On Groups in which every two Conjugate Operations are Permutable. *By* W. Burnside.

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In a paper published in the *Quarterly Journal of Mathematics* (1902), "On an Unsettled Question in the Theory of Discontinuous Groups," I have determined the order of a group with given generating operations when subject to the condition that the order of every operation shall be 3. If P and Q are any two operations of such a group, the relations

$$P^3 = 1, \quad Q^3 = 1, \quad (PQ)^3 = 1, \quad (PQ^2)^3 = 1$$

lead at once to $$P \, . \, QPQ^{-1} \, . \, Q^2PQ^{-2} = 1$$

and $$P \, . \, Q^2PQ^{-2} \, . \, QPQ^{-1} = 1;$$

so that P and QPQ^{-1} are permutable. The condition that every operation is of order 3 involves therefore that every two conjugate operations are permutable.

In the present paper I have considered the general problem thus presented; viz., the nature of a group generated by a finite number of operations when every two conjugate operations of the group are permutable. It will be seen that the general problem is closely connected with the more special one above referred to. When no further limitation is imposed on the operations, it is found that every operation of the group is given once and only once by a form

$$P^x Q^y \ldots R^z,$$

where P, Q, ..., R are a finite number of operations belonging to the group; and of the indices x, y, z, ... a certain number take all values from $-\infty$ to $+\infty$, while the remainder take the values 0, 1, 2.

The sufficient and necessary conditions that the group shall be of finite order are that the generating operations be of finite order. When this is the case, the group is the direct product of groups whose orders are powers of primes. In general for such a group the commutator of any two operations is a self-conjugate operation; but the case in which the order is a power of 3 is, as might be expected, exceptional.

1. In dealing with groups in which every two conjugate operations are permutable, the following notation will be used.

If $$P_a,\ P_b,\ P_c,\ \ldots$$

are any operations of such a group, the result of transforming P_a by P_b will be written $P_a P_{ab}$, so that

$$P_a^{-1} P_b^{-1} P_a P_b = P_{ab}.$$

Similarly, $$P_{ab}^{-1} P_c^{-1} P_{ab} P_c = P_{abc},$$

$$P_{abc}^{-1} P_d^{-1} P_{abc} P_d = P_{abcd},$$

and so on.

Further, the notation will be extended so that

$$P_{abc}^{-1} P_{de}^{-1} P_{abc} P_{de} = P_{(abc)(de)}.$$

The use of brackets in the suffixes will prevent any ambiguity; thus $P_{(abcd)e}$ is the same as P_{abcde}; but these are not necessarily the same as $P_{a(bcde)}$ or $P_{(ab)(cde)}$. From the definition of P_{ab} it follows that

$$P_{ba} = P_b^{-1} P_a^{-1} P_b P_a = P_{ab}^{-1}.$$

The operation P_{ab} may be regarded as the product of P_a^{-1} and $P_b^{-1} P_a P_b$; or as that of $P_a^{-1} P_b^{-1} P_a$ and P_b. Hence, since every operation is permutable with its conjugates, P_{ab} is permutable with both P_a and P_b.

Now, from $P_b^{-1} P_a P_b = P_a P_{ab}$ and $P_b^{-1} P_{ab} P_b = P_{ab}$,

it follows that $$P_b^{-y} P_a P_b^y = P_a P_{ab}^y,$$

and $$P_b^{-y} P_a^x P_b^y = P_a^x P_{ab}^{xy},$$

or $$P_a^{-x} P_b^{-y} P_a^x P_b^y = P_{ab}^{xy}.$$

Since P_{ab} is the product of P_a^{-1} and $P_b^{-1} P_a P_b$, it follows that P_{abc}, P_{abcd}, ... are products of powers of operations which are conjugate to P_a. Similarly, P_{abcd} may be expressed as products of powers of operations which are conjugate to P_b (or to P_c, or to P_d). Hence

$$P_{abcda} = 1,$$

$$P_{abcdb} = 1,$$

&c.

