Sin}\,ia\log[(n-x-i).\text{Tang}\,\tfrac{1}{2}(x+i)\pi],$$

sive denique, posito $n-\imath$ loco i,

$$22.\qquad L(a,x)+L(a,n-x) = 2\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\,\text{Sin}\,ia\log[(x-i)\,\text{Cotang}\,\tfrac{(x-i)\pi}{2n}].$$

$(m+n = \text{num. imp.})$

Constat igitur e formulis (21. et 22.), functionem $L(a,x)$ pro quovis ipsius x valore cognitam esse, si modo per totam periodum ab $x=0$ ad $x=\frac{1}{2}n$ cognita sit, exsistente $m+n$ numero impari.

Posito in (22.) $x=\frac{1}{2}n$, habemus formulam

$$23.\qquad L(a,\tfrac{1}{2}n) = \overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\,\text{Sin}\,ia\log[(\tfrac{1}{2}n-i)\,\text{Cotang}\,(\tfrac{\pi}{4}-\tfrac{i\pi}{2n})],$$

quae quidem pro $i=\frac{1}{2}n$ expressionem $\log 0.\infty$ praesentat; cujus tamen verus valor facile invenitur $\log\left(\frac{2n}{\pi}\right)$. Haec formula una cum (21.) docet, functionem $L(a,x)$, si $m+n$ numerus impar est, finite per logarithmos et functiones trigonometricas exprimi posse, posito $x=\frac{1}{2}n(2i+1)$, designante i numerum quemvis integrum.

Consimili fere modo relationes analogas e posteriori formularum (7.) derivare possumus. Nam substituatur ibi $x=n$ loco x, erit

$$L(a,x+n) = \text{Tang}\,\tfrac{1}{2}a\log.n + 2\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\,\text{Sin}\,ia\log\left\{\frac{\Gamma\left(\frac{2n+x-i}{n}\right)}{\Gamma\left(\frac{n+x+i}{n}\right)}\right\},$$

unde, subtrahendo,

$$24.\qquad L(a,x+n)-L(a,x) = 2\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\,\text{Sin}\,ia.\log\left\{\frac{x+n-i}{x+i}\right\}$$

$(m+n = \text{num. par.})$

Ceterum, si in eadem formula (7.) $n-x$ pro x ponitur, fit

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$$L(a, n-x) = \operatorname{Tang} \tfrac{1}{2} a . \log n + 2 \underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}} (-1)^{i-1} \operatorname{Sin} ia \log \left\{ \frac{\Gamma\left(\frac{2n-x-i}{n}\right)}{\Gamma\left(\frac{n+i+x}{n}\right)} \right\},$$

unde subtractione e cognita functionis Γ ratione

25. $$L(a, x) - L(a, n-x) = 2 \underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}} (-1)^{i} \operatorname{Sin} ia \log \left\{ \frac{x-i}{n-x-i} \cdot \frac{\operatorname{Sin} \frac{(x+i)\pi}{n}}{\operatorname{Sin} \frac{(x-i)\pi}{n}} \right\}.$$

$(m+n = \text{num. par.})$

Formularum (24. et 25.) beneficio sequitur, ut, exsistente $m+n$ numero pari, functionem $L(a, x)$ pro quovis ipsius x valore cognitam omnino habeamus, si modo per totam periodum ab $x=0$ ad $x=\frac{1}{2}n$ cognita sit.

Ponamus jam successive loco x:

$$x, \quad x+\frac{2n}{r}, \quad x+\frac{4n}{r}, \quad x+\frac{6n}{r}, \ldots\ldots x+\frac{(r-1).2n}{r}$$

in priori formularum (7.), et

$$x, \quad x+\frac{n}{r}, \quad x+\frac{2n}{r}, \quad x+\frac{3n}{r}, \ldots\ldots x+\frac{(r-1).n}{r}$$

in posteriori; tunc beneficio cognitae formulae

$$\Gamma(y) . \Gamma\left(y+\frac{1}{r}\right) . \Gamma\left(y+\frac{2}{r}\right) \ldots\ldots \Gamma\left(y+\frac{r-1}{r}\right) = \Gamma(ry) . (2\pi)^{\frac{1}{2}(r-1)} . r^{\frac{1}{2}-ry}$$

summando invenimus

26. $$\left\{ \begin{array}{l} L(a,x) + L\left(a, x+\frac{2n}{r}\right) + L\left(a, x+\frac{4n}{r}\right) + \ldots\ldots + L\left(a, x+\frac{(r-1)2n}{r}\right) \\ \quad = r \operatorname{Tang} \tfrac{1}{2} a \log . 2n + 2 \underset{i=1}{\overset{i=n-1}{S}} (-1)^{i-1} \operatorname{Sin} ia \log \left\{ \dfrac{\Gamma\left(\frac{1}{2}r + \frac{r(x+i)}{n}\right)}{r^{\frac{1}{2}r}\, \Gamma\left(\frac{r(x+i)}{2n}\right)} \right\} \\ \qquad (m+n = \text{num. imp.}) \\ L(a,x) + L\left(a, x+\frac{n}{r}\right) + L\left(a, x+\frac{2n}{r}\right) + \ldots\ldots L\left(a, x+\frac{(r-1).n}{r}\right) \\ \quad = r \operatorname{Tang} \tfrac{1}{2} a . \log n + 2 \underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}} (-1)^{i-1} \operatorname{Sin} ia \log \left\{ \dfrac{\Gamma\left(r + \frac{r(x-i)}{n}\right)}{r^{r-\frac{2ri}{n}} . \Gamma\left(\frac{r(x+i)}{n}\right)} \right\} \\ \qquad (m+n = \text{num. par.}) \end{array} \right.$$

Facile apparet, priorem harum summarum per logarithmos exprimi posse, cum r numerus par est.

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§. 4.

Cognita est formula

27. $$\int_0^\infty z^{s-1} e^{-xz} \operatorname{Cos} uz\, dz = \frac{\Gamma(s)}{(x^2+u^2)^{\frac{1}{2}s}} . \operatorname{Cos}\left(s \operatorname{ArcTang} \frac{u}{x}\right),$$

quae, exsistente $1 > s > 0$, pro $x = 0$ etiam legitima, praebet

28. $$\int_0^\infty z^{s-1} \operatorname{Cos} uz\, dz = \frac{\operatorname{Cos}.\frac{1}{2}s\pi . \Gamma(s)}{u^s}.$$

Multiplicemus (27.) utrimque per

$$\frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} . du \qquad [a < \pi];$$

integratione inter $u = 0$ et $u = \infty$ instituta, fit

$$\int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} . \frac{\operatorname{Cos}(s . \operatorname{ArcTang} \frac{u}{x})}{(x^2+u^2)^{\frac{3}{2}}} . du$$

$$= \frac{1}{\Gamma(s)} . \int_0^\infty e^{-xz} . z^{s-1} dz . \int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} . \operatorname{Cos} uz . du$$

$$= \frac{\operatorname{Sin} a}{\Gamma(s)} . \int_0^\infty \frac{e^{-xz} z^{s-1} . e^{-z} dz}{1 + 2e^{-z} \operatorname{Cos} a + e^{-2z}},$$

unde, in dextero membro $e^{-z} = y$ ponendo, provenit

29. $$\int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} . \frac{\operatorname{Cos}(s . \operatorname{ArcTang} \frac{u}{x})}{(x^2+u^2)^{\frac{1}{2}s}} du = \frac{\operatorname{Sin} a}{\Gamma(s)} . \int_0^1 \frac{y^x \left(\log \frac{1}{y}\right)^{s-1} dy}{1 + 2y \operatorname{Cos} a + y^2}$$

et pro $x = 0$, si modo $1 > s > 0$,

30. $$\int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} . \frac{du}{u^s} = \frac{\operatorname{Sin} a}{\operatorname{Cos} \frac{1}{2} s\pi . \Gamma(s)} . \int_0^1 \frac{\left(\log \frac{1}{y}\right)^{s-1} dy}{1 + 2y \operatorname{Cos} a + y^2}.$$

Ex hac vero formula, si $a = \frac{m\pi}{n}$ et $e^{-\frac{\pi u}{n}} = y$ ponitur, unde fit

$$u = \frac{n}{\pi} . \log \frac{1}{y} \quad \text{et} \quad du = -\frac{n}{\pi} . \frac{dy}{y},$$

transformatione facta eruitur

31. $$\int_0^1 \frac{y^{m-1} - y^{-m-1}}{y^n - y^{-n}} . \frac{dy}{\left(\log \frac{1}{y}\right)^s} = \frac{\left(\frac{\pi}{n}\right)^{1-s} . \operatorname{Sin} \frac{m\pi}{n}}{\operatorname{Cos} \frac{1}{2} s\pi . \Gamma(s)} . \int_0^1 \frac{\left(\log \frac{1}{y}\right)^{s-1} dy}{1 + 2y \operatorname{Cos} \frac{m\pi}{n} + y^2},$$

et pro $m = 1$,

32. $$\int_0^1 \frac{y^{n-2} . dy}{1 + y^2 \ldots + y^{2(n-1)}} . \frac{1}{\left(\log \frac{1}{y}\right)^s} = \frac{\left(\frac{\pi}{n}\right)^{1-s} . \operatorname{Sin} \frac{\pi}{n}}{\operatorname{Cos} \frac{1}{2} s\pi . \Gamma(s)} . \int_0^1 \frac{\left(\log \frac{1}{y}\right)^{s-1} dy}{1 + 2y \operatorname{Cos} \frac{\pi}{n} + y^2}.$$

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Appellemus jam

$$33.\qquad G(s)=\int_0^1 \frac{(\log\frac{1}{y})^{s-1}dy}{1+y+y^2},\qquad G_1(s)=\int_0^1 \frac{(\log\frac{1}{y})^{s-1}dy}{1+y^2};$$

formula (32.) dabit, posito $n=3$ et mutato y^2 in y:

$$34.\qquad G(1-s)=\frac{(\frac{2}{3}\pi)^{1-s}\operatorname{Sin}\frac{1}{3}\pi}{\operatorname{Cos}\frac{1}{2}s\pi.\Gamma(s)}\,.\,G(s),$$

atque eâdem pro $n=2$ immediate colligitur

$$35.\qquad G_1(1-s)=\frac{(\frac{1}{2}\pi)^{1-s}}{\operatorname{Cos}\frac{1}{2}s\pi.\Gamma(s)}\,.\,G_1(s).$$

Ecce simplices et notandae aequationes, quae functiones $G(s)$ et $G_1(s)$ earumque complementarias $G(1-s)$ et $G_1(1-s)$ inter se conjungunt; in quo respectu analogae sunt cum hac cognita functionis Γ formula

$$\Gamma(a)\,.\,\Gamma(1-a)=\frac{\pi}{\operatorname{Sin} a\pi}.$$

Ex formulis (34. et 35.) logarithmando obtinebimus

$$\log G(1-s)-\log G(s)=(1-s)\log\tfrac{2}{3}\pi+\log\operatorname{Sin}\tfrac{1}{3}\pi-\log\operatorname{Cos}\tfrac{1}{2}s\pi-\log\Gamma(s),$$

$$\log G_1(1-s)-\log G_1(s)=(1-s)\log\tfrac{1}{2}\pi-\log\operatorname{Cos}\tfrac{1}{2}s\pi-\log\Gamma(s)$$

unde, si brevitatis causa ponimus

$$36.\qquad \begin{cases} F(s)=\dfrac{d.G(s)}{ds}=\displaystyle\int_0^1\frac{(\log\frac{1}{y})^{s-1}\log(\log\frac{1}{y})\,dy}{1+y+y^2}, \\ F_1(s)=\dfrac{dG_1(s)}{ds}=\displaystyle\int_0^1\frac{(\log\frac{1}{y})^{s-1}.\log(\log\frac{1}{y}).dy}{1+y^2}, \end{cases}$$

differentiando habebimus has novas relationes

$$37.\qquad \begin{cases} \dfrac{F(s)}{G(s)}+\dfrac{F(1-s)}{G(1-s)}=\log\tfrac{2}{3}\pi+Z'(s)-\tfrac{1}{2}\pi\operatorname{Tang}\tfrac{1}{2}s\pi, \\ \dfrac{F_1(s)}{G_1(s)}+\dfrac{F_1(1-s)}{G_1(1-s)}=\log\tfrac{1}{2}\pi+Z'(s)-\tfrac{1}{2}\pi\operatorname{Tang}\tfrac{1}{2}s\pi, \end{cases}$$

si cum *Legendre* $\dfrac{d.\log\Gamma(s)}{ds}$ per $Z'(s)$ signamus.

Supponamus in (37.) $s=\frac{1}{2}$; tunc erit

$$\int_0^1\frac{\log(\log\frac{1}{y})}{1+y+y^2}\cdot\frac{dy}{\sqrt{(\log\frac{1}{y})}}=\tfrac{1}{2}(\log.\frac{\pi}{6}-\tfrac{1}{2}\pi-C)\,.\int_0^1\frac{1}{1+y+y^2}\cdot\frac{dy}{\sqrt{(\log\frac{1}{y})}}$$

$$\int_0^1\frac{\log(\log\frac{1}{y})}{1+y^2}\cdot\frac{dy}{\sqrt{(\log\frac{1}{y})}}=\tfrac{1}{2}(\log\frac{\pi}{8}-\tfrac{1}{2}\pi-C)\,.\int_0^1\frac{1}{1+y^2}\cdot\frac{dy}{\sqrt{(\log\frac{1}{y})}},$$

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seu ponendo $\log\frac{1}{y} = x$:

$$38.\quad \begin{cases} \int_0^\infty \frac{\log x}{e^x+1+e^{-x}}\cdot\frac{dx}{\sqrt{x}} = \frac{1}{2}(\log\frac{\pi}{6} - \frac{1}{2}\pi - C)\,.\int_0^\infty \frac{dx}{e^x+1+e^{-x}}\cdot\frac{1}{\sqrt{x}}, \\ \int_0^\infty \frac{\log x}{e^x+e^{-x}}\cdot\frac{dx}{\sqrt{x}} = \frac{1}{2}(\log\frac{\pi}{8} - \frac{1}{2}\pi - C)\,.\int_0^\infty \frac{dx}{e^x+e^{-x}}\cdot\frac{1}{\sqrt{x}}, \end{cases}$$

exsistente $C = -Z'(1)$ cognita illa *Euleri* Constante $0,577\,216\ldots..$ Hae formulae, quas non vidimus hucusque propositas, haud indignae nobis videntur attentione Geometrarum.

