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"Bulletin des Sciences Mathématiques," Tome XXVI., Jan., 1902; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VIII., Fasc. 1; Napoli, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. XI., Fasc. 3, 4; Roma, 1902.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Bd. LIII., Pts. 4-6; 1901.

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The Solutions of a System of Linear Congruences. By the Rev. J. CULLEN, S.J. Received March 4th, 1902. Read March 13th, 1902.

The author is indebted to Lt.-Col. Allan Cunningham, R.E., not only for having carefully revised the original draft, but also for many important suggestions embodied in the paper.

1. The object of the present paper is to give a graphical process for obtaining solutions satisfying a system of linear congruences within a given limit. The process is adapted to the factorization of large composites, or the determination of large primes and other problems in the theory of numbers, such, for instance, as the representation of high numbers in binary quadratic forms, &c.

Incidentally some properties of a system of linear congruences are given that are believed to be new.

The scope of the paper consists in proving and explaining four simple rules to be employed in the application of the process, together with an example showing its working.

2. Let a quantity H be such that

$$\left.\begin{aligned} H &\equiv \alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_m \pmod{P} \\ &\equiv \beta_1,\ \beta_2,\ \beta_3,\ \ldots,\ \beta_n \pmod{Q} \end{aligned}\right\} \quad \text{(A)}$$

$$\left.\begin{aligned} &\equiv \rho_1',\ \rho_2',\ \rho_3',\ \ldots,\ \rho_f' \pmod{p'} \\ &\equiv \rho_1'',\ \rho_2'',\ \rho_3'',\ \ldots,\ \rho_g'' \pmod{p''} \\ &\equiv \rho_1''',\ \rho_2''',\ \rho_3''',\ \ldots,\ \rho_h''' \pmod{p'''} \\ &\quad \vdots \quad \vdots \quad \vdots \qquad\quad \vdots \\ &\equiv \rho_1^{(\sigma)},\ \rho_2^{(\sigma)},\ \rho_3^{(\sigma)},\ \ldots,\ \rho_s^{(\sigma)} \pmod{p^{(\sigma)}} \end{aligned}\right\}, \quad \text{(B)}$$

where P and Q are prime or composite moduli, but prime to each other, $p', p'', p''', \ldots, p^{(\sigma)}$ different odd primes, not contained in P or Q; $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m$; $\beta_1, \beta_2, \ldots, \beta_3, \ldots, \beta_n$, and the ρ's least residues for the corresponding moduli. Our object then is to obtain integral solutions or values of H simultaneously satisfying these $\sigma+2$ congruences under a given limit of H.

3. We shall first combine the congruences (A) and give results on which the subsequent work depends. Now we know that on combination there arise mn cases with the modulus PQ, but, as will presently appear, we need only solve the n congruences

$$P\lambda + \alpha_1 \equiv \beta_1, \beta_2, \beta_3, \ldots, \beta_n \pmod{Q},$$

and the $m-1$ congruences

$$P\lambda + \alpha_2, \alpha_3, \ldots, \alpha_m \equiv \beta_1 \pmod{Q},$$

altogether $m+n-1$ instead of mn. The result of combining is, of course, that we may take $H = xPQ+r$, where r has mn values less than PQ. Hence we may denote any one of these mn values of r by $r_{\varpi,\kappa}$, since it is capable of representing mn values as ϖ ranges from 1 to m, and κ from 1 to n independently. Further, we may restrict $r_{\varpi,\kappa}$ to that case arising out of

$$P\lambda + \alpha_\varpi \equiv \beta_\kappa \pmod{Q}, \tag{1}$$

observing that ϖ always refers to the subscript of one of the α's, and κ to that of one of the β's. It will also be useful to denote the corresponding value of λ by $\lambda_{\varpi,\kappa}$.

We have then $$H = xPQ + P\lambda_{\varpi,\kappa} + \alpha_\varpi \tag{2}$$

or $$r_{\varpi,\kappa} = P\lambda_{\varpi,\kappa} + \alpha_\varpi. \tag{3}$$

since $H \equiv a_\varpi \pmod{P} \equiv \beta_\kappa \pmod{Q}$ by (1) and (2); so we may take our solution to be

$$H = xPQ + r_{\varpi,\kappa}. \tag{4}$$

4. We now proceed to express (4) in another form. Let u be an integral solution (which is always possible) of

$$Qv - Pu = 1. \tag{5}$$

Then, multiplying by $a_\varpi - \beta_\kappa$, we have

$$Pu\,(a_\varpi - \beta_\kappa) \equiv \beta_\kappa - a_\varpi \pmod{Q}. \tag{6}$$

Hence, by (2), we see that

$$\lambda_{\varpi,\kappa} \equiv u\,(a_\varpi - \beta_\kappa) \pmod{Q}; \tag{7}$$

so that

$$\left.\begin{aligned} \lambda_{1,\kappa} &\equiv u\,(a_1 - \beta_\kappa) \\ \lambda_{\varpi,1} &\equiv u(a_\varpi - \beta_1) \\ \lambda_{1,1} &\equiv u\,(a_1 - \beta_1) \end{aligned}\right\} \pmod{Q}.$$

Therefore $\lambda_{\varpi,1} + \lambda_{1,\kappa} - \lambda_{1,1} \equiv u\,(a_\varpi - \beta_\kappa) \equiv \lambda_{\varpi,\kappa} \pmod{Q}$

or

$$\lambda_{\varpi,\kappa} \equiv \lambda_{\varpi,1} + \lambda_{1,\kappa} - \lambda_{1,1} \pmod{Q}; \tag{8}$$

so that, on multiplying (8) by P and adding a_ϖ, we have

$$P\lambda_{\varpi,\kappa} + a_\varpi \equiv (P\lambda_{\varpi,1} + a_\varpi) + (P\lambda_{1,\kappa} + a_1) - (P\lambda_{1,1} + a_1) \pmod{PQ}$$

or, by (3),

$$r_{\varpi,\kappa} \equiv r_{\varpi,1} + r_{1,\kappa} - r_{1,1} \pmod{PQ}. \tag{9}$$

Hence, finally, we may take our solution to be

$$H = xPQ + r_{\varpi,1} + r_{1,\kappa} - r_{1,1}. \tag{10}$$

On this equation the whole process is based.

5. If we assign any value to ϖ between 1 and m inclusive, and ranges from 1 to n, we obtain the n quantities

$$r_{1,1} + (r_{\varpi,1} - r_{1,1}),\ r_{1,2} + (r_{\varpi,1} - r_{1,1}),\ r_{1,3} + (r_{\varpi,1} - r_{1,1}),\ \ldots,\ r_{1,n} + (r_{\varpi,1} - r_{1,1}),$$

together with a multiple of PQ by (10). We shall speak of these n quantities as belonging to the *ϖ-th arrangement*. The first arrangement consists simply of $r_{1,1}, r_{1,2}, r_{1,3}, \ldots, r_{1,n}$, together with the multiple of PQ. Hence it is important to observe that the ϖ-th arrangement is obtained from the first arrangement by the mere addition of the known quantity $(r_{\varpi,1} - r_{1,1})$.

6. The first step then in the application of the process which gives rise to Rule I. is to solve the n congruences

$$P\lambda+a_1\equiv\beta_1,\ \beta_2,\ \beta_3,\ \ldots,\ \beta_n\ (\text{mod } Q)$$

and the $m-1$ congruences

$$P\lambda+a_2,\ a_3,\ \ldots,\ a_m\equiv\beta_1\ (\text{mod } Q),$$

giving the cases $r_{1,1}, r_{1,2}, r_{1,3}, \ldots, r_{1,n}$ and $r_{2,1}, r_{3,1}, \ldots, r_{m,1}$ respectively, which values are easily found by applying (3), (5) and (7).

