variables e and e'', and canonical parameters e'; then, by the so-called exponential theorem,

$$\Delta_{e''} = \Delta_{e'}\Delta_{e},$$

and hence $E'' = \Delta_{e'}\Delta_{e} - 1 - \frac{1}{2}(\Delta_{e'}\Delta_{e} - 1)^2 + \frac{1}{3}(\Delta_{e'}\Delta_{e} - 1)^3 - \dots$.

Here everything on the right side is directly calculable from the constants of structure. When the group has no special infinitesimal transformations, we know that among the r^2 linear forms in $e_1'', \dots, e_r''$ which constitute the elements of the matrix E'', there are r linearly independent; hence by solution of only linear equations we obtain the finite equations of the group. When there are $r-m$ special infinitesimal transformations we do not in this way obtain more than m independent equations; but the solution can be completed by quadratures, as explained in the note referred to.

The formula $E = \Delta - 1 - \frac{1}{2}(\Delta - 1)^2 + \frac{1}{3}(\Delta - 1)^3 - \dots$

may be proved in another way, without differentiation, by using (1) the general formula for the sum of such a series of matrices, (2) the fact that, if $\theta_1, \theta_2, \dots$ be the roots of the determinantal equation

$$|E - \theta| = 0,$$

the roots of $|\Delta - \phi| = 0$

are given by $\phi = e^{\theta}$;

in fact to every invariant factor of the first equation there is one of the latter of the same exponent. The formula is only an algebraical formula of inversion involving the roots of two related algebraical equations.

On the Integration of Linear Differential Equations. *By* H. F. Baker.

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PART I.

Let u_1, u_2, ... each denote square matrices of the same number of rows and columns whose elements are analytic functions of the complex variable t; let barriers be made joining the singularities of these functions to $t = \infty$, so that within the star region bounded thereby each function is developable about every point in an ordinary power series; let t_0 be an arbitrary fixed point in the region, and Qu_i denote the matrix of which any element is obtained by integrating the corresponding element of u_i from t_0 to t along a path lying within the region; further, let a_1, a_2, ... be such constants that the series

$$1 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \dots$$

converges for all finite values of t, however great, and let the symbols $Qu_1 Qu_2$, $Qu_1 Qu_2 Qu_3$, ... denote $Q(u_1 Qu_2)$, $Q\left[u_1 Q(u_2 Qu_3)\right]$, ..., so that each integration denoted by Q affects all that follow it in any symbol.

Then form the series of matrices

$$\nabla = 1 + a_1 Qu_1 + a_2 Qu_1 Qu_2 + a_3 Qu_1 Qu_2 Qu_3 + \dots,$$

defining a single matrix of which any element is the infinite series of corresponding elements, one from each term of the series ∇. It can be shown, as in *Proc. Lond. Math. Soc.*, Vol. XXXIV., 1902, pp. 354, 359, that each of these infinite series converges over the whole of the infinite star region described. The matrix ∇ thus exists, and is finite and developable over the whole of this region.

Such a matrix as ∇ may be called a *matrizant;* we shall restrict the term to the case when each of the matrices $u_1, u_2, \ldots$ is the same matrix u, and each of the constants $a_1, a_2, \ldots$ is unity, and shall denote the matrizant in that case by $\Omega(u)$, or sometimes, in more detail, by $\Omega^{t, t_0}(u)$. Its most fundamental property is that

$$\frac{d}{dt}\Omega(u) = u\Omega(u),$$

so that each of its columns furnishes a set of solutions of the system of linear differential equations

$$\frac{dx_i}{dt} = u_{i1}x_1 + \ldots + u_{in}x_n \quad (i = 1, \ldots, n),$$

and, since for $t = t_0$ we have

$$\Omega(u) = 1,$$

these sets of solutions are independent.

There are various other properties, partly of a formal nature, which must be clearly stated.

1. It is necessary to repeat the process for summing a series of matrices in order to show clearly how the infinite series of logarithms which may arise in the terms of a matrizant may disappear in the sum. If $f(t)$ denote an integral function

$$f(t) = 1 + a_1\frac{t}{1!} + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \ldots,$$

and M be any matrix satisfying the equation

$$M^\mu = \lambda + \lambda_1 M + \ldots + \lambda_{\mu-1}M^{\mu-1},$$

and no equation of lower order, and the roots of the algebraic equation in θ,

$$\theta^\mu = \lambda + \lambda_1\theta + \ldots + \lambda_{\mu-1}\theta^{\mu-1}$$

