ON THE NUMBERS OF REPRESENTATIONS OF A NUMBER AS A SUM OF $2r$ SQUARES, WHERE $2r$ DOES NOT EXCEED EIGHTEEN

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1. In a series of papers which are in course of publication in the *Quarterly Journal of Mathematics* * I have investigated systematically, by means of formulæ derived from elliptic functions, the numbers of representations of any number as a sum of 2, 4, 6, 8, 10, 12, 14, 16, and 18 squares, determining also in many cases the numbers of representations when the numbers of uneven and even squares in the representations are assigned.

The papers are somewhat lengthy, and the main results are partially obscured by the investigation of subsidiary formulæ, of recurring series required for the calculation of new functions introduced, and other collateral matters, so that I have thought I might be justified in submitting to the Society this short paper which contains a statement of the principal formulæ and results obtained relating to numbers of representations, separated from the mass of detail by which they are surrounded in the original papers.

Formulæ for the Numbers of Representations and Definitions of the Arithmetical Functions involved.

2. Using $R^{(t)}(n)$ to denote the number of representations of n as a sum of t squares, the following formulæ express the values of $R^{(t)}(n)$ for even

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* These papers are :—

(1) "On the Representations of a Number as the Sum of two, four, six, eight, ten, and twelve Squares," Vol. XXXVIII., pp. 1–62.

(2) "On the Representations of a Number as the Sum of fourteen and sixteen Squares," *ib.*, pp. 178–236.

(3) "On the Representations of a Number as the Sum of eighteen Squares," *ib.*, pp. 289–351.

(4) "On Elliptic Function Expansions in which the Coefficients are Powers of the Complex Numbers having n as Norm," Vol. XXXIX.

Connected also with the same subject are two other papers which had previously appeared in the *Quarterly Journal*, viz. : (i.) "On the Representations of a Number as a Sum of four Squares and on some Allied Arithmetical Functions," Vol. XXXVI., pp. 305–358 ; (ii.) "The Arithmetical Functions $P(m)$, $Q(m)$, $\Omega(m)$," Vol. XXXVII., pp. 36–48.

values of t up to $t = 18$:—

(i.) $$R^{(2)}(n) = 4E_0(n),$$

(ii.) $$R^{(4)}(n) = (-1)^{n-1}\, 8\xi_1(n),$$

(iii.) $$R^{(6)}(n) = 4\,\{4E_2'(n) - E_2(n)\},$$

(iv.) $$R^{(8)}(n) = (-1)^{n-1}\, 16\zeta_3(n),$$

(v.) $$R^{(10)}(n) = \tfrac{4}{5}\,\{E_4(n) + 16E_4'(n) + 8\chi_4(n)\},$$

(vi.) $$\begin{cases} R^{(12)}(2n) = -8\xi_5(2n), \\ R^{(12)}(2n+1) = 8\,\{\Delta_5(2n+1) + 2\Omega(2n+1)\}, \end{cases}$$

(vii.) $$R^{(14)}(n) = \tfrac{4}{61}\,\{64E_6'(n) - E_6(n) + 364\,W(n)\},$$

(viii.) $$R^{(16)}(n) = (-1)^{n-1}\tfrac{32}{17}\,\{\zeta_7(n) + 16\Theta(n)\},$$

(ix.) $$R^{(18)}(n) = \tfrac{4}{18005}\,\{13E_8(n) + 3328E_8'(n) + 97504\chi_8(n) - 61200G(n) - 6120G(2n)\}.$$

3. Five of the arithmetical functions which occur in the expression of these results, viz., $E_r(n)$, $E_r'(n)$, $\Delta_r(n)$, $\zeta_r(n)$, $\xi_r(n)$, depend upon the real divisors of n; one, $\chi_8(n)$, depends upon certain complex divisors of n, and, for the other four, I have obtained no useful arithmetical definition. Two of them, $\Omega(n)$ and $\Theta(n)$, may be defined by means of the representations of n as a sum of 4 squares; but, in the case of $W(n)$ and $G(n)$, I have had to content myself with their definitions as coefficients.