Ex formulis (38.) duas jam deducemus relationes, quae in serierum transformationibus haud infimo loco dignae videntur. Nimirum, exsistentibus identico modo

$$39.\quad \begin{cases} \frac{\sin\frac{1}{3}\pi}{e^x+1-e^{-x}} = \overset{i=n}{\underset{i=1}{\mathrm{S}}}(-1)^{i-1}.e^{-ix}.\sin\tfrac{1}{3}i\pi + \frac{(-1)^n.e^{-nx}(\sin\frac{1}{3}(n+1)\pi + e^{-x}\sin\frac{1}{3}n\pi)}{1+e^{-x}+e^{-2x}} \\ \frac{1}{e^x+e^{-x}} = \overset{i=n-1}{\underset{i=0}{\mathrm{S}}}(-1)^i.e^{-(2i+1)x} + \frac{(-1)^n.e^{-2nx}}{e^x+e^{-x}}, \end{cases}$$

ex cognita formula

$$40.\quad \int_0^\infty e^{-kx}.\log x.\frac{dx}{\sqrt{x}} = -\frac{\sqrt{\pi}}{\sqrt{k}}(\log k + 2\log 2 + C)$$

habebimus

$$\sin\tfrac{1}{3}\pi\,.\int_0^\infty \frac{\log x}{e^x+1+e^{-x}}\cdot\frac{dx}{\sqrt{x}}$$
$$= \sqrt{\pi}.\overset{i=n}{\underset{i=1}{\mathrm{S}}}(-1)^i.\sin\tfrac{1}{3}i\pi.\frac{\log i + 2\log 2 + C}{\sqrt{i}}$$
$$+ (-1)^n\{P(n).\sin\tfrac{1}{3}(n+1)\pi + P(n+1)\sin\tfrac{1}{3}n\pi\},$$

$$\int_0^\infty \frac{\log x}{e^x+e^{-x}}\cdot\frac{dx}{\sqrt{x}} = \sqrt{\pi}.\mathrm{S}(-1)^{i+1}.\frac{\log(2i+1)+2\log 2+C}{\sqrt{(2i+1)}} + (-1)^n.Q(n),$$

ubi brevitatis causa posuimus

$$P(n) = \int_0^\infty \frac{e^{-nx}\log x}{1+e^{-x}+e^{-2x}}\cdot\frac{dx}{\sqrt{x}} = \Theta\,.\int_0^\infty \frac{e^{-nx}\log x.dx}{\sqrt{x}} = -\frac{\Theta.\sqrt{\pi}}{\sqrt{n}}(\log n + 2\log 2 + C),$$

$$Q(n) = \int_0^\infty \frac{e^{-2nx}\log x}{e^x+e^{-x}}\cdot\frac{dx}{\sqrt{x}} = \Theta_1.\int_0^\infty \frac{e^{-2nx}\log x.dx}{\sqrt{x}} = -\frac{\Theta_1.\sqrt{\pi}}{\sqrt{(2n)}}(\log 2n + 2\log 2 + C),$$

exsistente $1 > \frac{\Theta}{\Theta_1} > 0$. Facile vero aparet esse

$$\lim P(n) = 0, \qquad \lim Q(n) = 0 \qquad [n=\infty],$$

unde concludi licet

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41. $$\begin{cases} \sin\frac{1}{3}\pi \cdot \int_0^\infty \frac{\log x}{e^x+1+e^{-x}} \cdot \frac{dx}{\sqrt{x}} = \sqrt{\pi} \cdot \underset{i=1}{\overset{i=\infty}{S}} (-1)^i \sin\frac{1}{3}i\pi \cdot \frac{\log i + 2\log 2 + C}{\sqrt{i}} \quad \text{et} \\ \int_0^\infty \frac{\log x}{e^x+e^{-x}} \cdot \frac{dx}{\sqrt{x}} = \sqrt{\pi} \cdot \underset{i=0}{\overset{i=\infty}{S}} (-1)^{i+1} \cdot \frac{\log(2i+1) + 2\log 2 + C}{\sqrt{(2i+1)}}. \end{cases}$$

Beneficio vero formularum (39.) facillimo etiam negotio deducere possumus

$$\sin\frac{1}{3}\pi \cdot \int_0^\infty \frac{dx}{e^x+1+e^{-x}} \cdot \frac{1}{\sqrt{x}} = \sqrt{\pi} \cdot \underset{i=1}{\overset{i=\infty}{S}} (-1)^{i-1} \cdot \frac{\sin\frac{1}{3}i\pi}{\sqrt{i}},$$

$$\int_0^\infty \frac{dx}{e^x+e^{-x}} \cdot \frac{1}{\sqrt{x}} = \sqrt{\pi} \cdot \underset{i=1}{\overset{i=\infty}{S}} \frac{(-1)^i}{\sqrt{(2i+1)}},$$

quae formulae una cum (41.), respectu ad (38.) habito, post reductiones quasdam facillimas praebent

42. $$\begin{cases} \frac{\log 1}{\sqrt{1}} - \frac{\log 2}{\sqrt{2}} + \frac{\log 4}{\sqrt{4}} - \frac{\log 5}{\sqrt{5}} + \frac{\log 7}{\sqrt{7}} - \text{etc.} \\ = \frac{1}{2}(\frac{1}{2}\pi - C - \log\frac{8}{3}\pi)\left\{\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \text{etc.} \ldots\right\} \\ \frac{\log 1}{\sqrt{1}} - \frac{\log 3}{\sqrt{3}} + \frac{\log 5}{\sqrt{5}} - \frac{\log 7}{\sqrt{7}} + \frac{\log 9}{\sqrt{9}} - \text{etc.} \\ = \frac{1}{2}(\frac{1}{2}\pi - C - \log 2\pi)\left\{\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{9}} - \text{etc.}\right\} \end{cases}$$

quae relationes etiam ita scribi possunt:

43. $$\begin{cases} \dfrac{1^{\frac{1}{\sqrt{1}}} \cdot 4^{\frac{1}{\sqrt{4}}} \cdot 7^{\frac{1}{\sqrt{7}}} \cdot 10^{\frac{1}{\sqrt{10}}} \ldots}{2^{\frac{1}{\sqrt{2}}} \cdot 5^{\frac{1}{\sqrt{5}}} \cdot 8^{\frac{1}{\sqrt{8}}} \cdot 13^{\frac{1}{\sqrt{13}}} \ldots} = e^{\frac{1}{2}(\frac{1}{2}\pi - C - \log\frac{8}{3}\pi)A} \\ \dfrac{1^{\frac{1}{\sqrt{1}}} \cdot 5^{\frac{1}{\sqrt{4}}} \cdot 9^{\frac{1}{\sqrt{9}}} \cdot 13^{\frac{1}{\sqrt{13}}} \ldots}{3^{\frac{1}{\sqrt{3}}} \cdot 7^{\frac{1}{\sqrt{7}}} \cdot 11^{\frac{1}{\sqrt{11}}} \cdot 15^{\frac{1}{\sqrt{15}}} \ldots} = e^{\frac{1}{2}(\frac{1}{2}\pi - C - \log 2\pi) \cdot B} \end{cases}$$

exsistentibus

$$A = \frac{1}{\sqrt{\pi}} \cdot \int_0^\infty \frac{dx}{e^x+1+e^{-x}} \cdot \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \ldots \text{etc.}$$

$$B = \frac{1}{\sqrt{\pi}} \cdot \int_0^\infty \frac{dx}{e^x+e^{-x}} \cdot \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{9}} - \ldots \text{etc.}$$

§. 5.

Si $\frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}}$ in seriem evolvimus, exsistente identico modo

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$$\frac{1}{e^{\pi u}-e^{-\pi u}} = e^{-\pi u}\,.\,\overset{i=n-1}{\underset{i=0}{\mathbf{S}}}\, e^{-2i\pi u} + \frac{e^{-2n\pi u}}{e^{\pi u}-e^{-\pi u}},$$

fit utique

$$\frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}} = \overset{i=n-1}{\underset{i=0}{\mathbf{S}}}\,[e^{-[(2i+1)\pi-a]u} - e^{-[(2i+1)\pi+a]u}] + \frac{e^{-2n\pi u}(e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}$$

unde

$$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\cdot\frac{du}{u^s} = \Gamma(1-s)\,.\,\overset{i=n-1}{\underset{i=0}{\mathbf{S}}}\,[\frac{1}{((2i+1)\pi-a)^{1-s}} - \frac{1}{((2i+1)\pi+a)^{1-s}}] + \varphi(n)$$

ubi brevitatis causa posuimus

$$\varphi(n) = \int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\cdot\frac{e^{-2n\pi u}.\,du}{u^s} = M\,.\int_0^\infty \frac{e^{-2n\pi u}\,du}{u^s} = \frac{M\Gamma(1-s)}{(2n\pi)^{1-s}},$$

exsistente M quantitate quadam finata. Hinc vero facile apparet

$$\lim.\,\varphi(n) = 0 \qquad [n=\infty]$$

atque igitur

44. $$\int_0^\infty \frac{e^{au}-e^{-au}}{e^{\pi u}-e^{-\pi u}}\cdot\frac{du}{u^s} = \Gamma(1-s)\,.\,\overset{i=n-1}{\underset{i=0}{\mathbf{S}}}\,[\frac{1}{((2i+1)\pi-a)^{1-s}} - \frac{1}{((2i+1)\pi+a)^{1-s}}].$$

Jam vero, exsistente identico modo

$44\frac{1}{2}$. $$\frac{\text{Sin}\,a}{1+2y\,\text{Cos}\,a+y^2} = \overset{i=n}{\underset{i=1}{\mathbf{S}}}(-1)^{i-1}y^{i-1}\,\text{Sin}\,ia + \frac{(-1)^n.y^n(\text{Sin}\,(n+1)a+y\,\text{Sin}\,na)}{1+2y\,\text{Cos}\,a+y^2},$$

habebimus etiam

45. $$\int_0^1 \frac{\text{Sin}\,a}{1+2y\,\text{Cos}\,a+y^2}\cdot\frac{dy}{(\log\frac{1}{y})^{1-s}}$$

$$= \Gamma(s)\,.\,\overset{i=n}{\underset{i=1}{\mathbf{S}}}(-1)^{i-1}\frac{\text{Sin}\,ia}{i^s} + (-1)^n(W(n)\,\text{Sin}\,(n+1)a + W(n+1)\,\text{Sin}\,na),$$

ubi brevitatis causa posuimus

$$W(n) = \int_0^1 \frac{y^n(\log\frac{1}{y})^{s-1}dy}{1+2y\,\text{Cos}\,a+y^2} = \Theta\,.\int_0^1 y^n(\log\frac{1}{y})^{s-1}dy = \frac{\Theta.\Gamma(s)}{(n+1)^s},$$

exsistente $1>\Theta>0$. Facile igitur apparet esse

$$\lim W(n) = 0 \qquad [n=\infty]$$

unde ex (45.) obtinebitur

46. $$\int_0^1 \frac{\text{Sin}\,a}{1+2y\,\text{Cos}\,a+y^2}\cdot\frac{dy}{(\log\frac{1}{y})^{1-s}} = \Gamma(s)\,.\,\overset{i=\infty}{\underset{i=1}{\mathbf{S}}}(-1)^{i-1}\,.\,\frac{\text{Sin}\,ia}{i^s}.$$

Substitutis vero in (29.) valoribus, quos formulae (44. et 46.) praebent, hanc notandam inter duas series infinitas relationem habemus, si s in $1-s$ mutatur:

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$$47.\quad\begin{cases}\frac{1}{(\pi-a)^s}-\frac{1}{(\pi+a)^s}+\frac{1}{(3\pi-a)^s}-\frac{1}{(3\pi+a)^s}+\frac{1}{(5\pi-a)^s}-\frac{1}{(5\pi+a)^s}+\text{etc.}\\ =\frac{1}{\sin\frac{1}{2}s\pi.\Gamma(s)}\left\{\frac{\sin a}{1^{1-s}}-\frac{\sin 2a}{2^{1-s}}+\frac{\sin 3a}{3^{1-s}}-\frac{\sin 4a}{4^{1-s}}+\frac{\sin 5a}{5^{1-s}}-\text{etc.}\right\}\end{cases}$$

et si $\pi-a$ loco a ponimus:

$$48.\quad\begin{cases}\frac{1}{a^s}-\frac{1}{(2\pi-a)^s}+\frac{1}{(2\pi+a)^s}-\frac{1}{(4\pi-a)^s}+\frac{1}{(4\pi+a)^s}-\frac{1}{(6\pi-a)^s}+\text{etc.}\\ =\frac{1}{\sin\frac{1}{2}s\pi.\Gamma(s)}\left\{\frac{\sin a}{1^{1-s}}+\frac{\sin 2a}{2^{1-s}}+\frac{\sin 3a}{3^{1-s}}+\frac{\sin 4a}{4^{1-s}}+\frac{\sin 5a}{5^{1-s}}+\text{etc.}\right\}\end{cases}$$