We next tabulate the *least* residues (θ) of these quantities for the prime moduli $p', p'', p''', \ldots, p^{(\sigma)}$, and also the *least* residues (t) of PQ for the same moduli as follows:—

Modulus	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$	......	$r_{1,n}$	$r_{2,1}$	$r_{3,1}$	......	$r_{m,1}$	PQ
p'	$\theta'_{1,1}$	$\theta'_{1,2}$	$\theta'_{1,3}$	......	$\theta'_{1,n}$	$\theta'_{2,1}$	$\theta'_{3,1}$	......	$\theta'_{m,1}$	t'
p''	$\theta''_{1,1}$	$\theta''_{1,2}$	$\theta''_{1,3}$	......	$\theta''_{1,n}$	$\theta''_{2,1}$	$\theta''_{3,1}$	......	$\theta''_{m,1}$	t''
p'''	$\theta'''_{1,1}$	$\theta'''_{1,2}$	$\theta'''_{1,3}$	......	$\theta'''_{1,n}$	$\theta'''_{2,1}$	$\theta'''_{3,1}$	......	$\theta'''_{m,1}$	t'''
$\vdots$	$\vdots$	$\vdots$	$\vdots$	......	$\vdots$	$\vdots$	$\vdots$	......	$\vdots$	$\vdots$
$p^{(\sigma)}$	$\theta^{(\sigma)}_{1,1}$	$\theta^{(\sigma)}_{1,2}$	$\theta^{(\sigma)}_{1,3}$	......	$\theta^{(\sigma)}_{1,n}$	$\theta^{(\sigma)}_{2,1}$	$\theta^{(\sigma)}_{3,1}$	......	$\theta^{(\sigma)}_{m,1}$	$t^{(\sigma)}$

It will be useful to speak of this table as the *elements table*. The object of this table will be best understood when considered in conjunction with what follows and what has been already stated.

7. We shall now confine our attention for the moment to the first arrangement ($\varpi=1$) which gives us

$$H=xPQ+r_{1,\kappa},$$

κ ranging from 1 to n inclusive (since these numbers κ form the same set, viz., 1, 2, 3, ..., n in each arrangement, we shall speak of them as *set numbers*, and they will be denoted by κ), and we now proceed to show how values of x may be graphically obtained where

$$H=xPQ+r'_{1,\kappa}\equiv\rho'_d\quad(\text{mod } p'),$$

d ranging from 1 to f (*cf.* § 2). Now, from the elements table, we have

$$H\equiv xt'+\theta'_{1,\kappa}\equiv\rho'_d\quad(\text{mod } p').$$

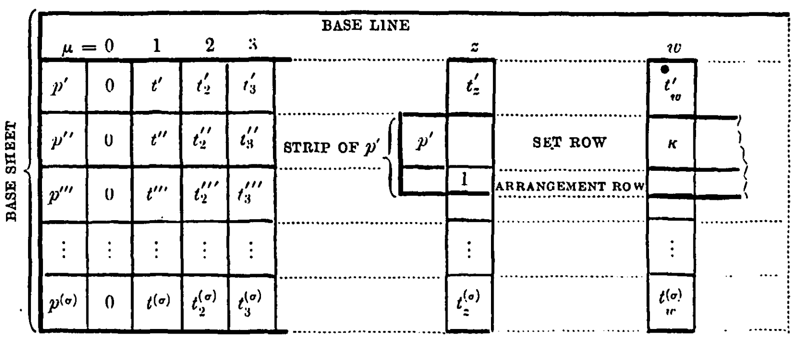
Let $$\theta'_{1,\kappa} \equiv zt' \pmod{p'}$$

and $$\rho'_d \equiv wt' \pmod{p'};$$

so that $$H \equiv t'(x+z) \equiv wt' \pmod{p'} \quad \text{or} \quad x = w - z. \qquad (11)$$

It is, however, quite needless in practice to solve these congruences, for, if we take a sheet of paper with vertical rules at *equal* horizontal spacing, and write down the *least* residues t'_μ of the successive multiples of t' for the modulus p' as follows with

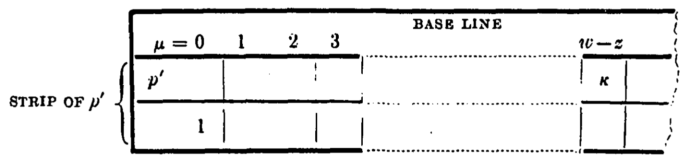
$$t'_\mu \equiv \mu t' \pmod{p'},$$



we obtain what we will call the *p′-line of the base sheet.* Then, since every integer less than p' will occur among the t'_μ's, and if we put a dot over $t'_\mu = \rho'_d$ (*i.e.*, f dots since d ranges from 1 to f), we have only to take a narrow strip of paper spaced as the base sheet and divided into two rows (the upper row called the *set row*, as it contains the set numbers κ, and the lower row or *arrangement row* containing, as we shall see presently, the arrangement numbers ϖ). We denote it by p', and we write 1 in the arrangement row to give the initial division of the first arrangement. Then, if we place the strip with this division at t'_z $(= \theta'_{1,\kappa})$ of the p'-line, and write the values of κ in the divisions of the set-row, under the dotted figures (*i.e.*, t'_w), we obtain the values of x by simply placing the strip under the *base line** so that the initial division marked 1 is at the 0 (or any multiple of p') of the base line, and read off the integers in the base line that are over the κ's, since, if we compare the preceding figure with the

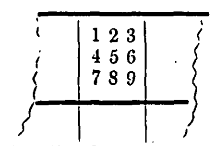
* The *base line* is the top line of the base sheet, and consists merely of successive integers in each division; its use is to give the value of x as explained above.

following, we see at once that the number over κ is $w-z$, which is the required value for x by (11).



8. In applying the principle just described, it is evident that as we placed the initial division of the strip at t'_z ($=\theta'_{1,\kappa}$), and wrote κ under each dotted figure, so we first place the initial division at t'_μ ($=\theta'_{1,1}$), and write 1 in the set row under each dotted figure; then it is placed at t'_μ ($=\theta'_{1,2}$), and 2 is written in the same row under each dotted figure and so on till t'_μ ($=\theta'_{1,n}$) has been dealt with and n written.

In filling up the divisions in this manner, it is necessary that each number should always occupy a fixed relative position with respect to the other set numbers: *e.g.*, if $n=9$, then we should follow the scheme



so that 3, for instance, should always occupy the right-hand top corner in those divisions in which it is to appear.

9. In the last paragraph we have shown how the strip for the prime p' is to be drawn up, and in a precisely similar manner strips are drawn up for the primes p'', p''', ..., $p^{(\sigma)}$. All the *least* residues (t'_μ) of the successive multiples of PQ are arranged in the base sheet as shown in the first figure, § 7, and dots are placed over the t_μ's of each line that equal the ρ's of the corresponding congruence of the system (B). It should be noticed that the residues t'_μ recur at intervals of p', t''_μ at intervals of p'', and so on. Hence the base sheet may be of any convenient length so long as the number of divisions in it exceeds $p^{(\sigma)}$, the highest prime. A similar remark applies to the strips.

10. So far we have been dealing only with the first arrangement, and the question that now arises is whether the strips already drawn

up for this arrangement suit for the other arrangements. We will now show that they do suit.

Taking the ϖ-th arrangement and attending to the remark in § 5, we have, by (10),

$$H = xPQ + r_{1,\kappa} + (r_{\varpi,1} - r_{1,1}) \equiv \rho'_d \pmod{p'}. \qquad (12)$$

Now, let $$(r_{\varpi,1} - r_{1,1}) \equiv yt' \pmod{p'}.$$

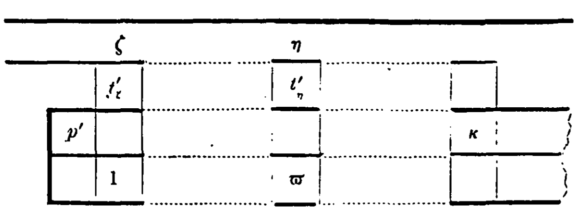
Then, by definition, y is constant for this arrangement and independent of w and z. Hence, as in § 7, we have

$$(x+z+y)\, t' \equiv wt' \pmod{p'},$$

or $$x = w - z - y; \qquad (13)$$

but $w-z$ is the distance between the initial division and κ in the first arrangement, and, by (13), we see that for the ϖ-th arrangement the effect of adding the quantity $(r_{\varpi,1} - r_{1,1})$ is only to shorten or lengthen the distance between the initial division of the first arrangement and that containing κ by y divisions, and further, since κ stands successively for the set of numbers 1, 2, 3, ..., n (*cf.* § 8), we see that all these numbers are affected in the same manner by the addition of $(r_{\varpi,1} - r_{1,1})$ to the first arrangement; hence we have only to find a new initial division for each arrangement.