The functions $E_r(n)$, ... which depend upon the real divisors of n (1 and n included) are defined as follows:—

(i.) $E_r(n)$ denotes the sum of the r-th powers of the divisors of n which are of the form $4k+1$ diminished by the sum of the r-th powers of the divisors of n which are of the form $4k+3$.

Thus $E_0(n)$ denotes the number of divisors of n of the form $4k+1$ diminished by the number of those of the form $4k+3$.

(ii.) $E_r'(n)$ denotes the sum of the r-th powers of the divisors of n whose conjugates are of the form $4k+1$ diminished by the sum of the r-th powers of those whose conjugates are of the form $4k+3$.

Thus, if n is of the form $4k+1$, $E_r'(n) = E_r(n)$; and, if n is of the form $4k+3$, $E_r'(n) = -E_r(n)$.

(iii.) $\Delta_r(n)$ denotes the sum of the r-th powers of the uneven divisors of n.

(iv.) $\zeta_r(n)$ denotes the sum of the r-th powers of the uneven divisors of n diminished by the sum of the r-th powers of the even divisors of n.

(v.) $\xi_r(n)$ denotes the sum of the r-th powers of the even divisors of n whose conjugates are even and of the uneven divisors of n whose conjugates are uneven diminished by the sum of the r-th powers of the even divisors of n whose conjugates are uneven and of the uneven divisors of n whose conjugates are even.

Thus, if d is any divisor of n, and $dd' = n$, then

$$\zeta_r(n) = \Sigma(-1)^{d-1} d^r, \qquad \xi_r(n) = \Sigma(-1)^{d+d'} d^r.$$

When n is uneven, $\Delta_r(n)$, $\zeta_r(n)$, and $\xi_r(n)$ become equal.

The definition of the function $\chi_r(n)$ is as follows:—

(vi.) $\chi_r(m)$, m being any uneven number,* denotes the sum of the r-th powers of the primary complex numbers having m as norm.

When r is a multiple of 4 (as is the case with both the χ's used in the expression of the representations by 10 and 18 squares), we may define $\chi_r(n)$, n being any number, as one-fourth part of the sum of the r-th powers of *all* the complex numbers having m as norm.

The functions $\Omega(n)$, $W(n)$, $\Theta(n)$, $G(n)$ may be defined as coefficients by the elliptic function expansions—

$$\text{(vii.)} \qquad k^2 k'^2 \rho^6 = 16\, \Sigma_1^\infty\, \Omega(n) q^n,$$

$$\text{(viii.)} \qquad k^2 k'^2 \rho^7 = 16\, \Sigma_1^\infty\, W(n) q^n,$$

$$\text{(ix.)} \qquad k^2 k'^4 \rho^8 = 16\, \Sigma_1^\infty\, \Theta(n) q^n,$$

$$\text{(x.)} \qquad k^2 k'^2 (k'^2 - k^2) \rho^9 = 16\, \Sigma_1^\infty\, G(n) q^n,$$

where ρ denotes $2K/\pi$.

The functions $\Omega(n)$ and $\Theta(n)$ may be defined arithmetically as follows:—

If $a^2+b^2+c^2+d^2$ is any representation of n as a sum of 4 squares, then

$$\Omega(n) = \tfrac{1}{8} \Sigma^{(r)} \{[a^4] - 2[a^2 b^2]\},$$

$$\Theta(n) = e\, \Sigma^{(r)} \{[a^6] - 5[a^4 b^2] + 30[a^2 b^2 c^2]\},$$

where e is $\frac{1}{8}$ or $\frac{1}{24}$ according as n is uneven or even. The square brackets signify that the sum of all the quantities of the type indicated is to be taken: *e.g.*, $[a^4]$ denotes $a^4+b^4+c^4+d^4$; and $\Sigma^{(r)}$ refers to all the representations of n as a sum of 4 squares. $\Omega(m)$ and $\Theta(m)$ can also

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* Throughout this paper, unless otherwise stated, n denotes any number and m any uneven number.

be defined by means of the compositions* of $4m$ as a sum of 4 uneven squares.