Hinc si $s=\frac{1}{2}$ facimus fit utique

$$49.\quad\begin{cases}\frac{1}{\sqrt{(\pi-a)}}-\frac{1}{\sqrt{(\pi+a)}}+\frac{1}{\sqrt{(3\pi-a)}}-\frac{1}{\sqrt{(3\pi+a)}}+\frac{1}{\sqrt{(5\pi-a)}}-\frac{1}{\sqrt{(5\pi+a)}}+\text{etc.}\\ =\sqrt{\frac{2}{\pi}}\left\{\frac{\sin a}{\sqrt{1}}-\frac{\sin 2a}{\sqrt{2}}+\frac{\sin 3a}{\sqrt{3}}-\frac{\sin 4a}{\sqrt{4}}+\frac{\sin 5a}{\sqrt{5}}-\text{etc.}\right\}\\ \frac{1}{\sqrt{a}}-\frac{1}{\sqrt{(2\pi-a)}}+\frac{1}{\sqrt{(2\pi+a)}}-\frac{1}{\sqrt{(4\pi-a)}}+\frac{1}{\sqrt{(4\pi+a)}}-\frac{1}{\sqrt{(4\pi-6)}}+\text{etc.}\\ =\sqrt{\frac{2}{\pi}}\left\{\frac{\sin a}{\sqrt{1}}+\frac{\sin 2a}{\sqrt{2}}+\frac{\sin 3a}{\sqrt{3}}+\frac{\sin 4a}{\sqrt{4}}+\frac{\sin 5a}{\sqrt{5}}+\text{etc.}\right.\end{cases}$$

Ponamus in (47. et 48.), a esse in ratione commensurabili ad π, i. e. $a=\frac{m\pi}{n}$ ($m<n$ num. integr.); facile tunc obtinebitur

$$50.\quad\begin{cases}\frac{1}{(n-m)^s}-\frac{1}{(n+m)^s}+\frac{1}{(3n-m)^s}-\frac{1}{(3n+m)^s}+\frac{1}{(5n-m)^s}-\frac{1}{(5n+m)^s}+\text{etc.}\\ =\frac{\left(\frac{\pi}{n}\right)^s}{\sin\frac{1}{2}s\pi.\Gamma(s)}\cdot\left\{\frac{\sin\frac{m\pi}{n}}{1^{1-s}}-\frac{\sin\frac{2m\pi}{n}}{2^{1-s}}+\frac{\sin\frac{3m\pi}{n}}{3^{1-s}}-\frac{\sin\frac{4m\pi}{n}}{4^{1-s}}+\text{etc.}\right\}\\ \frac{1}{m^s}-\frac{1}{(2n-m)^s}+\frac{1}{(2n+m)^s}-\frac{1}{(4n-m)^s}+\frac{1}{(4n+m)^s}-\frac{1}{(6n-m)^s}+\text{etc.}\\ =\frac{\left(\frac{\pi}{n}\right)^s}{\sin\frac{1}{2}s\pi.\Gamma(s)}\left\{\frac{\sin\frac{m\pi}{n}}{1^{1-s}}+\frac{\sin\frac{2m\pi}{n}}{2^{1-s}}+\frac{\sin\frac{3m\pi}{n}}{3^{1-s}}+\frac{\sin\frac{4m\pi}{n}}{4^{1-s}}+\text{etc.}\right\}\end{cases}$$

Ex. 1. Exsistentibus $m=1$, $n=2$, fit

$$51.\quad\frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}-\text{etc.}\ldots$$
$$=\frac{(\frac{1}{2}\pi)^s}{\sin\frac{1}{2}s\pi.\Gamma(s)}\cdot\left\{\frac{1}{1^{1-s}}-\frac{1}{3^{1-s}}+\frac{1}{5^{1-s}}-\frac{1}{7^{1-s}}+\text{etc.}\right\}$$

Ex. 2. Exsistentibus in priori formularum (50.) $m=1$, $n=3$, fit

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$$52. \quad \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \text{etc.} \ldots$$
$$= \frac{(\frac{2}{3}\pi)^s . \operatorname{Sin}\frac{1}{3}\pi}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} \left\{ \frac{1}{1^{1-s}} - \frac{1}{2^{1-s}} + \frac{1}{4^{1-s}} - \frac{1}{5^{1-s}} + \frac{1}{7^{1-s}} - \text{etc.} \right\}$$

Formulas (51. et 52.) memini (ni fallor) me vidisse ab *Eulero* alicubi per inductionem inventas, omni demonstratione carentes; neque apud quemquam alium demonstrationem earum invenimus, quamquam formâ suâ attentione Geometrarum digna videantur.

Ex. 3. Exsistentibus in posteriore formularum (50.) $m = 1$, $n = 3$, posito brevitatis causa

$$f(s) = \frac{1}{1^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{11^s} + \frac{1}{13^s} - \frac{1}{17^s} + \frac{1}{19^s} - \text{etc.}$$
$$\varphi(s) = \frac{1}{1^s} + \frac{1}{2^s} - \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{8^s} - \frac{1}{10^s} - \frac{1}{11^s} + \text{etc.}$$

fit utique

$$f(s) = \frac{(\frac{1}{3}\pi)^s . \operatorname{Sin}\frac{1}{3}\pi}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} . \varphi(1-s).$$

Cum autem sit

$$\varphi(s) = f(s) + 2^{-s} \left\{ \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \text{etc.} \right\}$$

atque etiam

$$\varphi(s) = f(s) + 2^{-s} f(s) - 2^{-2s} \left\{ \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \text{etc.} \right\}$$

unde facile prodit

$$(1 + 2^{-s})\,\varphi(s) = (1 + 2^{1-s})\, f(s),$$

has duas relationes habebimus:

$$53. \quad \begin{cases} f(s) = \dfrac{(\frac{1}{3}\pi)^s . \operatorname{Sin}\frac{1}{3}\pi}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} \cdot \dfrac{1+2^s}{1+2^{s-1}} . f(1-s) \\ \varphi(s) = \dfrac{(\frac{1}{3}\pi)^s . \operatorname{Sin}\frac{1}{3}\pi}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} \cdot \dfrac{2(1+2^{s-1})}{1+2^s} . \varphi(1-s). \end{cases}$$

Ex. 4. Exsistentibus in (50.) $m = 1$, $n = 4$, posito brevitatis causa

$$F(s) = \frac{1}{1^s} + \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{15^s} + \text{etc.}$$
$$W(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \text{etc.}$$
$$\psi(s) = \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{19^s} - \frac{1}{21^s}$$
$$P(s) = \frac{1}{1^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{15^s} + \frac{1}{17^s} - \text{etc.}$$

fit utique

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$$\psi(s) = \frac{\left(\frac{\pi}{4}\right)^s \operatorname{Sin}\frac{\pi}{4}}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} . F(1-s) - \frac{\frac{1}{2}.(\frac{1}{2}\pi)^s}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} . W(1-s),$$

$$P(s) = \frac{\left(\frac{\pi}{4}\right)^s \operatorname{Sin}\frac{\pi}{4}}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} . F(1-s) + \frac{\frac{1}{2}(\frac{1}{2}\pi)^s}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} . W(1-s),$$

unde addendo, cum sit

$$\psi(s) + P(s) = F(s),$$

erit

$$54. \qquad F(s) = \frac{2.\left(\frac{\pi}{4}\right)^s \operatorname{Sin}\frac{\pi}{4}}{\operatorname{Sin}\frac{1}{2}s\pi . \Gamma(s)} . F(1-s).$$

§. 6.

Differentiemus jam formulam (44.) respectu s tamquam variabilis; tum erit

$$\int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi n} - e^{-\pi u}} . u^{-s} \log u . du$$

$$= Z'(1-s) . \int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} . \frac{du}{u^s} - \Gamma(1-s) . \overset{i=\infty}{\underset{i=0}{S}} \left[\frac{\log((2i+1)\pi-a)}{((2i+1)\pi-a)^{1-s}} - \frac{\log((2i+1)\pi+a)}{((2i+1)\pi+a)^{1-s}}\right],$$

unde prò $s = 0$, exsistente

$$Z'(1) = -C \quad \text{et} \quad \int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} . du = \operatorname{Tang}\tfrac{1}{2}a,$$

habebimus

$$\overset{i=\infty}{\underset{i=0}{S}} \left[\frac{\log((2i+1)\pi-a)}{((2i+1)\pi-a)^{1-s}} - \frac{\log((2i+1)\pi+a)}{((2i+1)\pi-a)^{1-s}}\right]$$

$$= -\tfrac{1}{2}C . \operatorname{Tang}\tfrac{1}{2}a - \int_0^\infty \frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} \log u . du.$$

Utcumque vero a est in ratione commensurabili ad π, valorem integralis, quod in dextero membro occurrit, formulae (7.) (pro $x = 0$) praebent. Sit igitur $a = \frac{m\pi}{n}$ (m et n numeri integri, $n > m$); tunc obtinebimus

$$\frac{n}{\pi} . \overset{i=\infty}{\underset{i=0}{S}} \left[\frac{\log((2i+1)n-m)}{(2i+1)n-m} - \frac{\log((2i+1)n+m)}{(2i+1)n+m}\right]$$

$$+ \log\frac{\pi}{n} \overset{i=\infty}{\underset{i=0}{S}} \left[\frac{1}{(2i+1)\pi - \frac{m\pi}{n}} - \frac{1}{(2i+1)\pi + \frac{m\pi}{n}}\right]$$

$$= -\tfrac{1}{2}C . \operatorname{Tang}\frac{m\pi}{2n} - \frac{e^{\frac{m\pi}{n}.u} - e^{-\frac{m\pi}{n}.u}}{e^{\pi u} - e^{-\pi u}} . \log u . du,$$

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unde, cum sit ex (44.) (pro $s=0$)

$$\overset{i=\infty}{\underset{i=0}{S}}\left[\frac{1}{(2i+1)\pi-\frac{m\pi}{n}}-\frac{1}{(2i+1)\pi+\frac{m\pi}{n}}\right]=\int_0^\infty\frac{e^{\frac{m\pi}{n}.u}-e^{-\frac{m\pi}{n}.u}}{e^{\pi u}-e^{-\pi u}}.du=\tfrac{1}{2}\text{Tang}\,\frac{m\pi}{2n},$$

fit denique ex formulis (7.) citatis

55.
$$\left\{\begin{aligned}&\frac{\log(n-m)}{n-m}-\frac{\log(n+m)}{n+m}+\frac{\log(3n-m)}{3n-m}-\frac{\log(3n+m)}{3n+m}+\frac{\log(5n-m)}{5n-m}-\text{etc.}\\&=-\frac{\pi}{2n}.\text{Tang}\,\frac{m\pi}{2n}(C+\log 2\pi)-\frac{\pi}{n}.\overset{i=n-1}{\underset{i=1}{S}}(-1)^{i-1}\,\text{Sin}\,\frac{im\pi}{n}\log\left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\}\\&\qquad(m+n=\text{num. imp.})\\&\frac{\log(n-m)}{n-m}-\frac{\log(n+m)}{n+m}+\frac{\log(3n-m)}{3n-m}-\frac{\log(3n+m)}{3n+m}+\frac{\log(5n-m)}{5n-m}-\text{etc.}\\&=-\frac{\pi}{2n}.\text{Tang}\,\frac{m\pi}{2n}(C+\log\pi)-\frac{\pi}{n}.\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\,\text{Sin}\,\frac{im\pi}{n}\log\left\{\frac{\Gamma\left(\frac{n-i}{2n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\}\\&\qquad(m+n=\text{num. par}),\end{aligned}\right.$$

atque si $n-m$ loco m ponimus,

56.
$$\left\{\begin{aligned}&\frac{\log m}{m}-\frac{\log(2n-m)}{2n-m}+\frac{\log(2n+m)}{2n+m}-\frac{\log(4n-m)}{4n-m}+\frac{\log(4n+m)}{4n+m}-\text{etc.}\\&=-\frac{\pi}{2n}\text{Cotang}\,\frac{m\pi}{2n}(C+\log 2\pi)+\frac{\pi}{n}.\overset{i=n-1}{\underset{i=1}{S}}\,\text{Sin}\,\frac{im\pi}{n}\log\left\{\frac{\Gamma\left(\frac{n+i}{2n}\right)}{\Gamma\left(\frac{i}{2n}\right)}\right\}\\&\qquad(m=\text{num. imp.})\\&\frac{\log m}{m}-\frac{\log(2n-m)}{2n-m}+\frac{\log(2n+m)}{2n+m}-\frac{\log(4n-m)}{4n-m}+\frac{\log(4n+m)}{4n+m}-\text{etc.}\\&=-\frac{\pi}{2n}\text{Cotang}\,\frac{m\pi}{2n}(C+\log\pi)+\frac{\pi}{n}.\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}.\,\text{Sin}\,\frac{im\pi}{n}\log\left\{\frac{\Gamma\left(\frac{n+i}{n}\right)}{\Gamma\left(\frac{i}{n}\right)}\right\}\\&\qquad(m=\text{num. par}).\end{aligned}\right.$$

Ex. 1. Posito $n=2$, $m=1$, fit

$$\frac{\log 1}{1}-\frac{\log 3}{3}+\frac{\log 5}{5}-\frac{\log 7}{7}+\text{etc.}=\frac{\pi}{4}(\log\pi-C)-\pi\log\Gamma(\tfrac{3}{4})$$

atque inde

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$$\frac{1.5^{\frac{1}{5}}.9^{\frac{1}{9}}.13^{\frac{1}{13}}\dots}{3^{\frac{1}{3}}.7^{\frac{1}{7}}.11^{\frac{1}{11}}.15^{\frac{1}{15}}\dots} = \left\{\frac{\pi e^{-C}}{(\Gamma(\frac{3}{4}))^4}\right\}^{\frac{\pi}{4}}.$$