Again, P_{abc} and P_{ad} can both be represented as products of powers of operations which are conjugate to P_a. They are therefore permutable, and

$$P_{(abe)(ad)} = 1.$$

Hence, generally, $$P_{(abe\ldots)(de\ldots)(gh\ldots)\ldots} = 1,$$

if in the multiple suffix any simple suffix occurs more than once.

Since P_{ab} is permutable with both P_a and P_b, every substitution of the sub-group generated by P_a and P_b can be represented in the form

$$P_a^x P_b^y P_{ab}^z.$$

In fact $$P_a^x P_b^y P_{ab}^z P_a^{x'} P_b^{y'} P_{ab}^{z'} = P_a^{x+x'} P_a^{-x'} P_b^y P_a^{x'} P_b^{-y'} P_b^{y+y'} P_{ab}^{z+z'}$$

$$= P_a^{x+x'} P_{ab}^{-x'y} P_b^{y+y'} P_{ab}^{z+z'}$$

$$= P_a^{x+x'} P_b^{y+y'} P_{ab}^{z+z'-x'y}.$$

Let P_c be any operation which does not belong to this sub-group. Then

$$P_a P_c P_a^{-1} = P_c P_{ca}^{-1},$$

$$P_b P_a P_c P_a^{-1} P_b^{-1} = P_c P_{cb}^{-1} P_{ca}^{-1} P_{cab},$$

$$P_a^{-1} P_b P_a P_c P_a^{-1} P_b^{-1} P_a = P_c P_{cb}^{-1} P_{cba}^{-1} P_{cab};$$

and therefore $$P_{ab}^{-1} P_c P_{ab} = P_c P_{cba}^{-1} P_{cab},$$

or $$P_{c(ab)} = P_{cba}^{-1} P_{cab}.$$

Now $$P_{c(ab)} = P_{abc}^{-1};$$

hence $$P_{abc} P_{cba}^{-1} P_{cab} = 1,$$

or, since $$P_{cba} = P_{bca}^{-1},$$

$$P_{abc} P_{bca} P_{cab} = 1. \qquad \text{(i)}$$

Again, since $P_a P_b$ and $P_c^{-1} P_a P_b P_c$ $(= P_a P_{ac} P_b P_{bc})$ are conjugate, they are permutable. Hence

$$P_a P_{ac} P_b P_{bc} = P_b^{-1} P_a^{-1} (P_a P_{ac} P_b P_{bc}) P_a P_b$$

$$= P_b^{-1} (P_a P_{ac} P_b P_{ba} P_{bc} P_{bca}) P_b$$

$$= P_a P_{ab} P_{ac} P_{acb} P_b P_{ba} P_{bc} P_{bca}.$$

But P_a, P_{ab}, P_{ac}, P_{acb} are all permutable, as also are P_b, P_{ba}, P_{bc}, P_{bca}.

Hence $$P_a P_{ac} P_b P_{bc} = P_a P_{ac} P_{acb} P_{ab} P_{ba} P_{bca} P_b P_{bc},$$

or $$P_{acb} P_{bca} = 1. \qquad \text{(ii)}$$

If P_c belongs to the sub-group generated by P_a and P_b, the relations

(i) and (ii) become identities. They are therefore true in any case, and for any permutation of the suffixes. Now, (ii) may be written

$$P_{bca} = P_{cab},$$

or

$$P_{acb} = P_{cba}.$$

Hence from (i) and (ii) together it follows that

$$P_{abc} = P_{bca} = P_{cab} = P_{acb}^{-1} = P_{cba}^{-1} = P_{bac}^{-1}; \tag{iii}$$

and

$$P_{abc}^{3} = 1. \tag{iv}$$

The relations (iii) are equivalent to the statement that when any permutation of the suffixes is effected in the symbol P_{abc} the operation represented is unaltered or changed into its inverse, according as the permutation is an even one or an odd one. Since

$$P_{ab\dots def} = P_{(ab\dots d)ef} = P_{(ab\dots d)fe}^{-1} = P_{ab\dots dfe}^{-1},$$