11. It is here, then, that we make use of the second portion of the elements table, § 6, namely, the residues (θ) of $r_{2,1}, r_{3,1}, \ldots, r_{m,1}$. For, if we were to draw up a slip for the ϖ-th arrangement for the prime p', we should place its initial division (*i.e.*, the division having ϖ in the arrangement row of the strip) at the number $t'_\eta \,(= \theta'_{\varpi,1})$ of the base sheet, and write 1 in the set row under each dotted figure; but it is quite clear that, if we take the strip of p' already drawn up, the same result is obtained by placing the initial division (marked 1 in the arrangement row) at $t'_\zeta \,(= \theta'_{1,1})$ and writing ϖ in the arrangement row of that division, that is under $t'_\eta \,(= \theta'_{\varpi,1})$, since 1 of the set row is under each dotted figure. In fact, if we refer to the following figure, it is clear that the distance (a) between ϖ and κ is equal to



the distance (b) between 1 and κ less the distance (c) between 1 and ϖ; but

$$b = w-z,$$

by (11), and
$$c = \eta-\zeta = y,$$

since, by § 10,

$$r_{\varpi,1}-r_{1,1} \equiv yt' \pmod{p'} \equiv \theta_{\varpi,1}-\theta_{1,1} \equiv t'(\eta-\zeta);$$

therefore
$$a = w-z-y;$$

so that, if the strip of p' be placed with the division containing ϖ at the 0 of the base line, the number in the base line over κ would be $(w-z-y)$ which, by (13), is the value of x in

$$H = xPQ+r_{1,\kappa}+(r_{\varpi,1}-r_{1,1}) \equiv \rho'_d \pmod{p'}.$$

Thus we obtain our general solutions.

Hence the new initial divisions for the other arrangements are obtained by placing the strip p' so that the division containing 1 in the arrangement row is under $t_1 (= \theta'_{1,1})$ of the p' line of the base sheet, and writing 2, 3, ..., m in the arrangement row of the strip under the divisions (of the p' line of the base sheet) containing the numbers $t'_{\varpi,1}$ ($= \theta'_{2,1}, \theta'_{3,1}, \ldots, \theta'_{m,1}$ of the elements table).

12. Similarly the initial divisions for the different arrangements for the strips $p'', p''', \ldots, p^{(\sigma)}$ are obtained from their respective rows in the elements table and the base sheet.

13. We are now in a position to obtain values of x in (10) simultaneously satisfying the $\sigma+2$ congruences (A) and (B), supposing the σ strips to have been completed; also the values of ϖ and κ can now be found, and hence, by (10), and the top row of the elements table, our solutions can be obtained. For, if we place all the strips one under the other under the base line, so that the initial divisions containing ϖ in the arrangement rows form a column under 0,* and if we search the columns for a number κ appearing in the *set rows* of *all* the strips in a particular column, and read off the number

* Any strip for $p^{(r)}$ (say) may be placed so that the division with ϖ in its arrangement row is under any multiple of $p^{(r)}$ in the base line.

x in the base line over this column, a solution is

$$xPQ+r_{\varpi,1}+r_{1,\kappa}-r_{1,1},$$

while, if κ fails to appear in *any* strip, this cannot be a solution, as is clear from § 7.

Thus, in practice, we begin by placing all the strips so that the initial divisions containing 1 in the arrangement row of each are at the 0 of the base line; we then search up to the required limit of H; then we place the divisions containing 2 in the arrangement row of each strip at 0 of the base line, and continue the search; and so on till m has been dealt with.

If L be the upper limit of H, then $x \not> L/PQ$; yet we should search to the $(L/PQ+1)$-th column, since $(r_{\varpi,1}+r_{1,\kappa}-r_{1,1})$ may be negative, but $<PQ$; also this quantity may be $>PQ$, but $<2PQ$. Therefore x may $=-1$, and yield a positive solution. Thus the column of the strips to the left of the 0 of the base line should be searched.

14. Rules.—*Rule I.*—Apply (3), (5), and (7) to solve the n congruences

$$P\lambda+\alpha_1 \equiv \beta_1, \beta_2, \beta_3, \ldots, \beta_n \pmod{Q},$$

and the $m-1$ congruences

$$P\lambda+\alpha_2, \alpha_3, \ldots, \alpha_m \equiv \beta_1 \pmod{Q},$$

giving the cases $r_{1,1}, r_{1,2}, r_{1,3}, \ldots, r_{1,n}$ and $r_{2,1}, r_{3,1}, \ldots, r_{m,1}$, respectively. Then form the elements table, § 6.

Rule II.—Form the base sheet, § 7, and in the line p' place dots over the numbers $t'_\mu = \rho'_1, \rho'_2, \ldots, \rho'_f$, and so in like manner treat the lines $p'', p''', \ldots, p^{(\sigma)}$; then place the initial division (marked 1 in the arrangement row) of the strip p' at t'_{z_1} (of the base sheet) $= \theta'_{1,1}$ (of the elements table), and write 1 under each dotted figure; then place the same division at $t'_{z_2} = \theta'_{1,2}$, and write 2 under each dotted figure; and so on till m has been written in the set row of the strip. Thus, in a similar manner, complete the strips for $p'', p''', \ldots, p^{(\sigma)}$.

Rule III. — Now place the initial division (with 1 in the arrangement row) at $t'_{z_1} = \theta'_{1,1}$, as in Rule II., but now write $2, 3, \ldots, m$ in the arrangement row in the divisions under

$$t'_\varpi = \theta'_{2,1}, \theta'_{3,1}, \ldots, \theta'_{m,1}.$$

Thus also find the initial divisions of the different arrangements for the strips p'', p''', ..., $p^{(\sigma)}$.*

Rule IV.—Place all the strips one under the other so that the initial divisions (marked 1 in the arrangement rows) form a column under the 0 of the base line, and search for a number appearing in the *set row* of *every* strip throughout a particular column; having thus searched throughout $(L/PQ+2)$ columns ($x=-1$ to $x=L/PQ+1$), we place the strips with 2 of the arrangement rows under the 0 of the base line and continue the search, and so on till m of all the arrangement rows has been placed under the 0, and all the columns searched. If, then, when ϖ of the arrangement rows is under the 0 (or any multiple of $p^{(\tau)}$ for the strip $p^{(\tau)}$) of the base line, and the number κ appears throughout the divisions of the set rows which form a column under x of the base line, then $xPQ+r_{\varpi,1}+r_{1,\kappa}-r_{1,1}$ is a solution.