Representations by 2, 4, 6, *and* 8 *Squares.*

4. The first four formulæ of § 2 lead to the following theorems:—

(i.) The number of representations of any number n as a sum of 2 squares is equal to four times the excess of the number of divisors of n which are of the form $4k+1$ over the number of divisors which are of the form $4k+3$.

(ii.) The number of representations of n as a sum of 4 squares is equal to $(-1)^{n-1}8\xi_1(n)$, and is therefore equal to eight times the sum of the divisors of n, if n is uneven, and to twenty-four times the sum of the uneven divisors of n, if n is even.

(iii.) The number of representations of n as a sum of 6 squares is equal to $4\{4E_2'(n)-E_2(n)\}$.

It follows from this result that the number of representations of an uneven number m as a sum of 6 squares is equal to $12E_2(m)$, if m is of the form $4k+1$, and is equal to $-20E_2(m)$, if m is of the form $4k+3$.

(iv.) The number of representations of n as a sum of 8 squares is equal to $(-1)^{n-1}16\zeta_3(n)$, and is therefore equal to sixteen times the sum of the cubes of the divisors of n, if n is uneven, and to sixteen times the excess of the sum of the cubes of the even divisors over the sum of the cubes of the uneven divisors, if n is even.

The first of these theorems is a well known result due originally to Jacobi. The other three were given by Jacobi or Eisenstein in the case of n uneven, and by H. J. S. Smith in their generality.

Representations by 10 *Squares.*

5. The formula (v.) of § 2 shows that the number of representations of n as a sum of 10 squares is equal to $\frac{4}{5}\{E_4(n)+16E_4'(n)+8\chi_4(n)\}$. Now $\chi_4(n)$ vanishes when n is of the form $4k+3$, and, more generally, whenever n is divisible by a factor of the form $4k+3$, which does not occur raised to an even power.

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* Each method of expressing n as a sum of squares, when the order of the squares is taken into account, is called a composition of n. If, in addition, account is taken of the signs of the roots of the squares, so that a^2 may be either $(+a)^2$ or $(-a)^2$ but 0^2 has only one root, then each arrangement is called a representation of n (*Quarterly Journal*, Vol. XXXVI., p. 305). Thus the number of representations of a number as a sum of r uneven squares is equal to $2^r \times$ the number of compositions.

Taking the case of n uneven, the formula shows that, if m is of the form $4k+1$, the number of representations of m is $\frac{4}{5}\{17E_4(m)+8\chi_4(m)\}$, and that, if m is of the form $4k+3$, the number of representations is $-12E_4(m)$.

The second portion of this theorem (which relates to the case when m is of the form $4k+3$) is due to Eisenstein, who stated that he had found (by arithmetical methods) that "the number of representations of a number of the form $4n+3$ by 10 squares is equal to twelve times the difference between the sum of the fourth powers of those factors which have the form $4n+3$ and the sum of the fourth powers of those which have the form $4n+1$; but," he adds, "for numbers of the form $4n+1$ no similar theorem exists."*

The formula $\frac{4}{5}\{17E_4(m)+8\chi_4(m)\}$ shows that, for numbers of the form $4k+1$, the number of representations depends also upon the sum of the fourth powers of the complex divisors of m which have m as norm. For some values of m of the form $4k+1$, the number of representations depends upon real divisors only, viz., when m is divisible by a factor of the form $4k+3$ which is not raised to an even power as, *e.g.*, for

$$m = 21, 33, 57, 69, 77, 93, \ldots.$$

The number of representations of an even number will depend only upon real divisors of n whenever the largest uneven divisor of n is of the form $4k+3$, or, more generally, whenever n is divisible by any factor of the form $4k+3$ which is not raised to an even power.

It seems to me a very interesting result that the number of representations of n by 10 squares should depend upon the complex divisors of n, which, regarded as arithmetical quantities, are nearly as fundamental and as easy to determine as the real divisors.