Ex. 2. Posito $n=3$, $m=1$, exsistente

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\text{etc.}=\frac{\pi}{3\sqrt{3}},$$

fit utique

$$\frac{\log 1}{1}-\frac{\log 2}{2}+\frac{\log 4}{4}-\frac{\log 5}{5}+\frac{\log 7}{7}-\text{etc.}=\frac{\pi}{\sqrt{3}}\log\left\{\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}-\frac{\pi}{3\sqrt{3}}(C+\log 2\pi),\right.$$

$$\frac{\log 1}{1}-\frac{\log 5}{5}+\frac{\log 7}{7}-\frac{\log 11}{11}+\frac{\log 13}{13}-\text{etc.}$$
$$=\frac{\pi}{2\sqrt{3}}\left\{\log\frac{2\pi}{\sqrt{3}}-C-2\log\left[\Gamma(\tfrac{5}{6}).\Gamma(\tfrac{2}{3}),\right.\right.$$

atque inde

$$\frac{1.4^{\frac{1}{4}}.7^{\frac{1}{7}}.10^{\frac{1}{10}}\dots}{2^{\frac{1}{2}}.5^{\frac{1}{5}}.8^{\frac{1}{8}}.10^{\frac{1}{11}}\dots}=\left\{\frac{\Gamma(\frac{1}{3})e^{-\frac{1}{3}C}}{\Gamma(\frac{2}{3}).(2\pi)^{\frac{1}{3}}}\right\}^{\frac{\pi}{\sqrt{3}}},$$

$$\frac{1.7^{\frac{1}{7}}.13^{\frac{1}{13}}.19^{\frac{1}{19}}\dots}{5^{\frac{1}{5}}.11^{\frac{1}{11}}.17^{\frac{1}{17}}.23^{\frac{1}{23}}\dots}=\left\{\frac{(2\pi)^{\frac{1}{2}}.e^{-\frac{1}{2}C}}{3^{\frac{1}{4}}.\Gamma(\frac{2}{3}).\Gamma(\frac{5}{6})}\right\}^{\frac{\pi}{\sqrt{3}}}.$$

Ex. 3. Posito $n=4$, $m=1$, fit

$$\frac{\log 1}{1}-\frac{\log 3}{3}+\frac{\log 5}{5}-\frac{\log 11}{11}+\frac{\log 13}{13}-\text{etc.}=\frac{\pi}{2\sqrt{2}}\log\left\{\frac{\Gamma(\frac{5}{8}).\Gamma(\frac{3}{8})}{2^{\frac{1}{4}}.\pi^{\frac{1}{2}}.e^{-\frac{1}{2}C}}\right\}+\tfrac{1}{2}\pi\log\left\{\frac{e^{-\frac{1}{4}C}.\pi^{\frac{1}{4}}}{\Gamma(\frac{3}{4})}\right\},$$

$$\frac{\log 1}{1}-\frac{\log 7}{7}+\frac{\log 9}{9}-\frac{\log 15}{15}+\frac{\log 17}{17}-\text{etc.}=\tfrac{1}{2}\pi\log\left\{\frac{\pi^{\frac{1}{4}}e^{-\frac{1}{4}C}}{\Gamma(\frac{3}{4})}\right\}+\frac{\pi}{2\sqrt{2}}\log\left\{\frac{2^{\frac{1}{4}}.\pi^{\frac{1}{2}}e^{-\frac{1}{2}C}}{\Gamma(\frac{5}{8}).\Gamma(\frac{3}{8})}\right\},$$

atque inde

$$\frac{1.5^{\frac{1}{5}}.13^{\frac{1}{13}}.21^{\frac{1}{21}}\dots}{3^{\frac{1}{3}}.11^{\frac{1}{11}}.19^{\frac{1}{19}}.27^{\frac{1}{27}}\dots}=\left(\frac{e^{-\frac{1}{4}C}\pi^{\frac{1}{4}}}{\Gamma(\frac{3}{4})}\right)^{\frac{1}{2}\pi}.\left\{\frac{\Gamma(\frac{5}{8}).\Gamma(\frac{3}{8})}{2^{\frac{1}{4}}.\pi^{\frac{1}{2}}.e^{-\frac{1}{2}C}}\right\}^{\frac{\pi}{2\sqrt{2}}},$$

$$\frac{1.9^{\frac{1}{9}}.17^{\frac{1}{17}}.25^{\frac{1}{25}}\dots}{7^{\frac{1}{7}}.15^{\frac{1}{15}}.23^{\frac{1}{23}}.31^{\frac{1}{31}}\dots}=\left(\frac{e^{-\frac{1}{4}C}.\pi^{\frac{1}{4}}}{\Gamma(\frac{3}{4})}\right)^{\frac{1}{2}\pi}.\left\{\frac{2^{\frac{1}{4}}.\pi^{\frac{1}{2}}.e^{-\frac{1}{2}C}}{\Gamma(\frac{5}{8}).\Gamma(\frac{3}{8})}\right\}^{\frac{\pi}{2\sqrt{2}}}.$$

§. 7.

Si in formula (29.) $a=0$ ponitur, fit

57. $$\int_0^\infty \frac{2u.du}{e^{\pi u}-e^{-\pi u}}.\frac{\operatorname{Cos}(s.\operatorname{ArcTang}\frac{u}{x})}{(x^2+u^2)^{\frac{1}{2}s}}=\frac{1}{\Gamma(s)}.\int_0^1\frac{y^x(\log\frac{1}{y})^{s-1}dy}{(1+y)^2};$$

unde subtractione facile habebimus

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$$58.\quad \int_0^\infty \frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{\mathrm{Cos}(s.\mathrm{ArcTang}\frac{u}{x})}{(x^2+u^2)^{\frac{1}{2}s}}\cdot du$$
$$=\frac{1}{\Gamma(s)}\cdot\int_0^1\left(\frac{\mathrm{Sin}\,a}{1+2y\,\mathrm{Cos}\,a+y^2}-\frac{a}{(1+y)^2}\right)\cdot y^x\left(\log\frac{1}{y}\right)^{s-1}dy.$$

Differentiemus hanc formulam respectu s tamquam variabilis, posteaque $s=1$ ponamus; tunc exsistente

$$\mathrm{Arc\,Tang}\,\frac{u}{x}=\tfrac{1}{2}\pi-\mathrm{Arc\,Tang}\,\frac{x}{u},\qquad Z'(1)=-C,$$

obtinebimus

$$59.\quad\begin{cases}\tfrac{1}{2}\pi\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{udu}{x^2+u^2}-xF(x)+\tfrac{1}{2}x\,W(x)\\ =-\int_0^1\left(\frac{\mathrm{Sin}\,a}{1+2y\,\mathrm{Cos}\,a+y^2}-\frac{a}{(1+y)^2}\right)\cdot y^x\log\left(\log\frac{1}{y}\right)dy\\ \quad+C\int_0^1\left(\frac{\mathrm{Sin}\,a}{1+2y\,\mathrm{Cos}\,a+y^2}-\frac{a}{(1+y)^2}\right)\cdot y^x\,.\,dy,\end{cases}$$

positis brevitatis causa

$$F(x)=\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{\mathrm{Arc\,Tang}\,\frac{x}{u}}{\frac{x}{u}}\cdot\frac{dy}{x^2+u^2}$$
$$=\Theta\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{dy}{x^2+u^2},\qquad(1>\Theta>0)$$
$$W(x)=\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{\log(x^2+u^2)dy}{x^2+u^2}.$$

Cum autem sit

$$F(x)=\tfrac{1}{2}\Theta\,.\int_0^\infty\frac{e^{au}-e^{-au}-2au}{u(e^{\pi u}-e^{-\pi u})}\cdot\frac{2udu}{x^2+u^2}+\Theta\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{du}{x^2+u^2}$$
$$=M\int_0^1\frac{2udu}{x^2+u^2}+\frac{\Theta}{x^2+\xi^2}\cdot\int_0^1\frac{e^{au}-e^{-2au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot du$$
$$=M(\log(1+x^2)-\log x^2)+\frac{\Theta N}{x^2+\xi^2}$$

(ubi, quodcumque sit x, M et N numquam non finitum conservant valorem et ξ quantitas <1 est), atque

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$$W(x)=\tfrac{1}{2}\int_0^\infty \frac{e^{au}-e^{-au}-2au}{u(e^{\pi u}-e^{-\pi u})}\cdot\frac{\log(x^2+u^2).2udu}{x^2+u^2}+\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{\log(x^2+u^2)dy}{x^2+u^2}$$

$$=M_1\int_0^\infty\frac{\log(x^2+u^2).2u.du}{x^2+u^2}+\frac{\log(x^2+\xi_1^2)}{x^2+\xi_1^2}\cdot\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}du$$

$$=M_1[(\log(1+x^2))^2-(\log x^2)^2]+N.\frac{\log(x^2+\xi_1^2)}{x^2+\xi_1^2}$$

(ubi, quodcumque x sit, idem omnino de M_1 et ξ_1 valet, quod supra de M et ξ dictum est); facile concludi licet

$$\lim.xF(x)=0,\qquad \lim.xW(x)=0,\qquad [x=\infty],$$

unde ex (59.) habebimus, convergente x in nihilum,

$$\tfrac{1}{2}\pi\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{du}{u}$$

$$=-\int_0^\infty\Big(\frac{\operatorname{Sin}a}{1+2y\operatorname{Cos}a+y^2}-\frac{a}{(1+y)^2}\Big)\log(\log\frac{1}{y})dy+C.\int_0^1\Big(\frac{\operatorname{Sin}a}{1+2y\operatorname{Cos}a+y^2}-\frac{a}{(1+y)^2}\Big)dy$$

seu

60. $$\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{du}{u}=-\frac{2}{\pi}\int_0^1\Big(\frac{\operatorname{Sin}a}{1+2y\operatorname{Cos}a+y^2}-\frac{a}{(1+y)^2}\Big)\log(\log\frac{1}{y})dy,$$

exsistente

$$\int_0^1\Big(\frac{\operatorname{Sin}a}{1+2y\operatorname{Cos}a+y^2}\Big)-\frac{a}{(1+y)^2}\Big)dy=0.$$

Facimus

$$K=\int_0^\infty\frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}}\cdot\frac{du}{u};$$

differentiatione respectu a facta, habebimus

$$\frac{dK}{da}=\int_0^\infty\frac{e^{au}-e^{-au}-2}{e^{\pi u}-e^{-\pi u}}\cdot du=\frac{1}{\pi}\int_0^1\frac{y^{\frac{a}{\pi}}+y^{-\frac{a}{\pi}}-2}{1-y^2}dy,$$

unde

61. $$\frac{dK}{da}=\frac{1}{2\pi}\{2Z'(\tfrac{1}{2})-Z'(\tfrac{1}{2}+\frac{a}{2\pi})-Z'(\tfrac{1}{2}-\frac{a}{2\pi}),$$

cum in genere sit*)

$$\int_0^1\frac{y^a-y^b}{1-y^2}dy=\tfrac{1}{2}Z'\tfrac{1}{2}(b+1)-\tfrac{1}{2}Z'\tfrac{1}{2}(a+1).$$

Ex formula vero (61.), integratione ab $a=0$ instituta, redundat

*) Vid. *Legendre Exerc. du Calc. Int.* Tom. 11. pag. 156.

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$$\int_0^1 \frac{e^{au}-e^{-au}-2au}{e^{\pi u}-e^{-\pi u}} \cdot \frac{du}{u} = \frac{a}{\pi} \cdot Z'(\tfrac{1}{2}) - \log \left\{ \frac{\Gamma(\frac{1}{2}+\frac{a}{2\pi})}{\Gamma(\frac{1}{2}-\frac{a}{2\pi})} \right\},$$

quod cum (60.) comparatum praebet

$$62. \quad \int_0^1 \frac{\operatorname{Sin} a . \log(\log \frac{1}{y})\, dy}{1+2y \operatorname{Cos} a+y^2} = a \int_0^1 \frac{\log(\log \frac{1}{y})}{1+y^2} - \tfrac{1}{2} a Z'(\tfrac{1}{2}) + \tfrac{1}{2}\pi \log \left\{ \frac{\Gamma(\frac{1}{2}+\frac{a}{2\pi})}{\Gamma(\frac{1}{2}-\frac{a}{2\pi})} \right\}.$$

Pro $a = \frac{1}{2}\pi$ beneficio formulae (12.) (prioris) obtinebimus

$$\int_0^1 \frac{\log(\log \frac{1}{y})\, dy}{(1+y)^2} = \tfrac{1}{2} Z'(\tfrac{1}{2}) + \tfrac{1}{2} \log 2\pi,$$

quod in (62.) ductum praebet denique

$$63. \quad \int_0^1 \frac{\log(\log \frac{1}{y})\, dy}{1+2y \operatorname{Cos} a+y^2} = \frac{\pi}{2 \operatorname{Sin} a} \cdot \log \left\{ \frac{(2\pi)^{\frac{a}{\pi}} . \Gamma(\frac{1}{2}+\frac{a}{2\pi})}{\Gamma(\frac{1}{2}-\frac{a}{2\pi})} \right\};$$

quae nova formula admodum notanda videtur.

§. 8.