this statement may clearly be extended at once to any such symbol as $P_{ab\dots def}$. Again,

$$P_{a(ij\dots k)} = P_{(ij\dots k)a}^{-1} = P_{ij\dots ka}^{-1} = P_{aij\dots k}^{(-1)^{r+1}},$$

where r is the number of suffixes in the set $i, j, \dots, k$. Hence

$$P_{(l\dots m)(ij\dots k)} = P_{l\dots mij\dots k}^{(-1)^{r+1}};$$

and thus any symbol $P_{(\ )(\ )\dots}$ can be replaced by one in which there are no brackets in the suffix.

From the relation (iv) it follows that any symbol with three or more letters in its suffix is an operation of order 3, or else is the identical operation. Further, since

$$P_{abc}^{3} = 1$$

may be written

$$P_{ab}^{-3} P_c^{-1} P_{ab}^{3} P_c = 1,$$

the cube of every P with two letters in the suffix (*i.e.*, the cube of the commutator of any two operations) is a self-conjugate operation of the group. Again,

$$P_{abc}^{3} = 1$$

may be written

$$P_{ab}^{-1} P_c^{-3} P_{ab} P_c^{3} = 1.$$

Hence the cube of every operation of the group is permutable with every operation whose suffix contains two or more letters; *i.e.*, with every operation of the derived group.

2. Let $$P_1, P_2, \ldots, P_n$$
be n independent operations which generate a group G, and suppose that the only conditions to which they are subject are that every two conjugate operations of G are permutable.

The product of any two operations of the form

$$P_1^a P_2^b \ldots P_n^c P_{12}^d P_{13}^e \ldots P_{n-1,n}^f P_{123}^g \ldots P_{123\ldots n}^h, \qquad \text{(v)}$$

where every P with a multiple suffix occurs once, while the P's are written in a definite sequence, is another operation of the same form. In fact, from the preceding paragraph,

$$P_{a_1a_2\ldots a_r}^{-y} P_{b_1b_2\ldots b_s}^{x} P_{a_1a_2\ldots a_r}^{y}$$
$$= P_{b_1b_2\ldots b_s}^{x} P_{b_1b_2\ldots b_s a_1a_2\ldots a_r}^{(-1)^{r+1}xy};$$

so that the multiplication can be actually carried out, and in the result the P's can be re-arranged in the original sequence. Hence with suitably chosen indices every operation of G can be represented in the form (v.).

To specify all distinct operations of the group it remains to show under what conditions a symbol of the form (v) represents the identical operation. As the basis of an induction it will be assumed that when there are $n-1$ generating operations the conditions are that (α) the index of each P with a single or double suffix is zero, and (β) the index of each P with a triple or higher suffix is zero or a multiple of 3.

If to the conditions defining G we add

$$P_1 = 1,$$

a new group is defined, which is simply isomorphic with G/H, where H is the self-conjugate sub-group of G generated by P_1 and its conjugate operations. The latter is an Abelian group, and cannot therefore be identical with G.

Now $$P_1 = 1$$
involves $$P_{12} = 1,\ P_{13} = 1,\ \ldots,$$
$$P_{123} = 1,\ \ldots.$$

Hence G/H is simply isomorphic with the group generated by $P_2, P_3, \ldots, P_n$; and this sub-group of G can therefore have no operation, except identity, in common with H. Suppose now that

$$P_1^a P_2^b \dots P_n^c P_{12}^d P_{13}^e P_{23}^f \dots P_{n-1,n}^g P_{123}^h \dots P_{234}^i \dots = 1.$$

By preceding processes the factors on the left may be rearranged so that all the P's containing 1 in the suffix come at the end, the indices of the remaining P's being unaltered. Hence

$$P_2^b \dots P_n^c P_{23}^f \dots P_{n-1,n}^g P_{234}^i \dots = P_1^a P_{12}^{d'} P_{13}^{e'} \dots P_{123}^{h'} \dots .$$