15. We now give an example showing the working of the process. Let $N=1,886,601,653$, and, if we wish to determine whether N be prime or composite, we may seek the partition $N=H^2+G^2$, since $N=4k+1$: if this partition be unique, N is prime, while, if there is no partition or else two or more partitions, then N is composite. Taking H to be odd, we have $H^2=N-G^2$. Now

$$N\equiv 2 \pmod 3 \equiv 3 \pmod 5 \equiv 6 \pmod 7 \equiv 53 \pmod{64};$$

therefore

$$H\equiv \pm 1 \pmod 3 \equiv \pm 2 \pmod 5 \equiv \pm 2 \pm 3 \pmod 7 \equiv \pm 7 \pm 9 \pmod{32}.$$

We combine $H\equiv \pm 1 \pmod 3 \equiv \pm 2 \pm 3 \pmod 7$,

giving $$H\equiv \pm 2 \pm 4 \pm 5 \pm 10 \pmod{21},$$

and also $H\equiv \pm 2 \pmod 5 \equiv \pm 7 \pm 9 \pmod{32}$,

giving $$H\equiv \pm 7 \pm 23 \pm 57 \pm 73 \pmod{160}.$$

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* In the application of Rule II. for finding the set numbers κ, and of Rule III. for finding the arrangement numbers ϖ, it should be noticed that the $(t^{(\tau)})$ may be to the left of the initial division, but, if a second 1 be written in the arrangement row of the strip $p^{(\tau)}$ at a distance of $p^{(\tau)}$ divisions from the initial division, we have only to move the strip so that the latter division occupies the place of the former to give us a division in which κ or ϖ is to be written.

Hence, taking $P = 160$ and $Q = 21$, we have

$$\left.\begin{aligned} H &\equiv \left.\begin{matrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6, & \alpha_7, & \alpha_8 \\ 7, & 23, & 57, & 73, & 87, & 103, & 137, & 153 \end{matrix}\right\} \pmod{160} \\ &\equiv \left.\begin{matrix} \beta_1, & \beta_2, & \beta_3, & \beta_4, & \beta_5, & \beta_6, & \beta_7, & \beta_8 \\ 2, & 4, & 5, & 10, & 11, & 16, & 17, & 19 \end{matrix}\right\} \pmod{21} \end{aligned}\right\} \quad \text{(A)}$$

$$\left.\begin{aligned} &\equiv 1, 2, 3, 8, 9, 10 \pmod{11} && \text{since } N \equiv 2 \pmod{11} \\ &\equiv 0, 1, 2, 6, 7, 11, 12 \pmod{13} && \equiv 1 \pmod{13} \\ &\equiv 1, 4, 7, 8, 9, 10, 13, 16 \pmod{17} && \equiv 14 \pmod{17} \\ &\equiv 0, 3, 4, 7, 8, 9, 10, 11, 12, 15, 16 \pmod{19} && \equiv 16 \pmod{19} \\ &\equiv 1, 4, 6, 7, 8, 11, 12, 15, 16, 17, 19, 22 \pmod{23} && \equiv 19 \pmod{23} \\ &\equiv 4, 5, 6, 8, 10, 11, 12, 17, 18, 19, 21, 23, \\ &\qquad 24, 25 \pmod{29} && \equiv 12 \pmod{29} \\ &\equiv 1, 4, 5, 7, 9, 10, 14, 15, 16, 17, 21, 22, \\ &\qquad 24, 26, 27, 30 \pmod{31} && \equiv 26 \pmod{31} \\ &\equiv 2, 4, 5, 6, 9, 12, 13, 14, 18, 19, 23, 24, \\ &\qquad 25, 28, 31, 32, 33, 35 \pmod{37} && \equiv 32 \pmod{37} \end{aligned}\right\}. \quad \text{(B)}$$

Now $21v - 160u = 1$ gives $u = 8$; hence, applying (7) and then (3), we find

	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$	$r_{1,4}$	$r_{1,5}$	$r_{1,6}$	$r_{1,7}$	$r_{1,8}$	$r_{2,1}$	$r_{3,1}$	$r_{4,1}$	$r_{5,1}$	$r_{6,1}$	$r_{7,1}$	$r_{8,1}$	PQ
Modulus.	3047	487	2567	2887	1607	1927	647	1447	23	3257	233	1367	1703	1577	1913	3360
11	0	3	4	5	1	2	9	6	1	1	2	3	9	4	10	5
13	5	6	6	1	8	3	10	4	10	7	12	2	0	4	2	6
17	4	11	0	14	9	6	1	2	6	10	12	7	3	13	9	11
19	7	12	2	18	11	8	1	3	4	8	5	18	12	0	13	16
23	11	4	14	12	20	18	3	21	0	14	3	10	1	13	4	2
29	2	23	15	16	12	13	9	26	23	9	1	4	21	11	28	25
31	9	22	25	4	26	5	27	21	23	2	16	3	29	27	22	12
37	13	6	14	1	16	3	18	4	23	1	11	35	1	23	26	30

This is the elements table of § 6 and application of Rule I.

On the accompanying diagram we have the base sheet and the strips drawn up by Rules II. and III. They are in position for the first arrangement $\varpi = 1$, and we search the set rows in each column for a number appearing throughout up to the fourteenth column. Since $H \not> \sqrt{N} \not> 43435$ and $PQ = 3360$, which is the application of Rule IV., we find

x	6	10	12	13	1	6	0
ϖ	1	1	1	2	5	5	6
κ	6	3	3	5	8	5	3
$H =$	22087	36167	42887	42263	3127	20087	1223

The first three results $\varpi = 1$ are shown on the strips, viz., 6 appears in set rows in the column under 6, of the base line, and 3 in the columns under 10 and 12. On actual trial we have

$$N = 1{,}886{,}601{,}653 = 42887^2 + 6878^2 = 42263^2 + 10022^2$$
$$= 17837 \,.\, 105769.$$

The other five solutions would soon fail to conform to subsequent moduli if we used strips for 41, 43, &c.

Note (i.).—In searching any particular column we compare the numbers of any two adjacent set rows in the column and mentally carry the numbers common to the two rows into a third, and then those common to the three rows into a fourth, and so on. Thus, for instance, in the column under 7 of the base line the two top strips give 5 and 8 common, while 5 is only common to the third; so we need only look for 5 in the remaining set rows of the column, and, since 5 does not appear in strip 29, we pass on to the next column, 8.

Note (ii.).—It is well to observe the advantage of the graphical work over purely arithmetical, for each column deals with eight cases (in the example). Thus the column under 4 of the base line gives the eight cases of the first part of the elements table + 4.3360. Then in arithmetical work we should have to find their residues to (mod 11), then those of the five cases $r_{1,1}, r_{1,2}, r_{1,3}, r_{1,4}, r_{1,5} + 4.3360$ to (mod 13), then $r_{1,4}, r_{1,5} + 4.3360$ to (mod 17), and, finally, $r_{1,4}$ to (mod 19), in order to exclude these eight cases, in all $8+5+2+1 = 16$ residues; but in graphical work a glance at the column gives this information.

BASE SHEET.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
11	0	5	•10	4	•9	•3	•8	•2	7	•1	6	0																										
13	•0	•6	•12	5	•11	4	10	3	9	•2	8	•1	•7	•0																								
17	0	11	5	•16	•10	•4	15	•9	3	14	•8	2	•13	•7	•1	12	6	0																				
19	•0	•16	13	•10	•7	•4	1	17	14	•11	•8	5	2	18	•15	•12	•9	6	•3	•0																		
23	0	2	•4	•6	•8	10	•12	14	•16	18	20	•22	•1	3	5	•7	9	•11	13	•15	•17	•19	21	0														
29	0	•25	•21	•17	13	9	•5	1	26	22	•18	14	•10	•6	2	27	•23	•19	15	•11	7	3	28	•24	20	16	•12	•8	•4	0								
31	0	12	•24	•5	•17	29	•10	•22	3	•15	•27	8	20	•1	13	25	6	18	•30	11	23	•4	•16	28	•9	•21	2	•14	•26	•7	19	0						
37	0	30	•23	16	•9	•2	•32	•25	•18	11	•4	34	27	20	•13	•6	36	29	22	15	8	1	•31	•24	17	10	3	•33	26	•19	•12	•5	•35	•28	21	•14	7	0