—say, $\theta_1, \theta_2, \ldots, \theta_\mu$—be all different, the series

$$\nabla = 1 + a_1\frac{M}{1!} + a_2\frac{M^2}{2!} + a_3\frac{M^3}{3!} + \ldots,$$

wherein each term is to be reduced to a polynomial in M of order

$\mu-1$, will when rearranged in powers of M have for sum a form

$$\nabla = C_0 + C_1 M + \ldots + C_{\mu-1} M^{\mu-1},$$

in which the single quantities $C_0, C_1, \ldots, C_{\mu-1}$ have the values obtained by expressing the identity

$$f(\theta) = C_0 + C_1\theta + \ldots + C_{\mu-1}\theta^{\mu-1}$$

for all the roots $\theta_1, \ldots, \theta_\mu$ of a supposed irreducible equation

$$\theta^\mu = \lambda + \lambda_1\theta + \ldots + \lambda_{\mu-1}\theta^{\mu-1};$$

this gives at once (*Proc. Lond. Math. Soc.*, Vol. XXXIV., pp. 114, 359)

$$\nabla = \frac{D_0}{D} + \frac{D_1}{D} M + \ldots + \frac{D_{\mu-1}}{D} M^{\mu-1},$$

where

$$D = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \theta_1 & \theta_2 & \ldots & \theta_\mu \\ \ldots & \ldots & & \ldots \\ \theta_1^{\mu-1} & \theta_2^{\mu-1} & \ldots & \theta_\mu^{\mu-1} \end{vmatrix}$$

is the product of the differences of the roots $\theta_1, \ldots, \theta_\mu$, and D_{i-1} is the determinant obtained from D by replacing the elements of the i-th row of D by $f(\theta_1), \ldots, f(\theta_\mu)$.

This formula yields at once the corresponding formula when the roots $\theta_1, \ldots, \theta_\mu$ are not all different; for instance, by subtracting the first two columns of every one of the determinants $D, D_0, \ldots, D_{\mu-1}$, dividing by $\theta_2-\theta_1$, and proceeding to the limit, we obtain the case $\theta_2 = \theta_1$. The quantities $C_i = D_i/D$, which, in the case of unequal roots, are linear functions of $f(\theta_1), \ldots, f(\theta_\mu)$, become in general linear functions of these quantities and their differential coefficients in regard to $\theta_1, \ldots, \theta_\mu$; the coefficients in these linear functions are the same whatever be the form of the function f, and depend only on the matrix M; they may therefore be calculated from a suitably chosen simple form of $f(t)$, in particular $(1-t)^{-1}$. Thus we prove the formula of Frobenius and Stickelberger (references in Bromwich, *Proc. Camb. Phil. Soc.*, Vol. XI., 1901, p. 79), that, if the inverse matrix $(\theta-M)^{-1}$ be arranged in the form

$$(\theta-M)^{-1} = \sum_{i=1}^{\mu}\left[\frac{M_i}{\theta-\theta_i} + \frac{N_i}{(\theta-\theta_i)^2} + \ldots + \frac{K_i}{(\theta-\theta_i)^{\epsilon_i}}\right],$$

where $M_i, N_i, \ldots, K_i$ are matrices, then

$$\nabla = \sum_{i=1}^{\mu}\left[f(\theta_i) M_i + f'(\theta_i) N_i + \ldots + \frac{f^{(\epsilon_i-1)}(\theta_i)}{(\epsilon_i-1)!} K_i\right]$$

One of the simplest cases of $\Omega(u)$ is when u has the form $t^{-1}M$, where M is a matrix of constants; then

$$Qu = M\log t/t_0,\quad Qu\,Qu = M^2Q(t^{-1}\log t/t_0) = \frac{M^2}{2}(\log t/t_0)^2,\ \ldots,$$

or, if $\lambda = \log t/t_0$,

$$\Omega(u) = 1+\lambda M+\frac{\lambda^2}{2!}M^2+\frac{\lambda^3}{3!}M^3+\ldots;$$

thus, if the matrices of constants M_i, N_i, ... be determined from the matrix M, we have, putting $f(\theta) = e^{\lambda\theta}$,

$$\Omega(t^{-1}M) = \sum_{i=1}^{\mu}\left\{M_i\left(\frac{t}{t_0}\right)^{\theta_i}+N_i\left(\frac{t}{t_0}\right)^{\theta_i}\log\frac{t}{t_0}+\ldots\right.$$
$$\left.\ldots+K_i\left(\frac{t}{t_0}\right)^{\theta_i}\frac{1}{(\epsilon_i-1)!}\left(\log\frac{t}{t_0}\right)^{\epsilon_i-1}\right\},$$

the exponent ϵ_i-1 of the highest power of the logarithm corresponding to the root θ_i being one less than the multiplicity of this root in the equation satisfied by M, namely, than the exponent of the first invariant factor of the matrix $M-\theta$ corresponding to this root.