Now the operation on the right belongs to H, and that on the left to $\{P_2, P_3, \dots, P_n\}$. Hence each must be the identical operation, and therefore by the assumption made

$$b = \dots = c = f = \dots = g = 0,$$

$$i = \dots = 0, \text{ or a multiple of } 3.$$

Similar series of results follow by considering the self-conjugate sub-groups arising respectively from the suppositions

$$P_2 = 1, \text{ or } P_3 = 1, \dots .$$

Hence an operation of the form (v) can only represent the identical operation, in a group generated by n operations, when the indices of *all* the P's with less than n symbols in the suffix satisfy the assumed conditions. But when this is so the operation reduces to $P_{123\dots n}^m$. If m is neither zero nor a multiple of 3,

$$P_{123\dots n}^m = 1, \quad \text{and} \quad P_{123\dots n}^3 = 1$$

involve

$$P_{123\dots n} = 1,$$

which would constitute an additional limitation on the group, not contained in the original conditions. Hence m must be zero or a multiple of 3; and the induction is completed.

Finally, therefore, every operation of G is given once, and once only, by the form (v), if the indices of the P's with a single or double suffix take all values from $-\infty$ to $+\infty$, while the indices of the P's with a triple or higher suffix take all values from 0 to 2.

3. So far it has been supposed that the only limitation on the group considered is that every two conjugate operations are permutable. The group under these conditions necessarily contains operations whose order is not finite. It will still have this property under a variety of further limitations. For instance, the condition

$$P_{123} = 1$$

implies that every P with a multiple suffix in which 1, 2, and 3 occur is the identical operation. The generality of the group is thus reduced; but it still contains operations whose order is not finite.

From the form giving the operations of the group it is clear that, if the order of every operation of the group is finite, the group is one of finite order. Moreover, the necessary and sufficient conditions for this are that each one of the generating operations should be of finite order. That these conditions are necessary is clear from the form (v). To show that they are sufficient, it is only necessary to notice that, if P_1 is of finite order m, P_{12} is of finite order, equal to or a factor of m.

Suppose now that G is of finite order, and consider the operation $S^{-1}T^{-1}ST$ of G. As the product of S^{-1} and $T^{-1}ST$ its order must be equal to or a factor of that of S. Similarly its order is equal to or a factor of that of T. Hence, if the orders of S and T are relatively prime,

$$S^{-1}T^{-1}ST = 1,$$

or S and T are permutable. Since any two operations of G whose orders are relatively prime are permutable, G must be the direct product of a number of groups, the order of each of which is the power of a prime. In dealing with groups of finite order with the property considered, it is therefore sufficient to suppose the order to be p^a, where p is a prime.

The case $p = 3$ evidently stands by itself. Suppose first that p is not 3. Then P_{123} is an operation whose order is a power of p. But in any case it has been seen that

$$P_{123}^3 = 1.$$

Hence

$$P_{123} = 1,$$

or the commutator of any two operations of the group is a self-conjugate operation.

Every operation of the derived group is therefore a self-conjugate operation. That this condition is sufficient to ensure that in a group of order p^a every two conjugate operations are permutable is obvious.

If

$$P_1,\ P_2,\ \ldots,\ P_n,$$

of orders

$$p^{a_1},\ p^{a_2},\ \ldots,\ p^{a_n}$$

$$(a_1 \geqslant a_2 \geqslant a_3 \ldots \geqslant a_n),$$

are the generating operations of such a group, and if the only conditions beyond the given orders of the generating operations are that every two conjugate operations are permutable, the order of P_{rs} $(r<s)$ is p^{a_s}. Every operation of the group will then be given once by

$$\Pi P_r^{x_r} \, . \, \Pi P_{rs}^{x_{rs}}$$

$$(r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n;\ r<s),$$

where x_r takes all values from 0 to $p^{a_r}-1$ and x_{rs} from 0 to $p^{a_s}-1$. Hence the order of the group is