Si formula ($44\frac{1}{2}$) utrimque per $\log(\log \frac{1}{y})$ multiplicetur, integratione ab $y = 0$ ad $y = 1$ instituta, ex formula (63.) habebimus

$$\overset{i=n}{\underset{i=1}{S}}(-1)^{i-1} \operatorname{Sin} ia \int_0^1 y^{i-1} \log(\log \frac{1}{y})\, dy$$
$$+ (-1)^n \{\operatorname{Sin}(n+1)a . G(n) + \operatorname{Sin} na . G(n+1)\}$$
$$= \tfrac{1}{2}\pi \log \left\{ \frac{(2\pi)^{\frac{a}{\pi}} . \Gamma(\frac{1}{2}+\frac{a}{2\pi})}{\Gamma(\frac{1}{2}-\frac{a}{2\pi})} \right\},$$

posito brevitatis causa

$$G(n = \int_0^1 \frac{y^n \log(\log \frac{1}{y}) dy}{1+2y \operatorname{Cos} a+y^2} = \Theta . \int_0^1 y^n \log(\log \frac{1}{y})\, dy, \qquad (1 > \Theta > 0)$$

unde, exsistente

$$\int_0^1 y^{r-1} \log(\log \frac{1}{y}) dy = - \frac{\log r + C}{r}$$

(ubi C est constans ille *Euleri* 0,577 216 ....), obtinebimus

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$$\overset{i=n}{\underset{i=1}{S}}(-1)^{i-1}.\frac{\operatorname{Sin} ia \log i}{i} + C.\overset{i=n}{\underset{i=1}{S}}(-1)^{i-1}\frac{\operatorname{Sin} ia}{i}$$

$$+ (-1)^{n+1}\{\operatorname{Sin}(n+1)a.G(n) + \operatorname{Sin} na.G(n+1)\}$$

$$= \tfrac{1}{2}\pi \log\left\{\frac{\Gamma(\frac{1}{2}-\frac{a}{2\pi})}{\Gamma(\frac{1}{2}+\frac{a}{2\pi})}\right\} - \tfrac{1}{2}a \log 2\pi$$

et

$$G(n) = -\frac{\Theta}{n+1}\{\log(n+1) + C\}, \qquad (1 > \Theta > 0).$$

Cum autem facile inde appareat, esse

$$\lim G(n) = 0, \qquad [n = \infty]$$

fit omnino

$$\overset{i=\infty}{\underset{i=1}{S}}(-1)^{i-1}.\frac{\operatorname{Sin} ia.\log i}{i} + C.\overset{i=\infty}{\underset{i=1}{S}}(-1)^{i-1}.\frac{\operatorname{Sin} ia}{i} = \tfrac{1}{2}\pi \log\left\{\frac{\Gamma(\frac{1}{2}-\frac{a}{2\pi})}{\Gamma(\frac{1}{2}-\frac{a}{2\pi})}\right\} - \tfrac{1}{2}a \log 2\pi,$$

unde, exsistente

$$\overset{i=\infty}{\underset{i=1}{S}}(-1)^{i-1}.\frac{\operatorname{Sin} ia}{i} = \tfrac{1}{2}a, \qquad (a < \pi),$$

hanc denique notandam formulam habebimus:

64. $$\frac{\operatorname{Sin} a.\log 1}{1} - \frac{\operatorname{Sin} 2a.\log 2}{2} + \frac{\operatorname{Sin} 3a.\log 3}{3} - \frac{\operatorname{Sin} 4a.\log 4}{4} + \text{etc.}$$

$$= \tfrac{1}{2}\pi \log\left\{\frac{\Gamma(\frac{1}{2}-\frac{a}{2\pi})}{\Gamma(\frac{1}{2}+\frac{a}{2\pi})}\right\} - \tfrac{1}{2}a(C + \log 2\pi);$$

atque, si $\pi - a$ loco a ponitur:

65. $$\frac{\operatorname{Sin} a.\log 1}{1} + \frac{\operatorname{Sin} 2a.\log 2}{2} + \frac{\operatorname{Sin} 3a.\log 3}{3} + \frac{\operatorname{Sin} 4a.\log 4}{4} + \text{etc.}$$

$$= \tfrac{1}{2}\pi \log\left\{\frac{\Gamma(\frac{a}{2\pi})}{\Gamma(1-\frac{a}{2\pi})}\right\} - \tfrac{1}{2}(\pi - a)(C + \log 2\pi).$$

Ex his duabus formulis, addendo, hanc tertiam obtinebimus:

66. $$\frac{\operatorname{Sin} a.\log 1}{1} + \frac{\operatorname{Sin} 3a.\log 3}{3} + \frac{\operatorname{Sin} 5a.\log 5}{5} + \text{etc.}$$

$$= \tfrac{1}{2}\pi \log\left\{\frac{\Gamma(\frac{a}{2\pi})}{\Gamma(\frac{1}{2}+\frac{a}{2\pi})}\right\} - \tfrac{1}{4}\pi\left(C - \log\frac{2\pi}{\operatorname{Tang}\frac{1}{2}a}\right).$$

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Ex. 1. Posito $a = \frac{1}{2}\pi$, fit

$$\frac{\log 1}{1} - \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 7}{7} + \text{etc.} = \frac{\pi}{4}(\log \pi - C) - \pi \log \Gamma(\tfrac{3}{4})$$

atque inde

$$\frac{1.5^{\frac{1}{5}}.9^{\frac{1}{9}}.13^{\frac{1}{13}}\ldots.}{3^{\frac{1}{3}}.7^{\frac{1}{7}}.11^{\frac{1}{11}}.15^{\frac{1}{15}}\ldots.} = \left\{\frac{\pi e^{-C}}{(\Gamma(\frac{3}{4}))^4}\right\};$$

quod jam supra ivenimus.

Ex. 2. Posito $a = \frac{1}{3}\pi$, fit

$$\frac{\log 1}{1} - \frac{\log 2}{2} + \frac{\log 4}{4} - \frac{\log 5}{5} + \frac{\log 7}{7} - \text{etc.} = \frac{\pi}{\sqrt{3}} \log\left\{\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\right\} - \frac{\pi}{3\sqrt{3}}(C + \log 2\pi),$$

$$\frac{\log 1}{1} + \frac{\log 2}{2} - \frac{\log 4}{4} - \frac{\log 5}{5} + \frac{\log 7}{7} + \text{etc.} = \frac{\pi}{3\sqrt{3}}(\log 2\pi - 2C) - \frac{2\pi}{\sqrt{3}} \log \Gamma(\tfrac{5}{6}),$$

$$\frac{\log 1}{1} - \frac{\log 5}{5} + \frac{\log 7}{7} - \frac{\log 11}{11} + \frac{\log 13}{13} - \text{etc.}$$

$$= \frac{\pi}{2\sqrt{3}}\left\{\log \frac{2\pi}{\sqrt{3}} - C - 2\log\left[\Gamma(\tfrac{5}{6}) \,.\, \Gamma(\tfrac{2}{3})\right]\right\},$$

(ex quibus formulis primam et tertiam jam supra invenimus), atque inde

$$\frac{1.4^{\frac{1}{4}}.7^{\frac{1}{7}}.10^{\frac{1}{10}}\ldots.}{2^{\frac{1}{2}}.5^{\frac{1}{5}}.8_{8}.11^{\frac{1}{11}}\ldots.} = \left\{\frac{\Gamma(\frac{1}{3})e^{-\frac{1}{3}C}}{\Gamma(\frac{2}{3}).(2\pi)^{\frac{1}{3}}}\right\}^{\frac{\pi}{\sqrt{3}}},$$

$$\frac{1.2^{\frac{1}{2}}.7^{\frac{1}{7}}.8^{\frac{1}{8}}\ldots.}{4^{\frac{1}{4}}.5^{\frac{1}{5}}.10^{\frac{1}{10}}.11^{\frac{1}{11}}\ldots.} = \left\{\frac{(2\pi)^{\frac{1}{3}}.e^{-\frac{2}{3}C}}{(\Gamma(\frac{5}{6}))^2}\right\}^{\frac{\pi}{\sqrt{3}}},$$

$$\frac{1.7^{\frac{1}{7}}.13^{\frac{1}{13}}.19^{\frac{1}{19}}\ldots.}{5^{\frac{1}{5}}.11^{\frac{1}{11}}.17^{\frac{1}{17}}.23^{\frac{1}{23}}\ldots.} = \left\{\frac{(2\pi)^{\frac{1}{2}}.e^{-\frac{1}{2}C}}{3^{\frac{1}{4}}.\Gamma(\frac{2}{3})\,\Gamma(\frac{5}{6})}\right\}^{\frac{\pi}{\sqrt{3}}}.$$

Ex. 3. Posito in (66.) $a = \frac{1}{4}\pi$, fit utique

$$\frac{\log 1}{1} + \frac{\log 3}{3} - \frac{\log 5}{5} - \frac{\log 7}{7} - \text{etc.} = \frac{\pi}{\sqrt{2}} \log\left\{\frac{2^{\frac{1}{4}}.\Gamma(\frac{1}{8}).\Gamma(\frac{3}{8})}{e^{-\frac{1}{2}C}.(2\pi)^{\frac{3}{4}}}\right\},$$

atque inde

$$\frac{1.3^{\frac{1}{3}}.9^{\frac{1}{9}}.11^{\frac{1}{11}}\ldots.}{5^{\frac{1}{5}}.7^{\frac{1}{7}}.13^{\frac{1}{13}}.15^{\frac{1}{15}}\ldots.} = \left\{\frac{2^{\frac{1}{4}}.\Gamma(\frac{1}{8}).\Gamma(\frac{3}{8})}{e^{-\frac{1}{2}C}.(2\pi)^{\frac{3}{4}}}\right\}^{\frac{\pi}{\sqrt{2}}}.$$

§. 2.

Revocemus jam formulum (1.) (eam etiam pro $x = 0$ valere demonstravimus), quam utrimque per

$$\frac{e^{au} - e^{-au}}{e^{\pi u} - e^{-\pi u}} \,.\, du$$

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multiplicemus; tunc integratione inter $u=0$ et $u=\infty$ facta, fit

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\log(x^2+u^2)\,du = 2\int_0^\infty \frac{dz}{z}\cdot\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}(e^{-z}-e^{-xz}\operatorname{Cos} uz)\,du,$$

unde beneficio notae formulae*) sequitur

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\log(x^2+u^2)\,du = \int_0^\infty \frac{e^{-z}\,dz}{z}\left(\operatorname{Sec}\tfrac{1}{2}a - \frac{2e^{-(x-\frac{1}{2})z}(1+e^{-z})\operatorname{Cos}\frac{1}{2}a}{1+2e^{-z}\operatorname{Cos}a+e^{-2z}}\right),$$

sive, posito

$$e^{-z}=y,\qquad \text{unde}\qquad z=\log\left(\frac{1}{y}\right),\qquad e^{-z}dz=-dy:$$

67. $$\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot\log(x^2+u^2) = \int_0^1\left(\operatorname{Sec}.\tfrac{1}{2}a - \frac{2y^{x-1}(1+y)\operatorname{Cos}\frac{1}{2}a}{1+2y\operatorname{Cos}a+y^2}\right)\cdot\frac{dy}{\log\frac{1}{y}}$$
$$(a<\pi).$$

Appellemus

68. $$\mathfrak{L}(a,x) = \int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot\log(x^2+u^2)\,du.$$

Si in formula (2.) $x+\frac{1}{2}$ et $x-\frac{1}{2}$ loco x substituimus, addendo habebimus hanc simplicem inter $L(a,x)$ et $\mathfrak{L}(a,x)$ correlationem:

69. $$2\operatorname{Sin}\tfrac{1}{2}a.\mathfrak{L}(a,x) = L(a,x+\tfrac{1}{2}) + L(a,x-\tfrac{1}{2}).$$

His cognitis, si a est ad π in ratione qualibet commensurabili, i. e. pro $a=\frac{m\pi}{n}$, facile pro functione $\mathfrak{L}(a,x)$ invenire possumus formulas, quae cum iis, quae supra pro $L(a,x)$ proposuimus, analogae sunt. Etenim si ponitur in (7.) $x+\frac{1}{2}$ et $x-\frac{1}{2}$ loco x, inde addendo habebimus

$$L(a,x+\tfrac{1}{2}) + L(a,x-\tfrac{1}{2})$$
$$= 2\operatorname{Tang}\tfrac{1}{2}a\log 2n + 2\mathop{S}_{i=1}^{i=n-1}(-1)^{i-1}\operatorname{Sin} ia\log\left\{\frac{\Gamma\left(\frac{x+n+i+\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{2n}\right)}\right\}$$
$$= 2\mathop{S}_{i=1}^{i=n}(-1)^{i-1}\operatorname{Sin} ia\log\left\{\frac{\Gamma\left(\frac{x+n+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$
$$= 2\operatorname{Tang}\tfrac{1}{2}a\log 2n + 2\mathop{S}_{i=1}^{i=n-1}(-1)^{i-1}(\operatorname{Sin} ia-\operatorname{Sin}(i-1)a)\log\left\{\frac{\Gamma\left(\frac{x+n+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$
$$(m+n=\text{num. imp.})$$

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*) Vid. *Exerc. d. Calc. Intégr. par Legendre.* Tom. II. pag. 186.

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$$L(a, x + \tfrac{1}{2}) + L(a, x - \tfrac{1}{2})$$

$$= 2\,\mathrm{Tang}\,\tfrac{1}{2}a\,\log.n + 2\sum_{i=1}^{i=\frac{1}{2}(n-1)}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\log\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x-i-\frac{1}{2}}{n}\right)}\right\}$$

$$+ 2\sum_{i=1}^{i=\frac{1}{2}(n-1)}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\log\left\{\frac{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)}\right\}$$

$$= 2\,\mathrm{Tang}\,\tfrac{1}{2}a.\log n + 2\sum_{i=1}^{i=\frac{1}{2}(n-1)}(-1)^{i-1}\,(\mathrm{Sin}\,ia - \mathrm{Sin}\,(i-1)a)\,\log\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}\right\}$$

$$(m + n = \text{num. par.}),$$

cum sit

$$\sum_{i=1}^{i=\frac{1}{2}(n-1)}(-1)^{i-1}\,\mathrm{Sin}\,ia\,\log\left\{\frac{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)}\right\} = -\sum_{i=1}^{i=\frac{1}{2}(n-1)}(-1)^{i-1}\,\mathrm{Sin}\,(i-1)a\,\log\left\{\frac{\Gamma\left(\frac{x+n-i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i+\frac{1}{2}}{n}\right)}\right\}$$

Hinc autem, divisione per $2\,\mathrm{Sin}\,\frac{1}{2}a$ facta, quia est in genere

$$\mathrm{Sin}\,ia - \mathrm{Sin}\,(i-1)\,a = 2\,\mathrm{Sin}\,\tfrac{1}{2}a\,\mathrm{Cos}\,(i-\tfrac{1}{2})\,a,$$

beneficio formulae (69.) colligitur

70.
$$\left\{\begin{array}{l}
\mathfrak{L}(a,x) = \mathrm{Sec}\,\tfrac{1}{2}a\,\log 2n + 2\sum_{i=1}^{i=n}(-1)^{i-1}\,\mathrm{Cos}\,(i-\tfrac{1}{2})\,a\,\log\left\{\frac{\Gamma\left(\frac{x+n+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\} \\
\qquad\qquad (m + n = \text{num. imp.}) \\
\mathfrak{L}(a, x) = \mathrm{Sec}\,\tfrac{1}{2}a\,\log n + 2\sum_{i=1}^{i=\frac{1}{2}(n-1)}(-1)^{i-1}\,\mathrm{Cos}\,(i-\tfrac{1}{2})\,a\,\log\left\{\frac{\Gamma\left(\frac{x+n-i+\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{n}\right)}\right\}.
\end{array}\right.$$

§. 10.