STRIPS.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
11	2 5 6 7	2 3 4 7	1 2 3 6 7	3 4 7 8	1 2 3 4 5	1 4 7 8	1 3 4 5 6 8	1 5 8	2 4 5 6 8	1 5 6 7	2 3 6 8	2 5 6 7	2 3 4 7	1 2 3 6 7	3 4 7 8																							
	1		8	7	6	5		4		2, 3		1		8	7																							
13	2 3 4	1 2 3 4 5	4 5 6	2 3 4 5 7	4 5 6 8	5 6 7	1 4 6 7 8	5 6 7 8	1 2 3 6 7 8	1 7 8	1 2 3 6 8	1 2 3 4 7	1 2 3 5 8	2 3 4	1 2 3 4 5																							
	1		7	2			5, 8			3	6		4	1																								
17	1 5 7	4 8	1 2 8	2 3 4 5 8	2 3 4 6	1 3 4 5 6	2 5 6 7	1 3 5 7	1 6 7	1 2 8	3 7 8	2 4 6 8	2 3 4	2 3 4 5 6 7 8	3 5 6	1 4 5 6 7	1 7 8	1 5 7																				
	1		8	6				7	5		4	2					3	1																				
19	1 2 5 6 8	1 2 4 5 8	3 4 8	2 3 4 7	2 3 6 7 8	1 2 4 5 6 8	1 3 4 5 6 8	2 3 4 5	2 3 6 7	2 4 5 6 7	1 3 4 5 6 7 8	1 3 4 5 8	1 3 6 7	2 5 6 7	1 2 5 6 7	1 4 5 8	1 3 4 7 8	3 7 8	1 2 6 7	1 2 5 6 8																		
	1	2					3	4		5		6				7		8		1																		
23	1 2 4	2 3 5	1 2 4 5 6 7	1 6 8	1 2 3 7 8	3 4 5 8	2 4 6 7	5 7 8	1 3 6 7	1 2 4 5 8	1 2 3 5 6	4 5 6	1 3 6 7 8	2 3 4 7 8	1 3 4 7	2 4 5	5 6 7 8	1 2 5 6	1 2 3 6 7 8	2 3 4 5	3 4 6 8	1 4 5 7 8	3 6 7 8	1 2 4														
	1	7					2		8			5		3					6	4				1														
29	2 5	2 3 4 5 7	1 4 5 6 8	1 2 4	5 8	1 3 4 5 7 8	4 5 6	2 4 7	3 6 7 8	1 3 5 6 8	2 3 4	2 7 8	1 2 3 6 7	1 3 5 6	1 2 3 4 7	2 5 6 8	1 2 4 5	1 3 4	1 7 8	2 5 6 8	4 5 8	1 3 4 7	5 6 7 8	2 3 4 6 7 8	3 6 8	1 2 7	2 5 6 7	1 3 4 6 7 8	1 3 6	2 5								
	1		2			7			8						5			6			3		4							1								
31	1 2 4 5 6 7 8	1 4 5 6	2 8	1 2 3 4 6 7 8	1 4 6 8	1 5	2 3 4 5 6	3 4 5 6	4 7 8	1 3 5 8	1 3 5 6 8	1 2 7	3 4 5 7 8	1 3 4 5 8	1 2 3 4 7	2 6 7 8	1 4 5 8	1 2 4 7	2 3 6 7	3 4 6 7 8	1 2 3 4	2 5 6	2 3 6	3 4 7	5 6 7 8	1 3 5 6 7	2 3 6	2 5 7 8	1 2 4 5 7 8	1 3	2 5 6 7 8	1 2 4 5 6 7 8						
	1		3										6		8	5		7										2		4		1						
37	1 2 3 7 8	1 4 5 6	4 5 7	5 6	3 5 6 8	5 6 8	3 4 6 7	2 3 5 6 7	1 2 3 4	1 3 4 6	3 4	4 5	2 3 4 5 8	1 6 8	2 4 7	1 2 6 7	1 2 3 6	1 2 3 6 8	1 2 4 6	1 5 6 7 8	2 4 5 8	1 4 6 7 8	4 7 8	4 7 8	2 3 4 5 7	1 3 6 7 8	2 4 5 6	1 2 5 7	1 2 5	1 2 3 5 8	1 2 4 5	1 3 4 7 8	2 3 5 8	1 3 6 7 8	3 6 7 8	3 7 8	2 5 7	1 2 3 7 8
	1							3, 6							8				5							2, 7							4					1

To face p. 334.

The Errors in certain Quadrature Formulæ. By J. BUCHANAN, M.A., F.I.A. Received March 5th, 1902. Read March 13th, 1902. Received, in revised form,* May 18th, 1902.

1. In a recent paper† Mr. Sheppard has derived from the Maclaurin summation theorem many of the best known quadrature formulæ, with expressions for the errors in terms of differential coefficients. The use of differential coefficients, however, is inconvenient in cases where we do not know the form of the function, but only its value at stated intervals; and these are cases where the formulæ are of great practical value. Prof. Everett has recently given a new interpolation formula‡ involving only even central differences; and from it similar quadrature formulæ can be obtained by direct integration with expressions for the errors in terms of central differences.

Denoting by p the distance of the ordinate u_p in front of u_0, and by q its distance behind u_1, so that $p+q=1$, he writes

$$u_p = \left[q+\frac{q(q^2-1)}{3!}\delta^2+\frac{q(q^2-1)(q^2-4)}{5!}\delta^4+\dots\right]u_0$$

$$+\left[p+\frac{p(p^2-1)}{3!}\delta^2+\frac{p(p^2-1)(p^2-4)}{5!}\delta^4+\dots\right]u_1, \qquad (1)$$

where§ $$\delta u_0 = u_{\frac{1}{2}}-u_{-\frac{1}{2}},$$

so that δ^2, δ^4, ... are the even central differences.

If we integrate with respect to p between limits 0 and 1, we have, since $dq = -dp$,

$$\int_0^1 u_p\,dp = \int_0^1\left[p+\frac{p(p^2-1)}{3!}\delta^2+\frac{p(p^2-1)(p^2-4)}{5!}\delta^4+\dots\right](u_0+u_1)\,dp$$

$$= \tfrac{1}{2}\left[1-\tfrac{1}{12}\delta^2+\tfrac{11}{720}\delta^4-\tfrac{191}{60480}\delta^6+\tfrac{2497}{3628800}\delta^8-\dots\right](u_0+u_1)$$

$$= \tfrac{1}{2}\left[1+\lambda_2\delta^2+\lambda_4\delta^4+\lambda_6\delta^6+\dots\right](u_0+u_1),$$

where λ_2, λ_4, ... are written for shortness for the numerical coefficients.

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* [The title of the paper was changed in revision: *cf. Proceedings, supra*, p. 322.—SEC.]

† *Proceedings*, Vol. XXXII., p. 258.

‡ *Journal of the Institute of Actuaries*, Vol. XXXV., p. 452.

§ *Cf. Proceedings*, Vol. XXXI., pp. 459-60.

Hence
$$\int_0^n u_p\,dp = \int_0^1 u_p\,dp + \int_1^2 u_p\,dp + \dots + \int_{n-1}^n u_p\,dp$$
$$= (1+E+E^2+\dots+E^{n-1})\int_0^1 u_p\,dp$$
$$= \tfrac{1}{2}u_0 + u_1 + u_2 + \dots + \tfrac{1}{2}u_n$$
$$+ (\lambda_2\delta + \lambda_4\delta^3 + \dots)\,\mu\,(u_n - u_0), \qquad (2)$$

where*
$$\mu u_0 = \tfrac{1}{2}(u_{\frac{1}{2}} + u_{-\frac{1}{2}});$$
so that $\mu\delta$, $\mu\delta^3$, ... are the odd central differences.