$$p^{\Sigma r a_r}$$

When p is equal to 3, the only condition that P with a triple or higher suffix is subjected to is

$$P^3_{abc\ldots} = 1,$$

and it is no longer the case that the commutator of any two operations is a self-conjugate operation. If the group is generated by

$$P_1,\ P_2,\ \ldots,\ P_n,$$

of orders $3^{a_1},\ 3^{a_2},\ \ldots,\ 3^{a_n}$

$$(a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n),$$

every operation of the group will be represented once in the form (v), where the index of P_r lies between 0 and $3^{a_r}-1$, the index of P_{rs} $(r<s)$ lies between 0 and $3^{a_s}-1$, and all the other indices lie between 0 and 2; both limits included. The order of the group is therefore

$$3^{\Sigma r a_r + 2^n - 1 - \frac{1}{2}n(n+1)}$$

In particular, if the order of each of the generating operations is 3, the order of the group is 3^{2^n-1}.

This is the case already referred to in the introduction.

4. Groups of finite order p^a, of the kind here considered, possess two general properties in common with Abelian groups. First, the totality of the operations of the group whose orders are equal to or are less than p^r, where r is a given integer, constitute a sub-group. If P_1 and P_2 are any two operations of the group, of orders p^{a_1} and p^{a_2} $(a_1 \geqslant a_2)$, the order of P_{12} is equal to or is less than p^{a_2}, Now, by a repeated use of the formula

$$P_1^{-a} P_2^{-b} P_1^a P_2^b = P_{12}^{ab},$$

it follows that $$(P_1P_2)^x = P_1^x P_2^x P_{12}^{-\frac{1}{2}x(x-1)}.$$

Hence, if $$P_1^x = 1,\quad P_2^x = 1,\quad P_{12}^{\frac{1}{2}x(x-1)} = 1,$$

then $$(P_1P_2)^x = 1.$$

Now, if $p > 2$, the condition

$$P_{12}^{\frac{1}{2}x(x-1)} = 1$$

may be replaced by $$P_{12}^{x(x-1)} = 1.$$

It follows that, if $p > 2$, the order of P_1P_2 is equal to or is less than the order of P_1. The sub-group generated by all the operations whose orders are equal to or less than p^r consists therefore of these operations and of no others; for the product of any two of these operations is an operation whose order does not exceed p^r.

The case $p = 2$ clearly forms an exception to the general theorem; for, if P_1, P_2 are two non-permutable operations of the group of order 2, P_{12} is an operation of order 2. But

$$P_{12} = (P_1P_2)^2,$$

and P_1P_2 is therefore an operation of order 4.

Secondly, the totality of the distinct operations which arise, when every operation of the group is raised to the power p^r, where r is a given integer, constitute a sub-group. Consider the case in which $p > 3$, and the group is generated by two operations P_1 and P_2. The operations of the sub-group generated by P_1^p, P_2^p, and P_{12}^p are given by the form

$$P_1^{px} P_2^{py} P_{12}^{pz}.$$

The p-th power of any operation $P_1^a P_2^b P_{12}^c$ is given by the relation, obtained as above,

$$(P_1^a P_2^b P_{12}^c)^p = P_1^{pa} P_2^{pb} P_{12}^{pc-\frac{1}{2}abp(p-1)}.$$

Now $\frac{1}{2}(p-1)$ is an integer. Hence

$$P_1^{px} P_2^{py} P_{12}^{pz} = (P_1^x P_2^y P_{12}^{z+\frac{1}{2}(p-1)xy})^p;$$

i.e., every operation of $\{P_1^p, P_2^p, P_{12}^p\}$ is the p-th power of an operation of the group, and the p-th power of every operation of the group is contained in $\{P_1^p, P_2^p, P_{12}^p\}$. For more than two generating operations the proof will proceed on precisely similar lines. The theorem is therefore true when $r = 1$. But it must also be true for the resulting sub-group; so that it is true generally. The case $p = 2$ forms again an exception, since $\frac{1}{2}(p-1)$ is not then an integer. In fact, in this case $\{P_1^2, P_2^2, P_{12}^2\}$ will not contain P_{12}, which is the square of P_1P_2.