Si in formulis (70.) pro $x = 0$ faciamus

$$e^{\frac{\pi u}{n}} = y, \quad \text{unde} \quad u = \frac{n}{\pi}\log y,$$

habebimus utique, exsistente $a = \dfrac{m\pi}{n}$,

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$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}} \left\{\log\frac{n}{\pi}+\log(\log y)\right\}dy$$

$$=\frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n}\log 2n+\frac{\pi}{n}\cdot\underset{i=1}{\overset{i=n}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\cdot\log\left\{\frac{\Gamma(\frac{1}{2}+\frac{i-\frac{1}{2}}{2n})}{\Gamma(\frac{i-\frac{1}{2}}{2n})}\right\}$$

$$(m+n=\text{num. imp.})$$

$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}} \left\{\log\frac{n}{\pi}+\log(\log y)\right\}dy$$

$$=\frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n}\log n+\frac{\pi}{n}\cdot\underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\cdot\log\left\{\frac{\Gamma(1-\frac{i-\frac{1}{2}}{n})}{\Gamma(\frac{i-\frac{1}{2}}{n})}\right\}$$

$$(m+n=\text{num. par}).$$

Hinc vero, cum sit

$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}dy=\frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n},$$

sequitur

71.

$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\cdot\log(\log y)\,dy$$

$$=\frac{\pi}{2n}\operatorname{Sec}.\frac{m\pi}{2n}\log 2\pi+\frac{\pi}{n}\cdot\underset{i=1}{\overset{i=n}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\cdot\log\left\{\frac{\Gamma(\frac{1}{2}-\frac{i-\frac{1}{2}}{2n})}{\Gamma(\frac{i-\frac{1}{2}}{2n})}\right\}$$

$$(m+n=\text{num. imp.})$$

$$\int_1^\infty \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\cdot\log(\log y)\,dy$$

$$=\frac{\pi}{2n}\operatorname{Sec}\frac{m\pi}{2n}\log\pi+\frac{\pi}{n}\cdot\underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})\frac{m\pi}{n}\cdot\log\left\{\frac{\Gamma(1-\frac{i-\frac{1}{2}}{n})}{\Gamma(\frac{i-\frac{1}{2}}{n})}\right\}$$

$$(m+n=\text{num. par}).$$

Posito in posteriori $m=1$ et $n=3$, mutato y^2 in y, habebimus post reductionem quandam facillimam,

72. $$\int_1^\infty \frac{\log(\log y)\,dy}{1-y+y^2}=\frac{2\pi}{\sqrt{3}}\left(\tfrac{5}{6}\log 2\pi-\log\Gamma(\tfrac{1}{6})\right).$$

§. 11.

In suppositione $a=\frac{m\pi}{n}$, exsistente $m+n$ numero integro impari, constat

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$$\text{Tang}\,\tfrac{1}{2}a = \mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Sin}\,ia.$$

atque, posito $i-1$ loco i,

$$\text{Tang}\,\tfrac{1}{2}a = -\mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Sin}\,(i-1)\,a;$$

unde addendo, divisione per $2\,\text{Sin}\,\frac{1}{2}a$ facta, fit omnino

$$73.\qquad \text{Sec}\,\tfrac{1}{2}a = \mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Cos}\,(i-\tfrac{1}{2})\,a.$$

Quocirca prior formularum (70.) ita etiam exhiberi potest:

$$74.\qquad \mathfrak{L}(a,x) = 2\mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Cos}\,(i-\tfrac{1}{2})\,a\,\log\left\{\frac{(2n)^{\frac{1}{2}}.\,\Gamma\left(\frac{1}{2}+\frac{x+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$

$$(m+n = \text{num. imp.}).$$

Ponatur hic $x+n$ loco x; fit, utique addendo,

$$75.\qquad \mathfrak{L}(a,x+n)+\mathfrak{L}(a,x) = 2\mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Cos}\,(i-\tfrac{1}{2})a\,\log\,(x+i-\tfrac{1}{2})$$

$$(m+n = \text{num. imp.}).$$

Ceterum ex eâdem formula, si $n-x$ loco x substituitur, similiter addendo prodit:

$$\mathfrak{L}(a,x)+\mathfrak{L}(a,n-x) = 2\mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Cos}\,(i-\tfrac{1}{2})\,a\,.\log\left\{\frac{(2n)^{\frac{1}{2}}.\,\Gamma\left(\frac{1}{2}+\frac{x+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{x+i-\frac{1}{2}}{2n}\right)}\right\}$$

$$+\,2\mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Cos}\,(i-\tfrac{1}{2})\,a\,.\log\left\{\frac{(2n)^{\frac{1}{2}}.\,\Gamma\left(\frac{1}{2}+\frac{n+i-x-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{n+i-x-\frac{1}{2}}{2n}\right)}\right\},$$

sive, cum sit

$$\mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Cos}\,(i-\tfrac{1}{2})\,a\,.\log\left\{\frac{(2n)^{\frac{1}{2}}.\,\Gamma\left(\frac{1}{2}+\frac{n+i-x-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{n+i-x-\frac{1}{2}}{2n}\right)}\right\}$$

$$= \mathop{S}_{i=1}^{i=n}(-1)^{i-1}\,\text{Cos}\,(i-\tfrac{1}{2})\,a\,.\log\left\{\frac{(2n)^{\frac{1}{2}}.\,\Gamma\left(\frac{3}{2}-\frac{x+i-\frac{1}{2}}{2n}\right)}{\Gamma\left(1-\frac{x+i-\frac{1}{2}}{2n}\right)}\right\},$$

etiam, respectu ad cognitam functionis Γ proprietatem (eodem modo ac supra ad formulae (22.) deductionem):

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$$76.\quad \mathfrak{L}(a,x)+\mathfrak{L}(a,n-x)=2\overset{i=n}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a.\log\left[(i-x-\tfrac{1}{2}).\operatorname{Cotang}\frac{(i-1-\frac{1}{2})\pi}{2n}\right]$$

$$(m+n=\text{num. imp.})$$

E posteriore etiam formularum (70.) relationes analogas derivare possumus. Etenim, substituto ibi $x+n$ pro x, subtrahendo erit

$$77.\quad \mathfrak{L}(a,x+n)-\mathfrak{L}(a,x)=2\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a.\log\left\{\frac{x+n+\frac{1}{2}-i}{x+i-\frac{1}{2}}\right\}$$

$$(m+n=\text{num. par}).$$

Atque si in eadem formula $n-x$ loco x ponitur, fit etiam subtrahendo, calculo facto,

$$78.\quad \mathfrak{L}(a,x)-\mathfrak{L}(a,n-x)=2\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a.\log\left\{\frac{x+\frac{1}{2}-i}{n-x-i+\frac{1}{2}}\cdot\frac{\operatorname{Sin}\frac{(x+i-\frac{1}{2})\pi}{n}}{\operatorname{Sin}\frac{(x-i+\frac{1}{2})\pi}{n}}\right\}$$

$$(m+n=\text{num. par}).$$

E formulis (75. et 77.), una cum (76. et 78.) conjunctis, sequitur, ut, sive impar, sive par sit $m+n$, functionem $\mathfrak{L}(a,x)$ pro quovis ipsius x valore cognitam habeamus, si modo per totam periodum ab $x=0$ ad $x=\frac{1}{2}n$ cognita sit. Si in (76.) $x=\frac{1}{2}n$ ponitur, prodit

$$79.\quad \mathfrak{L}(a,\tfrac{1}{2}n)=\overset{i=n}{\underset{i=1}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a\log\left\{(\tfrac{1}{2}(n+1)-i).\operatorname{Cotang}\left(\frac{\pi}{4}-\frac{(i-\frac{1}{2}).\pi}{2n}\right)\right\}$$

$$(m+n=\text{num. imp.}),$$

unde, si simul formulam (75.) consideras, colligitur, exsistente $m+n$ numero impari, $\mathfrak{L}(a,x)$ pro $x=\frac{1}{2}(2i+1)n$ per logarithmos et functiones circulares numquam non exprimi posse. Haec formula (79.) pro $i=\frac{1}{2}(n+1)$ expressionem $\log 0.\infty$ praesentat; sed verus ejus valor facile invenitus est $\log\left(\frac{2n}{\pi}\right)$.

Ponamus jam successive loco x:

$$x,\quad x+\frac{2n}{r},\quad x+\frac{4n}{r},\quad x+\frac{6n}{r},\ \ldots\ x+\frac{(r-1).2n}{r},$$

in priori formularum (70.), et

$$x,\quad x+\frac{n}{r},\quad x+\frac{2n}{r},\quad x+\frac{3n}{r},\ \ldots\ x+\frac{(r-1).n}{r}$$

in posteriori; eodem modo ac supra in §. 3. e cognitae functionis Γ proprietate obtinebimus

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80. $$\left\{\begin{array}{l} \mathfrak{L}(a,x) + \mathfrak{L}(a, x+\frac{2n}{r}) + \mathfrak{L}(a, x+\frac{4n}{r}) + \ldots + \mathfrak{L}(a, x+\frac{(r-1).2n}{r}) \\ = r \operatorname{Sec}\tfrac{1}{2}a \log.2n + 2\underset{i=1}{\overset{i=n}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a \,.\, \log\left\{\dfrac{\Gamma(\frac{1}{2}r+\frac{r(x+i-\frac{1}{2})}{n})}{r^{\frac{1}{2}r}\,\Gamma(\frac{r(x+i-\frac{1}{2})}{2n})}\right\} \\ \qquad\qquad (m+n = \text{num. imp.}) \\ \mathfrak{L}(a,x) + \mathfrak{L}(a, x+\frac{n}{r}) + \mathfrak{L}(a, x+\frac{2n}{r}) + \ldots\ldots \mathfrak{L}(a, x+\frac{(r-1).n}{r}) \\ = r \operatorname{Sec}\tfrac{1}{2}a.\log n + 2\underset{i=1}{\overset{i=\frac{1}{2}(n-1)}{S}}(-1)^{i-1}\operatorname{Cos}(i-\tfrac{1}{2})a.\log\left\{\dfrac{\Gamma(r+\frac{r(x-i+\frac{1}{2})}{n})}{r^{r-\frac{r(2i-1)}{n}}.\Gamma(\frac{r(x+i-\frac{1}{2})}{n})}\right\} \end{array}\right.$$

unde apparet, priorem harum summarum, cum r numerus par est, finite semper per logarithmos posse signari.

§. 12.

Multiplicemus formulam (27.) utrimque per

$$\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}} \,.\, du;$$

tunc, integratione inter $u = 0$ et $u = \infty$ instituta, fit

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}} \cdot \frac{\operatorname{Cos}(s.\operatorname{ArcTang}\frac{u}{x})\,du}{(x^2+u^2)^{\frac{1}{2}s}}$$
$$= \frac{1}{\Gamma(s)} \cdot \int_0^\infty e^{-xz}.z^{s-1}dz \int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}} \operatorname{Cos} uz \,.\, du$$
$$= \frac{\operatorname{Cos}\frac{1}{2}a}{\Gamma(s)} \cdot \int_0^\infty \frac{z^{s-1}.e^{-(x-\frac{1}{2})}(1+e^{-z}).e^{-z}\,dz}{1+2e^{-z}\operatorname{Cos}a+e^{-2z}}$$

unde, si in dextero membro $e^{-z} = y$ ponimus, prodit

81. $$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}} \cdot \frac{\operatorname{Cos}(s.\operatorname{ArcTang}\frac{u}{x})}{(x^2+u^2)^{\frac{1}{2}s}}\,du = \frac{\operatorname{Cos}\frac{1}{2}a}{\Gamma(s)} \cdot \int_0^1 \frac{y^{x-\frac{1}{2}}(1+y)(\log\frac{1}{y})^{s-1}dy}{1+2y\operatorname{Cos}a+y^2},$$

et pro $x = 0$ (si modo $1 > s > 0$) mutato y in y^2,

82. $$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}} \cdot \frac{du}{u^s} = \frac{2^s\operatorname{Cos}\frac{1}{2}a}{\operatorname{Cos}\frac{1}{2}s\pi.\Gamma(s)} \cdot \int_0^1 \frac{(1+y^2)(\log\frac{1}{y})^{s-1}dy}{1+2y\operatorname{Cos}a+y^2}.$$

Ex hac vero formula, si $a = \frac{m\pi}{n}$ et $e^{-\frac{\pi u}{n}} = y$ ponitur, transformatione peracta, fit

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$$83.\quad \int_0^1 \frac{y^{m-1}+y^{-m-1}}{y^n+y^{-n}}\cdot\frac{dy}{(\log\frac{1}{y})^s}=\frac{2.\left(\frac{\pi}{n}\right)^{1-s}\operatorname{Cos}\frac{m\pi}{2n}}{\operatorname{Cos}\frac{1}{2}s\pi.\Gamma(s)}.\int_0^1\frac{(1+y^2)(\log\frac{1}{y})^{s-1}dy}{1+2y^2\operatorname{Cos}\frac{m\pi}{n}+y^4}.$$