2. Let $a, b, c, \dots$ be the factors, including unity, of n; and let $\mu_a\delta_a$, $\mu_a\delta_a^3$, ... be the odd central differences of the series of functions $\dots u_{-2ah}, u_{-ah}, u_0, u_{ah}, u_{2ah}, \dots$; then
$$\int_0^{nh} u_x\,dx = ah\left[\tfrac{1}{2}u_0 + u_{ah} + u_{2ah} + \dots + \tfrac{1}{2}u_{nh}\right]$$
$$+ ah\left[\lambda_2\delta_a + \lambda_4\delta_a^3 + \dots\right]\mu_a\,(u_{nh} - u_0).$$

Now
$$\mu\delta^{2n-1} = \cosh\tfrac{1}{2}hD\,(2\sinh\tfrac{1}{2}hD)^{2n-1},$$
$$\mu_a\delta_a^{2n-1} = \cosh\tfrac{1}{2}ahD\,(2\sinh\tfrac{1}{2}ahD)^{2n-1},$$
and since†
$$2\sinh a\phi\cosh a\phi = a\cosh\phi\left[2\sinh\phi + \frac{a^2-1}{3!}(2\sinh\phi)^3 + \frac{(a^2-1)(a^2-4)}{5!}(2\sinh\phi)^5 + \frac{(a^2-1)(a^2-4)(a^2-9)}{7!}(2\sinh\phi)^7 + \dots\right]$$
$$(2\sinh a\phi)^3\cosh a\phi = a^3\cosh\phi\left[(2\sinh\phi)^3 + \frac{a^2-1}{4}(2\sinh\phi)^5 + \frac{(a^2-1)(3a^2-7)}{4.5.6}(2\sinh\phi)^7 + \dots\right]$$
$$(2\sinh a\phi)^5\cosh a\phi = a^5\cosh\phi\left[(2\sinh\phi)^5 + \frac{a^2-1}{3}(2\sinh\phi)^7 + \dots\right],$$

... ...

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* *Proceedings*, Vol. XXXI., pp. 459-60.
† *Ibid.*, Vol. XXXI., p. 454.

we have at once

$$\mu_a\delta_a = a\mu\left[\delta + \frac{a^2-1}{3!}\delta^3 + \frac{(a^2-1)(a^2-4)}{5!}\delta^5 + \frac{(a^2-1)(a^2-4)(a^2-9)}{7!}\delta^7 + \dots\right],$$

$$\mu_a\delta_a^3 = a^3\mu\left[\delta^3 + \frac{a^2-1}{4}\delta^5 + \frac{(a^2-1)(3a^2-7)}{4.5.6}\delta^7 + \dots\right]$$

$$\mu_a\delta_a^5 = a^5\mu\left[\delta^5 + \frac{a^2-1}{3}\delta^7 + \dots\right],$$

$$\mu_a\delta_a^7 = a^7\mu\left[\delta^7 + \dots\right].$$

... ...

If we make these substitutions for $\mu_a\delta_a$, $\mu_a\delta_a^3$, ..., and replace λ_2, λ_4, ... by their numerical values, equation (2) reduces to

$$A = \int_0^{nh} u_x dx = A_a - \frac{a^2h}{720}\left[6\delta - (a^2+10)\,\delta^3 + \left(\frac{a^4}{42} + \frac{a^2}{4} + 2\right)\delta^5 - \left(\frac{a^6}{1680} + \frac{a^4}{126} + \frac{7a^2}{120} + \frac{3}{7}\right)\delta^7 + \dots\right]\mu\,(u_{nh} - u_0), \quad (3)$$

where $$A_a = ah\left[\tfrac{1}{2}u_0 + u_{ah} + u_{2ah} + \dots + \tfrac{1}{2}u_{nh}\right].$$

An approximate expression, together with an expression for the error in terms of differences, will be got by writing

$$(p+q+r+\dots)A = pA_a + qA_b + rA_c + \dots,$$

where p, q, r, ... are chosen to make the coefficients of the successive differences vanish.

The formulæ which can be got in this way are identical with those of Mr. Sheppard's paper,* as might be expected if we observe that the principal equation (2) obtained above is the central difference equivalent of the Maclaurin summation theorem. The method is, however, capable of extension.

3. The group of formulæ obtained by putting $n = 6$ is of special interest, as it includes a large number of those best known, and it is proposed to discuss it here in some detail. In what follows the letter h is for convenience omitted from the suffix of u.

If we choose p and q so that $pa^2+qb^2 = 0$, and put $a = 1$, $b = 2$, we get

$$A = \frac{h}{3}\,[4A_1 - A_2] = \frac{h}{3}\,[u_0 + u_6 + 4\,(u_1+u_3+u_5) + 2\,(u_2+u_4)], \quad \text{(i.)}$$

* *Proceedings*, Vol. XXXII., pp. 262-65.

which is Simpson's rule, with an error

$$-\frac{h}{180}\left[\delta^3-\tfrac{31}{84}\delta^5+\tfrac{557}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$$

Putting $a=1,\ b=3$, so that $p:q::9:-1$, we have

$$A=\frac{h}{8}\left[9A_1-A_3\right]=\frac{3h}{8}\left[u_0+u_6+3\,(u_1+u_2+u_4+u_5)+2u_3\right],\qquad\text{(ii.)}$$

which is Simpson's second rule, with an error

$$-\frac{h}{80}\left[\delta^3-\tfrac{41}{84}\delta^5+\tfrac{967}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0).$$

Put $a=1:b=6$; then $p:q::36:-1$, and

$$A=\frac{h}{35}\left[36A_1-A_6\right]=\frac{h}{35}\left[15\,(u_0+u_6)+36\,(u_1+u_2+\dots+u_5)\right],\qquad\text{(iii.)}$$

with an error $-\frac{h}{20}\left[\delta^3-\tfrac{95}{84}\delta^5+\tfrac{5773}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0)$.

Other formulæ involving only five terms can be obtained from the above by elimination. Thus the elimination of u_0 and u_6 between (i.) and (ii.) gives

$$A=h\left[3A_1-3A_2+A_3\right]=3h\left[u_1+u_5-(u_2+u_4)+2u_3\right],\qquad\text{(iv.)}$$

with an error $+\frac{h}{20}\left[\delta^3-\tfrac{51}{84}\delta^5+\tfrac{1377}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0)$.

The elimination of u_1 and u_5 between the same two formulæ gives

$$A=\frac{h}{5}\left[9A_2-4A_3\right]=\frac{3h}{5}\left[u_0+u_6+6\,(u_2+u_4)-4u_3\right],\qquad\text{(v.)}$$

with an error $-\frac{h}{20}\left[\delta^3-\tfrac{47}{84}\delta^5+\tfrac{1213}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0)$.

The result of eliminating u_2 and u_4 between (i.) and (ii.) is

$$A=\frac{h}{11}\left[18A_1-9A_2+2A_3\right]=\frac{3h}{11}\left[u_0+u_6+6\,(u_1+u_5)+8u_3\right],\qquad\text{(vi.)}$$

with an error $+\frac{h}{220}\left[\delta^3-\tfrac{71}{84}\delta^5+\tfrac{2197}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0)$,

while the elimination of u_3 gives

$$A=\frac{h}{7}\left[6A_1+3A_2-2A_3\right]=\frac{3h}{7}\left[u_0+u_6+2\,(u_1+u_5)+4\,(u_2+u_4)\right],\qquad\text{(vii.)}$$

with an error $-\frac{h}{140}\left[3\delta^3-\tfrac{133}{84}\delta^5+\tfrac{3311}{5040}\delta^7-\dots\right]\mu\,(u_6-u_0)$.

Several of the above will be found to give very good results considering the small number of terms used and the simplicity of the coefficients.