6. It was not till after the completion of the series of papers (and after this notice of them was written) that, on looking through Liouville's writings, I found that he had given in 1866† a formula for the number of representations of any number n as a sum of 10 squares. His result is that, if $n = 2^{\alpha}m$, m being uneven, the number of representations is

$$\tfrac{4}{5}\{16^{\alpha+1}+(-1)^{\frac{1}{2}(m-1)}\}\lambda+\tfrac{8}{5}n^2\mu-\tfrac{64}{5}\nu,$$

where λ is the excess (taken positively) of the sum of the fourth powers of the divisors of n of the form $4k+1$ over the sum of the fourth powers

* *Crelle's Journal*, Vol. xxxv., p. 135.

† *Liouville's Journal*, Ser. 2, Vol. xi., pp. 1–8.

of the divisors of n of the form $4k+3$; μ is the number of solutions of the equation $n = s^2+s'^2$, and ν is the sum of the products $s^2s'^2$ for all the solutions.

It is easy to identify the first two terms with $\frac{4}{5}\{E_4(n)+16E'_4(n)\}$ and the last two with $\frac{32}{5}\chi_4(n)$. Liouville writes these last two terms also in the form $\frac{16}{5}\Sigma(s^4-3s^2s'^2)$, but he does not appear to have noticed the connection of this quantity with the complex divisors of n which have n as norm.

I found also that Liouville had given, in 1864, an expression for the number of representations of an even number as a sum of 12 squares. (See § 8.)

Representations by 12 Squares.

7. From the formula (vi.) of § 2 we see that the number of representations of an even number $2n$ as a sum of 12 squares is equal to $-8\xi_5(2n)$, that is, to $8\Sigma(-1)^{d+d'-1}d^5$, where d is any divisor of $2n$ and $dd' = 2n$. We thus have the remarkable result that the number of representations of an even number as a sum of 12 squares can be expressed by means of real divisors only.

In the case of an uneven number m, the number of representations of m as a sum of 12 squares is equal to $8\{\Delta_5(m)+2\Omega(m)\}$. The function $\Omega(m)$ does not vanish for any form of m, so that, unless $\Omega(m)$ for some form of m is dependent upon real divisors of m, it would seem that the number of representations of an uneven number m as a sum of 12 squares cannot for any form of m be expressed by real divisors only.

8. Liouville's formula, referred to in § 6, for the number of representations of an even number $2^a m$ as a sum of 12 squares is

$$\tfrac{24}{31}(21+2^{5a+1}5)\,\Delta_5(m),^*$$

which can be readily identified with $-8\xi_5(2n)$ by means of the general formula that, if $n = 2^s m$,

$$\xi_t(n) = -\frac{2^{ts+t}-2^{ts+1}+2^{t+1}-1}{2^t-1}\,\Delta_t(n).\dagger$$

* *Liouville's Journal*, Ser. 2, Vol. IX., p. 296. Liouville uses the notation $\zeta_5(m)$ for what I have denoted by $\Delta_5'(m)$.

† *Messenger of Mathematics*, Vol. XVIII., p. 47. In the statement of arithmetical formulæ relating to an even number n, $= 2^s m$, I have always endeavoured to avoid the use of the exponent s, expressing the result by means of divisors of n.

Representations by 14, 16, 18 *Squares.*

9. The formulæ (vii.), (viii.), (ix.) of § 2 give the numbers of representations of any number n as a sum of 14, 16, and 18 squares. I do not find that the functions $W(n)$, $\Theta(n)$, $G(n)$ vanish for any form of n, so that (unless for some form of n any of these functions depend upon real divisors only) it would seem that for no form of n can the number of representations of n by 14, 16, or 18 squares be expressed by means of real divisors only.

Compositions by Uneven Squares.