Posito $m=1$ et $n=2$, si brevitatis causa appellamus

$$84.\quad Q(s)=\int_0^1\frac{1+y^2}{1+y^4}\cdot(\log\frac{1}{y})^{s-1}dy,$$

hanc inter functionem $Q(s)$ et ejus complementariam relationem habebimus:

$$85.\quad Q(1-s)=\frac{2.\left(\frac{\pi}{4}\right)^{1-s}.\operatorname{Sin}\frac{\pi}{4}}{\operatorname{Cos}\frac{1}{2}s\pi.\Gamma(s)}\cdot Q(s).$$

Posito porro in (83.) $m=1$ et $m=3$, reductionibus factis, provenit

$$86.\quad \int_0^1\frac{(\log\frac{1}{y})^{-s}dy}{1-y+y^2}=\frac{(\frac{2}{3}\pi)^{1-s}\operatorname{Sin}\frac{1}{3}\pi}{\operatorname{Cos}\frac{1}{2}s\pi.\Gamma(s)}.\int_0^1\frac{2(1+y^2)}{1+y^2+y^4}\cdot(\log\frac{1}{y})^{s-1}dy.$$

Cum autem sit

$$87.\quad \int_0^1\frac{2(1+y^2)}{1+y^2+y^4}\cdot(\log\frac{1}{y})^{s-1}dy=\int_0^1\frac{(\log\frac{1}{y})^{s-1}dy}{1+y+y^2}+\int_0^1\frac{(\log\frac{1}{y})^{s-1}dy}{1-y+y^2},$$

atque etiam

$$\int_0^1\frac{(\log\frac{1}{y})^{s-1}dy}{1+y+y^2}=\int_0^1\frac{2^{2s}.y^3(\log\frac{1}{y})^{s-1}dy}{1+y^4+y^8}$$

$$=2^{2s-2}\int_0^1\frac{2y(\log\frac{1}{y})^{s-1}dy}{1-y^2+y^4}-2^{2s-2}\int_0^1\frac{2y(\log\frac{1}{y})^{s-1}dy}{1+y^2+y^4},$$

i. e.

$$(1+2^{s-1}).\int_0^1\frac{(\log\frac{1}{y})^{s-1}dy}{1+y+y^2}=2^{s-1}.\int_0^1\frac{(\log\frac{1}{y})^{s-1}dy}{1-y+y^2};$$

fit etiam ex (87.),

$$\int_0^1\frac{2(1+y^2)}{1+y^2+y^4}\cdot(\log\frac{1}{y})^{s-1}dy=\frac{1+2^s}{1+2^{s-1}}.\int_0^1\frac{(\log\frac{1}{y})^{s-1}dy}{1-y+y^2},$$

quod in (86.) ductam, posito brevitatis causa

$$88.\quad R(s)=\int_0^1\frac{(\log\frac{1}{y})^{s-1}}{1-y+y^2},$$

hanc notandam relationem praebet:

$$89.\quad R(1-s)=\frac{(\frac{1}{3}\pi)^{1-s}\operatorname{Sin}\frac{1}{3}\pi}{\operatorname{Cos}\frac{1}{2}s\pi.\Gamma(s)}\cdot\frac{1+2^s}{1+2^{s-1}}\cdot R(s).$$

Ex formulis (85. et 89.) logarithmando obtinebimus

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$$\log Q(1-s) = \log 2 \operatorname{Sin} \tfrac{1}{4}\pi + (1-s) \log \tfrac{1}{4}\pi - \log \operatorname{Cos} \tfrac{1}{2}s\pi - \log \Gamma(s) + \log Q(s),$$
$$\log R(1-s) = (1-s) \log \tfrac{1}{3}\pi + \log \operatorname{Sin} \tfrac{1}{3}\pi + \log (1+2^s) - \log \operatorname{Cos} \tfrac{1}{2}s\pi - \log \Gamma(s) - \log(1+2^{s-1}) + \log Q(s),$$

unde, positis brevitatis causa

$$M(s) = \frac{dQ(s)}{ds} = \int_0^1 \frac{1+y^2}{1+y^4} \cdot \log(\log \frac{1}{y}) \cdot (\log \frac{1}{y})^{s-1} dy \quad \text{et}$$
$$N(s) = \frac{dR(s)}{ds} = \int_0^1 \cdot \frac{\log(\log \frac{1}{y}) \cdot (\log \frac{1}{y})^{s-1} dy}{1-y+y^2},$$

differentiando habebimus

90. $$\begin{cases} \dfrac{M(s)}{Q(s)} + \dfrac{M(1-s)}{Q(1-s)} = \log \tfrac{1}{4}\pi - \tfrac{1}{2}\pi \operatorname{Tang} \tfrac{1}{2}s\pi + Z'(s) \quad \text{et} \\ \dfrac{N(s)}{R(s)} + \dfrac{N(1-s)}{R(1-s)} = \log \tfrac{1}{3}\pi - \tfrac{1}{2}\pi \operatorname{Tang} \tfrac{1}{2}s\pi + Z'(s) - \dfrac{\log 2}{(1+2^s)(1+2^{1-s})}. \end{cases}$$

Supponamus $s = \frac{1}{2}$; tunc erit

$$\int_0^1 \frac{1+y^2}{1+y^4} \cdot \frac{\log(\log \frac{1}{y}) dy}{\sqrt{(\log \frac{1}{y})}} = \tfrac{1}{2}(\log \frac{\pi}{16} - \tfrac{1}{2}\pi - C) \cdot \int_0^1 \frac{1+y^2}{1+y^4} \cdot \frac{dy}{\sqrt{(\log \frac{1}{y})}},$$

$$\int_0^1 \frac{\log(\log \frac{1}{y}) dy}{1-y+y^2} = \tfrac{1}{2}(\log \tfrac{1}{3}\pi - \tfrac{1}{2}\pi - C - (5-2\sqrt{2}) \log 2) \cdot \int_0^1 \frac{1}{1+y+y^2} \cdot \frac{dy}{\sqrt{(\log \frac{1}{y})}},$$

seu, ponendo $\log \frac{1}{y} = x$:

91. $$\begin{cases} \displaystyle\int_0^\infty \frac{e^x+e^{-x}}{e^{2x}+e^{-2x}} \cdot \frac{\log x . dx}{\sqrt{x}} = \tfrac{1}{2}(\log \frac{\pi}{16} - \tfrac{1}{2}\pi - C) \cdot \int_0^\infty \frac{e^x+e^{-x}}{e^{2x}+e^{-2x}} \cdot \frac{dx}{\sqrt{x}}, \\ \displaystyle\int_0^\infty \frac{\log x}{e^x-1+e^{-x}} \cdot \frac{dx}{\sqrt{x}} = \tfrac{1}{2}(\log \tfrac{1}{3}\pi - \tfrac{1}{2}\pi - C - (5-2\sqrt{2})\log 2) \cdot \int_0^\infty \frac{dx}{e^{2x}-1+e^{-2x}} \cdot \frac{1}{\sqrt{x}}, \end{cases}$$

quae quidem ejusdem omnino generis sunt, ac formulae (38.).

Ex his formulis analogâ omnino methodo, qua ad formulas (42.) inveniendas usi sumus, has duas aeque notandas relationes deducere licet:

92. $$\begin{cases} \dfrac{\log 1}{\sqrt{1}} + \dfrac{\log 3}{\sqrt{3}} - \dfrac{\log 5}{\sqrt{5}} - \dfrac{\log 7}{\sqrt{7}} + \dfrac{\log 9}{\sqrt{9}} + \text{etc.} \\ = \tfrac{1}{2}(\tfrac{1}{2}\pi - C - \log \pi) \{ \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{5}} - \dfrac{1}{\sqrt{7}} + \dfrac{1}{\sqrt{9}} + \text{etc.} \} \quad \text{et} \\ \dfrac{\log 1}{\sqrt{1}} + \dfrac{\log 2}{\sqrt{2}} - \dfrac{\log 4}{\sqrt{4}} - \dfrac{\log 5}{\sqrt{5}} + \dfrac{\log 7}{\sqrt{7}} + \dfrac{\log 8}{\sqrt{8}} - \dfrac{\log 10}{\sqrt{10}} - \text{etc.} \\ = \tfrac{1}{2}(\tfrac{1}{2}\pi - C - 2\sqrt{2}\log 2 - \log \dfrac{\pi}{6}) \{ \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{4}} - \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \dfrac{1}{\sqrt{8}} - \text{etc.} \} \end{cases}$$

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§. 13.

Si

$$\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}$$

in seriem evolvimus, fit utique identice:

$$\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}=\underset{i=0}{\overset{i=n-1}{S}}(-1)^i\,[e^{-[(2i+1)\pi-a]u}+e^{-[(2i+1)\pi+a]u}]+\frac{(-1)^n e^{-2n\pi u}(e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}},$$

unde

$$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot\frac{du}{u^s}$$

$$=\Gamma(1-s)\,.\underset{i=0}{\overset{i=n-1}{S}}(-1)^i\,[\frac{1}{((2i+1)\pi-a)^{1-s}}+\frac{1}{((2i+1)\pi+a)^{1-s}}]+(-1)^n.\,\varphi_1(n),$$

ubi brevitatis causa posuimus

$$\varphi_1(n)=\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot\frac{e^{-2n\pi u}\,du}{u^s}=E\,.\int_0^\infty \frac{e^{-2n\pi u}\,du}{u^s}=\frac{M\Gamma(1-s)}{(2n\pi)^{1-s}},$$

(exsistente E quantitate quadam finita); atque, cum evidenter sit

$$\lim\varphi(n)=0 \qquad [n=\infty],$$

etiam

93. $$\int_0^\infty \frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot\frac{du}{u^s}=\Gamma(1-s).\underset{i=0}{\overset{i=\infty}{S}}(-1)^i[\frac{1}{((2i+1)\pi-a)^{1-s}}-\frac{1}{((2i+1)\pi+a)^{1-s}}].$$

Jam vero exsistente

$$\frac{(1+y^2)\operatorname{Cos}\frac{1}{2}a}{1+2y^2\operatorname{Cos}a+y^4}=\underset{i=0}{\overset{i=n-1}{S}}(-1)^i.y^{2i}.\operatorname{Cos}(i+\tfrac{1}{2})a+(-1)^n.y^{2n}.\frac{\operatorname{Cos}(n+\frac{1}{2})a+y^2\operatorname{Cos}(n-\frac{1}{2})a}{1+2y^2\operatorname{Cos}a+y^4},$$

habebimus etiam

$$\int_0^1 \frac{(1-y^2)\operatorname{Cos}\frac{1}{2}a}{1+2y^2\operatorname{Cos}a+y^4}\cdot\frac{dy}{(\log\frac{1}{y})^{1-s}}$$

$$=\Gamma(s).\underset{i=1}{\overset{i=n}{S}}(-1)^i\,.\,\frac{\operatorname{Cos}(i+\frac{1}{2})a}{(2i+1)^s}+(1)^n(p(n)\operatorname{Cos}(n+\tfrac{1}{2})a+p(n+1)\operatorname{Cos}(n-\tfrac{1}{2})a),$$

posito brevitatis causa

$$p(n)=\int_0^1\frac{y^{2n}.(\log\frac{1}{y})^{s-1}dy}{1+2y^2\operatorname{Cos}a+y^4}=\Theta\,.\int_0^1 y^{2n}(\log\frac{1}{y})^{s-1}dy=\frac{\Theta.\Gamma(s)}{(2n+1)^s}\quad(1>\Theta>0).$$

Manifestum igitur est

$$\lim p(n)=0 \qquad [n=\infty],$$

unde fit utique

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$$94. \quad \int_0^1 \frac{(1+y^2)\operatorname{Cos}\frac{1}{2}a}{1+2y^2\operatorname{Cos}a+y^4} \cdot \frac{dy}{(\log\frac{1}{y})^{1-s}} = \Gamma(s) \,.\, \underset{i=1}{\overset{i=\infty}{\mathrm{S}}}(-1)^i \,.\, \frac{\operatorname{Cos}(i+\frac{1}{2})a}{(2i+1)^s}\,.$$