4. From the above we can get other formulæ which are true to fifth differences. Thus, if we eliminate δ^3 between (i.) and (ii.), we get

$$A = \frac{h}{10}\left[15A_1 - 6A_2 + A_3\right] = \frac{3h}{10}\left[u_0 + u_2 + u_4 + u_6 + 5(u_1 + u_5) + 6u_3\right], \text{(viii.)}$$

which is Weddle's rule, with an error

$$-\frac{h}{840}\left[\delta^5 - \tfrac{41}{60}\delta^7 + \ldots\right]\mu\,(u_6 - u_0).$$

The elimination of δ^3 between (i.) and (iii.) gives

$$A = \frac{h}{280}\left[384A_1 - 105A_2 + A_6\right]$$

$$= \frac{h}{140}\left[45\,(u_0 + u_6) + 192\,(u_1 + u_3 + u_5) + 87\,(u_2 + u_4)\right], \qquad \text{(ix.)}$$

with an error $-\dfrac{h}{210}\left[\delta^5 - \tfrac{163}{120}\delta^7 + \ldots\right]\mu\,(u_6 - u_0)$,

while the elimination of δ^3 between (ii.) and (iii.) gives

$$A = \frac{h}{210}\left[243A_1 - 35A_3 + 2A_6\right]$$

$$= \frac{h}{70}\left[25\,(u_0 + u_6) + 81\,(u_1 + u_2 + u_4 + u_5) + 46u_3\right], \qquad \text{(x.)}$$

with an error $-\dfrac{9h}{840}\left[\delta^5 - \tfrac{801}{540}\delta^7 + \ldots\right]\mu\,(u_6 - u_0)$.

As before, we can, by elimination, get other formulæ involving only five terms: thus, if we eliminate u_0 and u_6 between (viii.) and (ix.), we get

$$A = \frac{h}{20}\left[66A_1 - 75A_2 + 30A_3 - A_6\right]$$

$$= \frac{3h}{10}\left[11\,(u_1 + u_5) - 14\,(u_2 + u_4) + 26u_3\right], \qquad \text{(xi.)}$$

with an error $h\left[\tfrac{41}{840}\delta^5 - \tfrac{3949}{50400}\delta^7 + \ldots\right]\mu\,(u_6 - u_0)$.

The elimination of u_1 and u_5 between the same two formulæ gives

$$A = \frac{h}{120}\left[243A_2 - 128A_3 + 5A_6\right]$$

$$= \frac{h}{20}\left[11\,(u_0+u_6) + 81\,(u_2+u_4) - 64u_3\right], \qquad \text{(xii.)}$$

with an error $\quad -h\left[\tfrac{36}{840}\delta^5 - \tfrac{561}{8400}\delta^7 + \dots\right]\mu\,(u_0 - u_6)$,

while the elimination of u_2 and u_4 gives

$$A = \frac{h}{300}\left[486A_1 - 243A_2 + 58A_3 - A_6\right]$$

$$= \frac{h}{50}\left[14\,(u_0+u_6) + 81\,(u_1+u_5) + 110u_3\right],^* \qquad \text{(xiii.)}$$

with an error $\quad h\left[\tfrac{3}{1400}\delta^5 - \tfrac{1}{224}\delta^7 + \dots\right]\mu\,(u_0 - u_6)$.

By eliminating δ^5 between any two of these fifth difference formulæ, we get

$$A = \frac{h}{840}\left[1296A_1 - 567A_2 + 112A_3 - A_6\right]$$

$$= \frac{h}{140}\left[41\,(u_0+u_6) + 216\,(u_1+u_5) + 27\,(u_2+u_4) + 272u_3\right], \qquad \text{(xiv.)}$$

with an error $\quad -\dfrac{3h}{2800}\mu\delta^7\,(u_0 - u_6) + \dots$.

5. Of the preceding formulæ some err in excess, and others in defect, of the true value, and by combining them in various ways the error can often be considerably reduced. For example, by taking the mean of (iv.) and (v.), both of which show a relatively large error, we get Weddle's rule. If we take the mean of (i.) and (vi.), we get

$$A = \frac{h}{33}\left[49A_1 - 19A_2 + 3A_3\right]$$

$$= \frac{h}{33}\left[10\,(u_0+u_6) + 49\,(u_1+u_5) + 11\,(u_2+u_4) + 58u_3\right], \qquad \text{(xv.)}$$

with an error $\; -\dfrac{h}{1980}\left[\delta^3 + \tfrac{149}{84}\delta^5 + \tfrac{6823}{5040}\delta^7 - \dots\right]\mu\,(u_0 - u_6)$,

* *Journal of the Institute of Actuaries*, Vol. XXIV., p. 107, where the formula is derived from Gauss's theorem. It has been pointed out by the referee that this is a particular case of formula (42) of Mr. Sheppard's paper (*Proceedings*, Vol. XXXII., p. 270); but with this exception the formulæ given on pp. 269–70 appear to be distinct from those of this paper.

while the mean of (xi.) and (xii.) gives

$$A = \frac{h}{240}[396A_1 - 207A_2 + 52A_3 - A_0]$$

$$= \frac{h}{40}[11(u_0 + u_6) + 66(u_1 + u_5) - 3(u_2 + u_4) + 92u_3], \qquad \text{(xvi.)}$$

with an error $h[\frac{1}{336}\delta^5 - \frac{583}{100800}\delta^7 + \ldots]\,\mu(u_6 - u_0)$.

Again, if to (iv.) we add twice (vii.) and take the mean, we get

$$A = \frac{h}{7}[11A_1 - 5A_2 + A_3]$$

$$= \frac{h}{7}[2(u_0 + u_6) + 11(u_1 + u_5) + u_2 + u_4 + 14u_3], \qquad \text{(xvii.)}$$

with an error $\frac{h}{420}[\delta^3 - \frac{91}{84}\delta^5 + \frac{3017}{5040}\delta^7 - \ldots]\,\mu(u_6 - u_0)$.

If we double (viii.) and add it to (xvi.), we get

$$A = \frac{h}{720}[1116A_1 - 495A_2 + 100A_3 - A_0]$$

$$= \frac{h}{120}[35(u_0 + u_6) + 186(u_1 + u_5) + 21(u_2 + u_4) + 236u_3], \qquad \text{(xviii.)}$$

with an error $\frac{h}{5040}[\delta^5 - \frac{419}{60}\delta^7 + \ldots]\,\mu(u_6 - u_0)$,

while the result of doubling (viii.) and adding it to (xiii.) is

$$A = \frac{h}{900}[1386A_1 - 603A_2 + 118A_3 - A_0]$$

$$= \frac{h}{150}[44(u_0 + u_6) + 231(u_1 + u_5) + 30(u_2 + u_4) + 290u_3], \qquad \text{(xix.)}$$

with an error $-\frac{h}{12600}[\delta^5 + \frac{143}{12}\delta^7 + \ldots]\,\mu(u_6 - u_0)$.

Formulæ of this kind can easily be extended.

6. By giving to n other values such as 8, 9, 10, ..., we get other groups; but in all these formulæ the earlier differences are got rid

of at the expense of increasing the coefficients of the later ones, and the larger the factors of n the greater is the increase in these coefficients. Thus, taking three factors a, b, c and p, q, r to satisfy the equations

$$pa^2+qb^2+rc^2=0,$$

$$pa^4+qb^4+rc^4=0,$$

we have

$$\frac{pa^6+qb^6+rc^6}{p+q+r}=-a^2b^2c^2,\quad \frac{pa^8+qb^8+rc^8}{p+q+r}=-a^2b^2c^2(a^2+b^2+c^2),$$

$$\frac{pa^{10}+qb^{10}+rc^{10}}{p+q+r}=-a^2b^2c^2(a^4+b^4+c^4+b^2c^2+c^2a^2+a^2b^2),$$

and so on; and these are elements of the coefficients in the expression for the error. The success of many of the formulæ involving six intervals appears to be due to some extent to the fact that *six* has as factors the first three natural numbers. If we put $n=12$, we should get all the preceding formulæ duplicated, and a large number of others due to the introduction of the other factors. The degree of approximation is increased, but in practical applications the calculation of the ordinates often involves considerable numerical work, and it is desirable to combine a good degree of approximation with facility of computation. It is well known too that these differences run with great irregularity; they often change sign, and after first decreasing numerically they often increase rapidly in proceeding to the higher orders; so that a formula which is true to third differences only may give a better result than one which is true to the fifth or higher orders. A preliminary examination of the differences may guide us as to which set of formulæ is the best to use. This point is illustrated by the numerical examples given at the end of this paper.