2. A second property to be remarked is that, if M be a matrix of constants, we have

$$M\Omega(u)\,M^{-1} = \Omega(MuM^{-1}).$$

3. If $s = \phi(t)$ be uniquely reversible, in a form $t = \phi_1(s)$, or we confine t to a region about t_0 for which this is so, and if $t_0 = \phi(h_0)$, and

$$\Omega^{t,t_0}[u(t)] = F(t),$$

then

$$F(s) = \Omega^{t,h_0}\left\{u[\phi(t)]\frac{d\phi(t)}{dt}\right\}.$$

If, for instance,

$$s = \phi(t) = \frac{at+b}{ct+d},\quad t_0 = h_0 = \frac{at_0+b}{ct_0+d},\quad \Omega(u) = F(t),$$

then

$$F\left(\frac{at+b}{ct+d}\right) = \Omega\left[\frac{ad-bc}{(ct+d)^2}u\left(\frac{at+b}{ct+d}\right)\right];$$

it may happen that a matrix of constants, μ, can be found such that

$$\mu^{-1}u(t)\,\mu = \frac{ad-bc}{(ct+d)^2}u\left(\frac{at+b}{ct+d}\right);$$

then

$$F\left(\frac{at+b}{ct+d}\right) = \mu^{-1}\Omega(u)\,\mu.$$

4. We clearly have

$$\Omega^{t, t_0}(u) = \Omega^{t, t_1}(u)\,\Omega^{t_1, t_0}(u)\,; \quad [\Omega^{t, t_0}(u)]^{-1} = \Omega^{t_0, t}(u).$$

We have, however,

$$[\Omega^{t, t_0}(u)]^{-1} = \bar{\Omega}^{t, t_0}(-\bar{u}),$$

where $\bar{u}$ denotes the matrix obtained from u by interchange of rows and columns, and the same transposition is to be carried out after $\Omega(-\bar{u})$ has been calculated.

For, if $U = \Omega(u)$,

$$\frac{dU^{-1}}{dt} = -U^{-1}\frac{dU}{dt}U^{-1} = -U^{-1}u,$$

and, if $V = \Omega(-\bar{u})$, $W = \bar{V}$,

$$\frac{dV}{dt} = -\bar{u}V, \quad \frac{dW}{dt} = -Wu,$$

and hence

$$\frac{d}{dt}(WU) = \frac{dW}{dt}U + W\frac{dU}{dt} = -WuU + WuU = 0,$$

while, for $t = t_0$, we have $W = 1$, $U = 1$;

thus, as stated, $U^{-1} = W$,

or $[\Omega(u)]^{-1} = \bar{\Omega}(-\bar{u}) = 1 - Qu + Q(Qu.u) - Q[Q(Qu.u)u] + \ldots,$

where each term is formed from the preceding by multiplying by u on the right and then integrating the product, or, if v_n denote the general term,

$$v_{n+1} = -Q(v_n u).$$

5. If u, h be arbitrary matrices of the same order, the latter of non-vanishing determinant, and a star region be constructed within which the elements of both the matrices

$$u, \quad huh^{-1} + \frac{dh}{dt}h^{-1}$$

are everywhere developable, we have

$$\Omega\left(huh^{-1} + \frac{dh}{dt}h^{-1}\right) = h\Omega(u)\,h_0^{-1},$$

where h_0 is the value of h at $t = t_0$. For, if

$$v = huh^{-1} + \frac{dh}{dt}h^{-1},$$

$$\frac{d}{dt}[\Omega^{-1}(u)\,h^{-1}\Omega(v)]$$

$$= -\Omega^{-1}(u)\,uh^{-1}\Omega(v) - \Omega^{-1}(u)\,h^{-1}\frac{dh}{dt}h^{-1}\Omega(v) + \Omega^{-1}(u)\,h^{-1}v\Omega(v)$$

$$= -\Omega^{-1}(u)\,h^{-1}\left\{huh^{-1} + \frac{dh}{dt}h^{-1} - v\right\}\Omega(v) = 0,$$

and for $t = t_0$ we have

$$\Omega^{-1}(u)\,h^{-1}\Omega(v) = h_0^{-1}.$$

Taking $$h = \Omega(w), \quad huh^{-1} = \sigma,$$

we obtain $$\Omega(w+\sigma) = \Omega(w)\,\Omega[\Omega^{-1}(w)\,\sigma\Omega(w)].$$