10. When the squares are all uneven we may conveniently use compositions instead of representations. Denoting by $C_r(n)$ the number of compositions of n as a sum of r uneven squares, the formulæ for $C_r(n)$, for $r = 2, 4, \ldots, 18$, are

$$C_2(8n+2) = E_0(4n+1),$$

$$C_4(8n+4) = \Delta(2n+1),$$

$$C_6(8n+6) = -\tfrac{1}{8}E_2(4n+3),$$

$$C_8(8n) = \Delta_3'(n),$$

$$C_{10}(8n+2) = \tfrac{1}{640}\{E_4(4n+1)-\chi_4(4n+1)\},$$

$$C_{12}(8n+4) = \tfrac{1}{256}\{\Delta_5(2n+1)-\Omega(2n+1)\},$$

$$C_{14}(8n+6) = \tfrac{1}{124928}\{-E_6(4n+3)+91W(4n+3)\},$$

$$C_{16}(8n) = \tfrac{1}{136}\{\Delta_7'(n)-\Theta(n)\},$$

$$C_{18}(8n+2) = \frac{1}{2^{15}.18005}\{13E_8(4n+1)+3047\chi_8(4n+1)-3060G(4n+1)\}.$$

In all these formulæ the argument is of the most general form possible, as a sum of r uneven squares must be of the form $8k+r$.

The function $\Delta_r'(n)$ denotes the sum of the r-th powers of those divisors of n whose conjugates are uneven.

The formulæ in the case of 2 and 4 squares are due to Jacobi, and those for 6 and 8 squares were given by H. J. S. Smith. The others I believe to be new.

We may pass from compositions to representations by multiplying by 2^r.

Classification of the Representations by 2, 4, 6, *and* 8 *Squares.*

11. I have obtained expressions for the numbers of representations of any number n by r uneven and $t-r$ even squares for $r = 0, 1, 2, 3, \ldots, t$

and $t = 2, 4, 6, 8, 10, 12$, and also for some values of r, when $t = 14, 16, 18$.

These results would occupy too much space if quoted in their entirety,* and so I confine myself here to the cases $t = 2, 4, 6, 8$.

Denoting by $R_{r,s}(n)$ the number of representations of n as a sum of r uneven and s even squares, the formulæ are

$$\begin{aligned}
R_{0,2}(4n) &= 4E_0(n),\\
R_{1,1}(4n+1) &= 4E_0(4n+1),\\
R_{2,0}(8n+2) &= 4E_0(4n+1);
\end{aligned}$$

$$\begin{aligned}
R_{0,4}(4n) &= (-1)^{n-1}8\xi_1(n),\\
R_{1,3}(4n+1) &= 8\Delta(4n+1),\\
R_{2,2}(4n+2) &= 24\Delta(4n+1),\\
R_{3,1}(4n+3) &= 8\Delta(4n+3),\\
R_{4,0}(8n+4) &= 16\Delta(2n+1);
\end{aligned}$$

$$\begin{aligned}
R_{0,6}(4n) &= 4\{4E_2'(n)-E_2(n)\},\\
R_{1,5}(4n+1) &= 6\{E_2(4n+1)+\chi_2(4n+1)\},\\
R_{2,4}(4n+2) &= (-1)^n 60E_2(2n+1),\\
R_{3,3}(4n+3) &= -20E_2(4n+3),\\
R_{4,2}(4n) &= 240E_2'(n),\\
R_{5,1}(4n+1) &= 6\{E_2(4n+1)-\chi_2(4n+1)\},\\
R_{6,0}(8n+6) &= -8E_2(4n+3);
\end{aligned}$$

$$\begin{aligned}
R_{0,8}(4n) &= (-1)^{n-1}16\zeta_3(n),\\
R_{1,7}(4n+1) &= 2\{\Delta_3(2n+1)+7P(4n+1)\},\\
R_{2,6}(4n+2) &= 56\{\Delta_3(2n+1)+P(4n+1)\},\\
R_{3,5}(4n+3) &= 14\{\Delta_3(4n+3)-P(4n+3)\},\\
R_{4,4}(4n) &= 1120\Delta_3'(n),\\
R_{5,3}(4n+1) &= 14\{\Delta_3(4n+1)-P(4n+1)\},\\
R_{6,2}(4n+2) &= 56\{\Delta_3(2n+1)-P(2n+1)\},\\
R_{7,1}(4n+3) &= 2\{\Delta_3(4n+3)+7P(4n+3)\},\\
R_{8,0}(8n) &= 256\Delta_3'(n).
\end{aligned}$$

* The complete lists up to $t = 16$ are given in the *Quarterly Journal*, Vol. XXXVIII., pp. 206–201.