The case $p=3$ requires separate treatment. It may be easily shown that in this case the cube of any operation of the group when expressed in the form (v) contains no P with a triple or higher suffix; and from this the truth of the theorem immediately follows.

Thursday, June 12th, 1902.

Dr. E. W. HOBSON, F.R.S., President, in the Chair.

Fifteen members present.

The President announced that the "De Morgan medal" for 1902 had been awarded to Prof. A. G. Greenhill.

Mr. A. C. Porter was admitted into the Society.

The following paper was communicated by Prof. Love:—

Prof. A. W. Conway: "The Principle of Huygens in a Uniaxal Crystal."

Lieut.-Col. A. Cunningham gave an account of "Some Investigations concerning the repetition of the Sum-Factor Operation."

The following papers were communicated from the Chair:—

Prof. E. Picard: "Sur un théorème fondamental dans la théorie des équations différentielles."

Mr. G. H. Hardy: "Some Arithmetical Theorems."

Prof. M. J. M. Hill: "On a Geometrical Proposition connected with the Continuation of Power Series."

Mr. J. H. Grace: "Types of Perpetuants."

The following presents were made to the Library:—

"Educational Times," June, 1902.

"Indian Engineering," Vol. xxxi., Nos. 16–20; 1902.

"Queen's College, Galway—Calendar for 1901–1902."

Penfield, S. L.—"The Stereographic Projection and its Possibilities from a Graphical Standpoint," 1901.

Penfield, S. L.—"On the Use of the Stereographic Projection for Geographical Maps and Sailing Charts," 1902.

"Periodico di Matematica," Serie 2, Vol. iv., Fasc. 6; Livorno, 1902.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 7; Livorno, 1902.

"L'Enseignement Mathématique," Année IV., No. 3; 1902.

"Mathematical Gazette," Vol. II., No. 33, 1902.

Guldberg, A.—"Ueber Integralinvarianten und Integralparameter bei Berührungs-Transformationsgruppen," 1902.

"Journal de l'Ecole Polytechnique—Hommage rendu à M. le Colonel Mannheim," 1902.

De Morgan, A.—"Theory of Probabilities" (extract from *Encyclopædia Metropolitana*). From Mr. R. Tucker.

Carvallo, E.—"L'Electricité" (*Scientia*, No. 19).

"Mathematical Questions and Solutions from the *Educational Times*," New Series, Vol. I.; 1902.

D. Ocagne, M.—"Sur quelques Travaux récents relatifs à la Nomographie."

The following exchanges were received:—

Académie Royale de Belgique:—

"Annuaire, 1902," Bruxelles.

"Bulletin de la Classe des Sciences," Nos. 1-3; Bruxelles, 1901-1902.

"Mémoires Couronnés," Bruxelles, 1901-1902.

"Mémoires," Tome LIV., Fasc. 1-4; Bruxelles, 1900-1901.

"Mémoires Couronnés et Mémoires des Savants Etrangers," Tome LIX., Fasc. 1, 2; Bruxelles, 1901.

"Proceedings of the Royal Society," Vol. LXX., Nos. 459, 460; 1902.

"Reports to the Evolution Committee of the Royal Society"; 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXVI., No. 5; Leipzig, 1902.

"Bulletin de la Société Mathématique de France," Tome XXX., Fasc. 1; Paris, 1902.

"Bulletin of the American Mathematical Society," Vol. VIII., No. 8; "Transactions," Vol. III., No. 2; New York, 1902.

"Bulletin des Sciences Mathématiques," Tome XXV., "Contents," 1901; Tome XXVI., Mars, 1902; Paris.

"Annali di Matematica," Tomo VII., Fasc. 2, 3; Milano, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. XI., Fasc. 8, 9, 10; Roma, 1902.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 1-22; 1902.