Valoribus vero, quos formulae (**93.** et **94.**) praebent, in (**82.**) substitutis, si s in $1-s$ mutatur, prodit,

$$95. \quad \begin{cases} \frac{1}{(\pi-a)^s}+\frac{1}{(\pi+a)^s}-\frac{1}{(3\pi-a)^s}-\frac{1}{(3\pi+a)^s}+\frac{1}{(5\pi-a)^s}+\frac{1}{(5\pi+a)^s}-\text{etc.} \\ =\frac{2^{1-s}}{\operatorname{Sin}\frac{1}{2}s\pi\,.\,\Gamma(s)}\left\{\frac{\operatorname{Cos}\frac{1}{2}a}{1^{1-s}}-\frac{\operatorname{Cos}\frac{3}{2}a}{3^{1-s}}+\frac{\operatorname{Cos}\frac{5}{2}a}{5^{1-s}}-\frac{\operatorname{Cos}\frac{7}{2}a}{7^{1-s}}+\frac{\operatorname{Cos}\frac{9}{2}a}{9^{1-s}}-\text{etc.}\right\} \end{cases}$$

et si $\pi-a$ loco a ponitur:

$$96. \quad \begin{cases} \frac{1}{a^s}+\frac{1}{(2\pi-a)^s}-\frac{1}{(2\pi+a)^s}-\frac{1}{(4\pi-a)^s}+\frac{1}{(4\pi+a)^s}+\frac{1}{(6\pi-a)^s}-\text{etc.} \\ =\frac{2^{1-s}}{\operatorname{Sin}\frac{1}{2}s\pi\,\Gamma(s)}\left\{\frac{\operatorname{Sin}\frac{1}{2}a}{1^{1-s}}+\frac{\operatorname{Sin}\frac{3}{2}a}{3^{1-s}}+\frac{\operatorname{Sin}\frac{5}{2}a}{5^{1-s}}+\frac{\operatorname{Sin}\frac{7}{2}a}{7^{1-s}}+\frac{\operatorname{Sin}\frac{9}{2}a}{9^{1-s}}+\text{etc.}\right\} \end{cases}$$

atque si $a=\frac{m\pi}{n}$ ($m<n$ num. integr.) supponamus:

$$97. \quad \begin{cases} \frac{1}{(n-m)^s}+\frac{1}{(n+m)^s}-\frac{1}{(3n-m)^s}-\frac{1}{(3n+m)^s}+\frac{1}{(5n-m)^s}+\frac{1}{(5n+m)^s}-\text{etc.} \\ =\frac{2.\left(\frac{\pi}{2n}\right)^s}{\operatorname{Sin}\frac{1}{2}s\pi\,.\,\Gamma(s)}\,.\left\{\frac{\operatorname{Cos}\frac{m\pi}{2n}}{1^{1-s}}-\frac{\operatorname{Cos}\frac{3m\pi}{2n}}{3^{1-s}}+\frac{\operatorname{Cos}\frac{5m\pi}{2n}}{5^{1-s}}-\frac{\operatorname{Cos}\frac{7m\pi}{2n}}{7^{1-s}}+\text{etc.}\right\} \\ \frac{1}{m^s}+\frac{1}{(2n-m)^s}-\frac{1}{(2n+m)^s}-\frac{1}{(4n-m)^s}+\frac{1}{(4n+m)^s}+\frac{1}{(6n-m)^s}-\text{etc.} \\ =\frac{2.\left(\frac{\pi}{2n}\right)^s}{\operatorname{Sin}\frac{1}{2}s\pi\,.\,\Gamma(s)}\left\{\frac{\operatorname{Sin}\frac{m\pi}{2n}}{1^{1-s}}+\frac{\operatorname{Sin}\frac{3m\pi}{2n}}{3^{1-s}}+\frac{\operatorname{Sin}\frac{5m\pi}{2n}}{5^{1-s}}+\frac{\operatorname{Sin}\frac{7m\pi}{2n}}{7^{1-s}}+\text{etc.}\right\} \end{cases}$$

Ex. 1. Posito $m=1$, $n=2$, fit

$$\frac{1}{1^s}+\frac{1}{3^s}-\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}-\text{etc.}$$

$$=\frac{2.\left(\frac{\pi}{4}\right)^s.\operatorname{Sin}\frac{\pi}{4}}{\operatorname{Sin}\frac{1}{2}s\pi\,.\,\Gamma(s)}\left\{\frac{1}{1^{1-s}}+\frac{1}{3^{1-s}}-\frac{1}{5^{1-s}}-\frac{1}{7^{1-s}}+\text{etc.}\right\};$$

quod jam supra in (**54.**) invenimus.

Ex. 2. Posito $m=1$, $n=3$, prior formularum (97.), factis quibusdam facillimis reductionibus, formulas (**53.**) reddit; e posteriore vero, positis brevitatis causa

$$T(s)=\frac{1}{1^s}+\frac{1}{5^s}-\frac{1}{7^s}-\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}-\text{etc.}$$

$$W(s)=\frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}-\frac{1}{11^s}+\text{etc.}$$

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fit

$$T(s)=\frac{2\left(\frac{\pi}{6}\right)^s}{\operatorname{Sin}\frac{1}{2}s\pi.\Gamma(s)}\{\tfrac{1}{2}T(1-s)+3^{s-1}W(1-s)\},$$

unde, cum sit

$$W(s)=T(s)-3^{-s}W(s),$$

habebimus

$$98.\qquad T(s)=\frac{(\frac{1}{2}\pi)^s}{\operatorname{Sin}\frac{1}{2}s\pi.\Gamma(s)}\cdot\frac{1+3^{-s}}{1+3^{s-1}}\cdot T(1-s).$$

§. 14.

Differentiemus jam formulam (93.) respectu s tamquam variabilis, tunc erit

$$\int_0^\infty\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot u^{-s}\log u\cdot du$$

$$=Z'(1-s).\int_0^\infty\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot\frac{du}{u^s}-\Gamma(1-s).\overset{i=\infty}{\underset{i=0}{S}}(-1)^i\left[\frac{\log((2i+1)\pi-a)}{((2i+1)\pi-a)^{1-s}}+\frac{\log((2i+1)\pi+a)}{((2i+1)\pi+a)^{1-s}}\right],$$

et pro $s=0$, exsistente $Z'(1)=-C$ et

$$99.\qquad\int_0^\infty\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\cdot du=\tfrac{1}{2}\operatorname{Sec}a,$$

habebimus

$$\overset{i=\infty}{\underset{i=0}{S}}(-1)^i\left[\frac{\log((2i+1)\pi-a)}{(2i+1)\pi-a}-\frac{\log((2i+1)\pi+a)}{(2i+1)\pi+a}\right]$$

$$=-\tfrac{1}{2}C\operatorname{Sec}\tfrac{1}{2}a-\int_0^\infty\frac{e^{au}+e^{-au}}{e^{\pi u}+e^{-\pi u}}\log u\,.\,du.$$

Ubicumque vero a est ad π in ratione commensurabili, integralia in dextero membro valorem formulae (70.) (pro $x=0$) praebent. Sit igitur $a=\frac{m\pi}{n}$; tunc obtinebimus

$$\frac{n}{\pi}.\overset{i=\infty}{\underset{i=0}{S}}(-1)^i\left[\frac{\log((2i+1)n-m)}{(2i+1)n-m}-\frac{\log((2i+1)n+m)}{(2i+1)n+m}\right]$$

$$+\log\frac{\pi}{n}.\overset{i=\infty}{\underset{i=0}{S}}(-1)^i\left[\frac{1}{(2i+1)\pi-\frac{m\pi}{n}}+\frac{1}{(2i+1)\pi+\frac{m\pi}{n}}\right]$$

$$=-\tfrac{1}{2}C.\operatorname{Sec}.\frac{m\pi}{2n}-\int_0^\infty\frac{e^{\frac{m\pi}{n}.u}+e^{-\frac{m\pi}{n}.u}}{e^{\pi u}+e^{-\pi u}}.\log u.du,$$

unde, cum sit ex (93.) (pro $s=0$)

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$$\overset{i=\infty}{\underset{i=0}{S}}(-1)^i\left[\frac{1}{(2i+1)\pi-\frac{m\pi}{n}}+\frac{1}{(2i+1)\pi+\frac{m\pi}{n}}\right]=\int_0^\infty \frac{e^{\frac{m\pi}{n}.u}+e^{-\frac{m\pi}{n}.u}}{e^{\pi u}+e^{-\pi u}}.du=\tfrac{1}{2}\text{Sec}\,\frac{m\pi}{2n},$$

fit denique ex formulis (70.) citatis:

100.

$$\left\{\begin{aligned}
&\frac{\log(n-m)}{n-m}+\frac{\log(n+m)}{n+m}-\frac{\log(3n-m)}{3n-m}-\frac{\log(3n+m)}{3n+m}+\frac{\log(5n-m)}{5n-m}+\frac{\log(5n+m)}{5n+m}-\text{etc.}\\
&=-\frac{\pi}{2n}\text{Sec}\,\frac{m\pi}{2n}(C+\log 2\pi)-\frac{\pi}{n}.\overset{i=n}{\underset{i=1}{S}}(-1)^{i-1}.\text{Cos}\,(2i-1).\frac{m\pi}{2n}.\log\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{2n}\right)}\right\}\\
&\qquad(m+n=\text{num. imp.})\\
&\frac{\log(n-m)}{n-m}+\frac{\log(n+m)}{n+m}-\frac{\log(3n-m)}{3n-m}-\frac{\log(3n+m)}{3n+m}+\frac{\log(5n-m)}{5n-m}+\frac{\log(5n+m)}{5n+m}-\text{etc.}\\
&=-\frac{\pi}{2n}\text{Sec}\,\frac{m\pi}{2n}.(C+\log 2\pi)-\frac{\pi}{n}\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}(-1)^{i-1}\text{Cos}\,(2i-1).\frac{m\pi}{n}.\log\left\{\frac{\Gamma\left(1-\frac{i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{n}\right)}\right\}\\
&\qquad(m+n=\text{num. par}),
\end{aligned}\right.$$

atque, si $n-m$ loco m ponitur,

101.

$$\left\{\begin{aligned}
&\frac{\log m}{m}+\frac{\log(2n-m)}{2n-m}-\frac{\log(2n+m)}{2n+m}-\frac{\log(4n-m)}{4n-m}+\frac{\log(4n+m)}{4n+m}+\frac{\log(6n-m)}{6n-m}-\text{etc.}\\
&=-\frac{\pi}{2n}\text{Cosec}.\frac{m\pi}{2n}.(C+\log 2\pi)-\frac{\pi}{n}.\overset{i=n}{\underset{i=1}{S}}\text{Sin}\,(2i-1).\frac{m\pi}{n}.\log\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{i-\frac{1}{2}}{2n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{2n}\right)}\right\}\\
&\qquad(m=\text{num. imp.})\\
&\frac{\log m}{m}+\frac{\log(2n-m)}{2n-m}-\frac{\log(2n+m)}{2n+m}-\frac{\log(4n-m)}{4n-m}+\frac{\log(4n+m)}{4n+m}+\frac{\log(6n-m)}{6n-m}-\text{etc.}\\
&=-\frac{\pi}{2n}\text{Cosec}.\frac{m\pi}{2n}(C+\log\pi)-\frac{\pi}{n}.\overset{i=\frac{1}{2}(n-1)}{\underset{i=1}{S}}\text{Sin}\,(2i-1).\frac{m\pi}{2n}.\log\left\{\frac{\Gamma\left(1-\frac{i-\frac{1}{2}}{n}\right)}{\Gamma\left(\frac{i-\frac{1}{2}}{n}\right)}\right\}\\
&\qquad(m=\text{num. par}).
\end{aligned}\right.$$

Ex. 1. Posito $m=1$, $n=2$, fit

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$$\frac{\log 1}{1}+\frac{\log 3}{3}-\frac{\log 5}{5}-\frac{\log 7}{7}+\frac{\log 9}{9}+\frac{\log 11}{11}-\text{etc.}=\frac{\pi}{\sqrt{2}}\log\left\{\frac{2^{\frac{1}{4}}.\Gamma(\frac{1}{8}).\Gamma(\frac{3}{8})}{e^{\frac{1}{2}C}.(2\pi)^{\frac{3}{4}}}\right\},$$

atque inde

$$\frac{1.3^{\frac{1}{3}}.9^{\frac{1}{9}}.11^{\frac{1}{11}}.17^{\frac{1}{17}}\ldots.}{5^{\frac{1}{5}}.7^{\frac{1}{7}}.13^{\frac{1}{13}}.15^{\frac{1}{15}}.21^{\frac{1}{21}}\ldots.}=\left\{\frac{2^{\frac{1}{4}}.\Gamma(\frac{1}{8}).\Gamma(\frac{3}{8})}{e^{\frac{1}{2}C}.(2\pi)^{\frac{3}{4}}}\right\}^{\frac{\pi}{\sqrt{2}}}.$$

Ex. $m=1$, $n=3$, fit

$$\frac{\log 1}{1}+\frac{\log 2}{2}-\frac{\log 4}{4}-\frac{\log 5}{5}+\frac{\log 7}{7}+\frac{\log 8}{8}-\text{etc.}=\frac{\pi}{3\sqrt{3}}(\log 2\pi-2C)-\frac{2\pi}{\sqrt{3}}\log\Gamma(\tfrac{5}{6}),$$

$$\frac{\log 1}{1}+\frac{\log 5}{5}-\frac{\log 7}{7}-\frac{\log 11}{11}+\frac{\log 13}{13}+\text{etc.}=\tfrac{1}{3}\pi\log\left\{\frac{3^{\frac{1}{4}}e^{-C}.\Gamma(\frac{1}{4})}{2^{\frac{1}{2}}.(\Gamma(\frac{3}{4}))^3}\right\},$$

atque inde

$$\frac{1.2^{\frac{1}{2}}.7^{\frac{1}{7}}.8^{\frac{1}{8}}\ldots.}{4^{\frac{1}{4}}.5^{\frac{1}{5}}.10^{\frac{1}{10}}.11^{\frac{1}{11}}\ldots.}=\left\{\frac{(2\pi)^{\frac{1}{3}}.e^{-\frac{2}{3}C}}{(\Gamma(\frac{5}{6}))^2}\right\}^{\frac{\pi}{\sqrt{3}}},$$

$$\frac{1.5^{\frac{1}{5}}.13^{\frac{1}{13}}.17^{\frac{1}{17}}\ldots.}{7^{\frac{1}{7}}.11^{\frac{1}{11}}.19^{\frac{1}{19}}.23^{\frac{1}{23}}\ldots.}=\left\{\frac{3^{\frac{1}{4}}.e^{-C}.\Gamma(\frac{1}{4})}{2^{\frac{1}{2}}.(\Gamma(\frac{3}{4}))^3}\right\}^{\frac{\pi}{3}}.$$

P. S. Ex antecedentibus hanc etiam demonstrare licet notandam formulam

$$\text{Cos}\tfrac{1}{2}a.\log 1-\tfrac{1}{3}\text{Cos}\tfrac{3}{2}a\log 3+\tfrac{1}{5}\text{Cos}\tfrac{5}{2}a.\log 5-\tfrac{1}{7}\text{Cos}\tfrac{7}{2}a\log 7+\tfrac{1}{9}\text{Cos}\tfrac{9}{2}a.\log 9+\text{etc.}$$
$$=\frac{\pi}{4}(\log\pi-C-\log\text{Cos}\tfrac{1}{2}a)-\tfrac{1}{2}\pi\log\left[\Gamma(\tfrac{3}{4}+\frac{a}{4\pi}).\Gamma(\tfrac{3}{4}-\frac{a}{4\pi})\right],$$

quae, quodcumque sit $a<\pi$, valet.

Upsaliae D. 1. Maji 1846.

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