In all the formulæ the argument on the left-hand side is the most general linear form that is capable of being represented by the quadratic form indicated by the suffixes, *e.g.*, if a number can be represented as the sum of 2 uneven and 6 even squares it must be of the form $4n+2$.

The expression of these results has required a new function $\chi_2(m)$ in the case of 6 squares and a new function $P(m)$ in the case of 8 squares, The function $\chi_2(m)$ is included among those defined in § 3, being the sum of the squares of the primary numbers having m for norm. The function $P(m)$ may be defined by means of the representations of m as a sum of 4 squares, or of $4m$ as a sum of 4 uneven squares.

The results in the case of 2 and 4 squares may be regarded as known: the other classified results are, I believe, new.

It is evident that $R_{0,s}(4n)$, the number of representations of $4n$ as a sum of s even squares, is necessarily equal to $R^{(s)}(n)$, the total number of representations of n, and that $R_{r,0}(n) = 2^r C_r(n)$.

Classification of the Representations by 10, 12, 14, 16, 18 *Squares.*

12. In the case of 10 squares, the functions $E_4(n)$, $E_4'(n)$, $\chi_4(n)$ suffice to express the values of

$$R_{0,10}(4n),\quad R_{2,8}(4n+2),\quad R_{4,6}(4n),\quad R_{5,5}(4n+1),\quad R_{6,4}(4n+2),$$
$$R_{8,2}(4n),\quad R_{10,0}(8n+2),$$

but a new function $Q(m)$ is required for

$$R_{1,9}(4n+1),\quad R_{3,7}(4n+3),\quad R_{7,3}(4n+3),\quad R_{9,1}(4n+1).$$

The function $Q(m)$ may be defined by means of the representations of m as a sum of 4 squares or of $4m$ as a sum of 4 uneven squares.

In the case of 12 squares the values of $R_{r,12-r}(n)$ for even values of r can be expressed by means of $\xi_5(n)$, $\Delta_5(m)$, $\Delta_5'(n)$, and $\Omega(m)$, but for the expression of $R_{r,12-r}(n)$ when r is uneven, a new function $\gamma(m)$ is required.

In the cases of 14, 16, and 18 squares, I have given the classified results only so far as they can be expressed without the introduction of additional functions, *i.e.*, the values given are those of $R_{2r,14-2r}(n)$ and $R_{7,7}(n)$, $R_{4r,16-4r}(n)$, $R_{2r,18-2r}(n)$ and $R_{9,9}(n)$ for all integral values of r.

General Remarks upon the Investigation.

13. The most interesting of the results obtained relate to 10 and 12 squares. So long ago as 1884 I had obtained the formulæ for the numbers

of representations of any even and uneven number as a sum of 12 squares, but I deferred the publication of the results in hopes of proving the relation $\Omega(m_1 m_2) = \Omega(m_1)\,\Omega(m_2)$, m_1 and m_2 being prime to one another, which I had observed to be satisfied by the values of $\Omega(m)$ that I had calculated. I resumed the question in 1889, and then obtained the classified results for 2, 4, 6, and 8 squares, using the functions $\chi_2(m)$ and $P(m)$, but I was still unsuccessful in proving the relation $\Omega(m_1 m_2) = \Omega(m_1)\,\Omega(m_2)$. The whole subject was then laid aside until the latter part of 1904, when I treated it more systematically, the results of the investigations which I then began being contained in the papers whose titles are given in the note to § 1. It was in 1905 that I worked out the formulæ for 10 squares, and was surprised to find that the number of representations depended upon the sum of the fourth powers of complex divisors. At that time I was not aware that the case of 10 squares had been considered by Eisenstein, and that he had found that the number of representations of a number of the form $4k+3$ could be expressed by real divisors only. The extension of the investigation to 14, 16, and 18 squares produced interesting results, but to some extent they were disappointing, as nothing so striking or unexpected was found as in the case of 10 or 12 squares.