7. If we take the ordinary central difference interpolation formula

$$u_x=u_0+\frac{x}{1!}\mu\delta u_0+\frac{x^2}{2!}\delta^2u_0+\frac{x(x^2-1)}{3!}\mu\delta^3u_0+\frac{x^2(x^2-1)}{4!}\delta^4u_0+\ldots,\quad (4)$$

and integrate with respect to x, between limits $-\frac{1}{2}$ and $+\frac{1}{2}$, we get

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}}u_x\,dx=\left[1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\tfrac{367}{967680}\delta^6-\ldots\right]u_0$$

or

$$\int_0^h u_x\,dx=h\left[1+\tfrac{1}{24}\delta^2-\tfrac{17}{5760}\delta^4+\tfrac{367}{967680}\delta^6-\ldots\right]u_{\frac{1}{2}h}.$$

Hence $$\int_0^{nh} u_x dx = h\,[1+E^h+E^{2h}+\ldots+E^{(n-1)h}] \int_0^h u_x dx$$
$$= h\,[u_{\frac{1}{2}h}+u_{\frac{3}{2}h}+\ldots+u_{(n-\frac{1}{2})h}]$$
$$+h\,[\lambda_2'\delta+\lambda_4'\delta^3+\lambda_6'\delta^5+\ldots](u_{nh}-u_0), \qquad (5)$$

where λ_2', λ_4', ... stand for the numerical coefficients.

This corresponds to the mid-ordinate formula* given by Mr. Sheppard; but, as pointed out by him, it is not adapted for finding more accurate formulæ. Proceeding as before, we should get

$$\int_0^{nh} u_x dx = ah\,[u_{\frac{1}{2}ah}+u_{\frac{3}{2}ah}+\ldots+u_{(n-\frac{1}{2}a)h}]$$
$$+\frac{ah}{144}\left[6\delta+\left(\frac{23a^2}{40}-1\right)\delta^3-\left(\frac{145a^4}{1344}+\frac{23a^2}{80}-\frac{9}{20}\right)\delta^5-\ldots\right](u_{nh}-u_0), \quad (6)$$

where a is one of the factors of n.

The numerical coefficients here are greater than those of equation (3), and, as $a, b, c, \ldots$ must be odd numbers, the coefficients of the differences in the expression for the error will be much larger. It may be noticed, however, that the coefficients in (5) are smaller than the corresponding coefficients in (2); so that, if greater accuracy be required, it will probably be better to compute the first few differences. This, of course, involves a knowledge of terms preceding $u_{\frac{1}{2}h}$, and following $u_{(n-\frac{1}{2})h}$.

8. As illustrations of the preceding formulæ the values of the integrals

$$\int_0^1 \frac{dx}{1+x} = \log_e 2 \qquad (a)$$

and

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \qquad (b)$$

have been computed, using six intervals, with ordinates $u_0, u_{\frac{1}{6}}, \ldots, u_1$. It may be noticed that the central differences of $1/(1+x)$ first decrease numerically, then increase and become infinite, since $u_{-1}=\infty$; so that after a certain point the expressions obtained above cease to represent the error. For u_0 the odd differences begin to increase with the seventh; for u_1 the increase begins much later. In the case of $1/(1+x^2)$ the odd central differences of u_0 are all zero, and those of u_1 begin to increase numerically after the third.

The true values of (a) and (b) to seven places of decimals are ·6931472, and ·7853982.

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* *Proceedings*, Vol. XXXII., p. 267.

The values of the ordinates are

x	$u = 1/(1+x)$	$u = 1/(1+x^2)$
0	1·	1·
$\frac{1}{6}$	·8571429	·9729730
$\frac{2}{6}$	·75	·9
$\frac{3}{6}$	·6666667	·8
$\frac{4}{6}$	·6	·6923077
$\frac{5}{6}$	·5454545	·5901639
1	·5	·5

and the errors in the computed values are as follows:—

Formula.	No. of terms in formula.	Error in (a) $\times 10^7$.	Error in (b) $\times 10^7$.
(i.) (Simpson's rule)	7	+226	− 3
(ii.) (Simpson's second rule)	7	+482	− 23
(vi.)	5	−146	+ 28
(viii.) (Weddle's rule)	7	+ 22	+ 14
(xiii.) (Hardy's formula)	5	− 15	−179
(xiv.)	7	+ 9	− 55
(xv.)	7	+ 40	+ 12
(xvii.)	7	− 66	+ 21
(xviii.)	7	+ 7	− 67
(xix.)	7	+ 10	− 50

It appears that, while any one of these formulæ will give a good approximation, the best results are not obtained by always using the same ones. When once the values of the ordinates have been obtained, that of the integral can be readily computed by several of these formulæ; and, as some err in excess and others in defect, we shall get very close limits within which the true value lies, and generally better results than would be attained by exclusive use of any one formula.

Thursday, April 10th, 1902.

Dr. HOBSON, F.R.S., President, in the Chair.

Eleven members present.

Prof. C. J. Joly, M.A., Dunsink Observatory, Ireland; Ganesh Prasad, D.Sc., Christ's College, Cambridge, and Miss Lilian Janie Whitley, B.A., Westfield College, Hampstead, N.W., were elected members.

The President (Dr. Larmor temporarily in the Chair) communicated a "Note on Divergent Series." Prof. Love next gave results he had arrived at in connection with "Stress and Strain in two-dimensional Elastic Systems." Discussions followed on both communications, in which the President and Messrs. Larmor and Love took part.

The President read the titles of the following papers:—

Further applications of Matrix Notation to Integration Problems: Dr. H. F. Baker.

On the Convergence of Series which represent a Potential: Prof. T. J. I'A. Bromwich.

On the Groups defined for an Arbitrary Field by the Multiplication Tables of certain Finite Groups: Dr. L. E. Dickson.

The following presents were made to the Library:—

"Educational Times," April, 1902.

"Indian Engineering," Vol. xxxi., March 15–April 5, 1902.

Gibbs, J. Willard.—"Elementary Principles in Statistical Mechanics," 8vo; London, 1902.

"Nautical Almanac for 1905," 8vo; Edinburgh, 1902.

"Mittheilungen der Mathematischen Gesellschaft," Bd. iv., Heft 2; Hamburg, 1902.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 5; Livorno, 1902.

"Il Pitagora," Anno viii., Nos. 1–5; Palermo, 1901–2.

"Memoirs of the National Academy of Sciences," Vol. viii.; Washington, 1898.

Dickson, Dr. L. E.—"College Algebra"; New York, 1902.

From the "Scientia" Series, presented by the publisher, M. C. Naud:—

- "Cryoscopie," par F. M. Raoult, No. 13.
- "Franges d'Interférences et leurs applications métrologiques," par J. Macé de Lépinaz, No. 14.
- "La Géométrie non-euclidienne," par P. Barbarin, No. 15.
- "Le Phénomène de Kerr," par E. Néculcéa, No. 16.
- "Théorie de la Lune," par H. Andoyer, No. 17.
- "Géométrographie," par E. Lemoine, No. 18.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. lxix., Nos. 457, 458; 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxvi., Heft 4; Leipzig, 1902.

"Rendiconti del Circolo Matematico di Palermo," Tomo xvi., Fasc. 1, 2; 1902.

"Bulletin of the American Mathematical Society," Vol. viii., Nos. 5, 6; New York, 1902.

"Bulletin des Sciences Mathématiques," Tome xxvi., Fév., 1902; Paris.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Vol. viii., Fasc. 2; Napoli, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. xi., Fasc. 5; Roma, 1902.

"Jahresbericht der Deutschen Mathematiker-Vereinigung in Monatsheften," herausgegeben von A. Gutzmer in Jena, Band xi., 1 and 2 (doppel-) Heft (Jan.–Feb.), Dec. 19, 1901; Band xi., 3 Heft (März), Feb., 1902.