6. Two particular cases of § 5 seem worth remark. First, if $w_1, w_2, \ldots, w_\mu$ be each single functions of t, and $A_1, \ldots, A_\mu$ be matrices of constants of the same order of which any two are commutable, so that $A_\rho A_\sigma = A_\sigma A_\rho$, and the star region be suitably constructed, we have

$$\Omega(w_1A_1 + \ldots + w_\mu A_\mu) = \Omega(w_1A_1)\,\Omega(w_2A_2)\ldots\Omega(w_\mu A_\mu).$$

For, if $$\sigma = w_2A_2 + \ldots + w_\mu A_\mu,$$

we have $$\Omega^{-1}(w_1A_1)\,\sigma\Omega(w_1A_1) = \sigma.$$

Second, if $A_1 = 1$, we have

$$\Omega(w_1+u) = e^{Qw_1}\,\Omega(u),$$

u being an arbitrary matrix, a result obtainable by multiplying the series

$$\Omega(u) = 1 + Qu + QuQu + \ldots,$$

$$e^{Qw_1} = 1 + Qw_1 + Qw_1Qw_1 + \ldots .$$

7. If ξ denote a matrix of which any column consists of the elements of one of n linearly independent sets of solutions of the system of linear differential equations

$$\frac{dx_i}{dt} = u_{i1}x_1 + \ldots + u_{in}x_n,$$

so that $$\frac{d\xi}{dt} = u\xi,$$

then, by the *cogredient* substitution, wherein h is an arbitrary matrix for which $|h| \neq 0$,

$$\eta = h\xi,$$

we obtain the system

$$\frac{d\eta}{dt} = v\eta, \quad v = huh^{-1} + \frac{dh}{dt}h^{-1},$$

and, by the *contragredient* substitution, wherein again $|k| \neq 0$,

$$\zeta = k(\bar{\xi})^{-1},$$

we obtain the linear system

$$\frac{d\xi}{dt} = w\zeta, \quad w = -k\bar{u}k^{-1} + \frac{dk}{dt}k^{-1}.$$

If the matrices u, v, w be assigned beforehand, these equations give for the forms of the matrices h, k, by §§ 4, 5, the equations

$$h = \Omega(v)\,h_0\Omega^{-1}(u), \quad k = \Omega(w)\,k_0\bar{\Omega}(\bar{u}),$$

where the constant matrices h_0, k_0 are arbitrary.

8. Thus any two linear systems can be transformed into one another either cogrediently or contragrediently. Now a single linear equation

$$\lambda_n \frac{d^n y}{dt^n} + \lambda_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + \lambda_0 y = 0$$

can be reduced to a linear system in various ways; taking here

$$x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad \ldots, \quad x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

for the elements of a column of a matrix ξ, whose n columns are obtained by putting y equal to the constituents of a fundamental system of integrals for the single linear equation, we have what we may for the present call the special linear system

$$\frac{d\xi}{dt} = u\xi,$$

where in the first $(n-1)$ rows of u every element is zero, except that one immediately to the right of the diagonal element, which is unity, while the n-th row consists of the elements

$$-\frac{\lambda_0}{\lambda_n}, \quad -\frac{\lambda_1}{\lambda_n}, \quad \ldots, \quad -\frac{\lambda_{n-1}}{\lambda_n}.$$

Any linear system can then, by § 7, be reduced to a special linear system, and so to a single linear equation, arbitrarily given, either cogrediently or contragrediently, the necessary explicit form of the matrix h or k used in the transformation requiring a knowledge of the solutions of the systems to be connected by the transformation.

If, however, we be given a linear system, which may be special, and require to transform this to some single linear equation, say contragrediently, by suitable choice of a matrix k, the equation

$$wk = -k\bar{u} + \frac{dk}{dt},$$

wherein w has the form just explained, arising for a special linear system, shows that the elements of k satisfy equations

$$k_{i+1,j} = \frac{dk_{i,j}}{dt} - (u_{j,1}k_{i,1} + \ldots + u_{jn}k_{in})$$
$$(j = 1 \ldots n;\ i = 1 \ldots n-1),$$

namely, that the $(i+1)$-th row of k is determined from the i-th by the rule

$$k^{(i+1)} = \frac{dk^{(i)}}{dt} - uk^{(i)}.$$

Conversely, if we take the first row of k arbitrarily and determine the subsequent rows by this rule from the first row, it is easily seen that w has the form for a special linear system, namely, the elements of any column of the matrix ζ satisfy equations

$$\frac{dz_1}{dt} = z_2,\ \frac{dz_2}{dt} = z_3,\ \ldots,\ \frac{dz_{n-1}}{dt} = z_n,\ \frac{dz_n}{dt} = w_{n1}z_1 + \ldots + w_{nn}z_n.$$