14. The principal difficulty in the investigations was the selection of the new functions to be introduced. It is obvious that the number of representations in any given case could be expressed by means of one or more new functions defined as coefficients and introduced for the purpose. It also appeared on investigation that the number of new functions which it was absolutely necessary to introduce was smaller than might have seemed likely. But my object was not to find the fewest functions which would suffice to express the numbers of representations, but to select suitable functions which had an arithmetical significance. The functions from which the choice had to be made were given in the first instance as coefficients, and this kind of definition throws no light whatever upon the arithmetical properties of the function (except that it might indicate classes of numbers for which the function vanished). The only method available to me for determining the arithmetical nature of a function was to calculate its numerical values and so find what relations it seemed to satisfy. For the calculation of the functions it was necessary to investigate suitable recurring formulæ; such formulæ affording the only convenient method of obtaining the numerical values of a function defined as a coefficient.

15. In all cases I have selected functions which were found (by trial) to satisfy the relation $\phi(m_1 m_2) = \phi(m_1)\,\phi(m_2)$, m_1 and m_2 being uneven numbers prime to each other, or a similar relation of slightly different form.

This relation is satisfied by

$$P(m), \quad \Omega(m), \quad \gamma(m), \quad \Theta(m),$$

and, except when m_1 and m_2 are both of the form $4k+3$, also by $Q(m)$, $W(m)$, and $G(m)$.*

The property $\phi(m_1 m_2) = \phi(m_1)\,\phi(m_2)$ would seem to indicate a product structure of some kind for the function $\phi(m)$. In the paper numbered (ii.) in the note to § 1, an imperfect proof† of this property is given in the case of the functions $P(m)$, $Q(m)$, $\Omega(m)$; and in the paper (4) expressions are obtained for these three functions as sums of the second, third, and fourth powers of certain complex numbers derived from the representations of m as a sum of 4 squares.

When I first found that $\Omega(m)$ satisfied the property

$$\Omega(m_1 m_2) = \Omega(m_1)\,\Omega(m_2),$$

it seemed to me a very special property of the function, but, as has just been mentioned, I have since found that it is shared by many other functions which, like $\Omega(m)$, have no apparent factorial structure.

16. In the series of papers to which this notice relates I have endeavoured to elicit the principal information connected with the numbers of representations of a number as a sum of even and uneven squares (up to 18) that can be derived from the formulæ of elliptic functions. The only complete and effective method of research in such investigations is afforded by the processes of the theory of numbers, and any method dependent upon elliptic functions or other algebraical processes is necessarily partial and inadequate. Still in spite of this incompleteness it seemed to me to be useful to investigate with some completeness the results afforded by elliptic functions, thereby at the same time

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* If m_1 and m_2 are both of the form $4k+3$,

$$Q(m_1 m_2) = -3\,Q(m_1)\,Q(m_2), \quad W(m_1 m_2) = -15\,W(m_1)\,W(m_2), \quad G(m_1 m_2) = -\tfrac{39}{25}G(m_1)\,G(m_2).$$

† The proof is imperfect because it depends upon an assumption which I was unable to prove, although I think there can be no doubt of its truth. The "proof" at all events affords a reason for the existence of the property.

ascertaining the limitations of this method of research. It seems fairly safe to conclude that when algebraical methods do yield results in the theory of numbers, they are so obtained in the simplest manner: and it is quite likely that the reduction to these forms from the state in which they are left by the theory of numbers might not be easy. It can also happen that results which are expressed in their generality by delicate and complicated rules may contain no indication of the cases in which they admit of simplification.