In particular, if the original system (u) is special, derived from

$$\lambda_n \frac{d^n x}{dt^n} + \lambda_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + \lambda_0 x = 0,$$

and we take $k_{11} = 0 = k_{12} = \ldots = k_{1n-1},\quad k_{1n} = -\dfrac{1}{\lambda_n}$,

it can be shown that the (w) system belongs to the so-called Lagrange adjoint equation

$$\lambda z - \frac{d}{dt}(\lambda_1 z) + \frac{d^2}{dt^2}(\lambda_2 z) + \ldots + (-1)^n \frac{d^n}{dt^n}(\lambda_n z) = 0.$$

Obviously the most general contragredient transformation

$$\zeta = k\,(\bar{\xi})^{-1}$$

may be obtained by first applying a particular contragredient transformation $\zeta_0 = k_0 (\bar{\xi})^{-1}$, and then a cogredient transformation $\zeta = kk_0^{-1}\zeta_0$. For instance, when we are transforming from a special linear system we may take the special contragredient transformation to be that to Lagrange's adjoint equation; or we may take $k_0 = 1$, though in that case the intermediate system will not be a special system. Also, as the equations of a contragredient transformation

$$wk + k\bar{u} = \frac{dk}{dt}, \quad k = \Omega(w)\, k_0 \overline{\Omega(u)},$$

are equivalent to

$$u\bar{k} + \bar{k}\bar{w} = \frac{d\bar{k}}{dt}, \quad \bar{k} = \Omega(u)\, \bar{k}_0 \,\Omega(w),$$

the inverse transformation, as applied to the new system to return to the original, is obtained by changing the rows of k into columns; in the case when we are transforming from a special system (u) to a special system (w), this gives the incidental result that, if the rows of a matrix k of non-vanishing determinant are determined from the first (arbitrary) row by the relation

$$k^{(i+1)} = \frac{dk^{(i)}}{dt} - uk^{(i)},$$

then the columns of this matrix are determined from the first column by the relation

$$k^{(j+1)} = \frac{dk^{(j)}}{dt} - wk^{(j)},$$

the special matrices u, w being connected by the equation

$$wk + k\bar{u} = \frac{dk}{dt}.$$

This explains the reciprocal relations of an equation, and its adjoint; for instance, for $n = 3$, the forms of the special matrices u, w and two forms of the matrix k are respectively

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\lambda}{\lambda_3} & -\frac{\lambda_1}{\lambda_3} & -\frac{\lambda_2}{\lambda_3} \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\mu}{\mu_3} & -\frac{\mu_1}{\mu_3} & -\frac{\mu_2}{\mu_3} \end{pmatrix},$$

$$k = \begin{pmatrix} 0 & 0 & & -\dfrac{1}{\lambda_3} \\ 0 & \dfrac{1}{\lambda_3} & & -\dfrac{\lambda_2+\lambda_3'}{\lambda_3^2} \\ -\dfrac{1}{\lambda_3} & \dfrac{\lambda_2-2\lambda_3'}{\lambda_3^2} & \dfrac{\lambda_1+\lambda_2'+\lambda_3''}{\lambda_3^2} & -\dfrac{(\lambda_2-2\lambda_3')(\lambda_2-\lambda_3')}{\lambda_3^3} \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & 0 & & -\dfrac{1}{\mu_3} \\ 0 & \dfrac{1}{\mu_3} & & \dfrac{\mu_2-2\mu_3'}{\mu_3^2} \\ -\dfrac{1}{\mu_3} & -\dfrac{\mu_2+\mu_3'}{\mu_3^2} & \dfrac{\mu_1-\mu_2'+\mu_3''}{\mu_3^2} & -\dfrac{(\mu_2-2\mu_3')(\mu_2-\mu_3')}{\mu_3^3} \end{pmatrix}.$$

Precisely similar remarks may be made as to cogredient transformations; in fact the relation

$$h = \Omega(v)\, h_0 \Omega^{-1}(u)$$

is the same as

$$h = \Omega(v)\, h_0 \bar{\Omega}(-\bar{u}),$$

and is a cogredient equation starting from equations in which u is replaced by $-\bar{u}$, that is, from the system obtained from the original by the particular contragredient transformation $\zeta_0 = (\bar{\xi})^{-1}$.

9. It has been remarked that there are various ways of reducing a single linear equation to a linear system. We explain now a way which is of great importance in the sequel of this paper.

If the single linear equation be

$$\frac{d^n y}{dt^n} = \frac{P_{n-1}}{\phi_n}\frac{d^{n-1}y}{dt^{n-1}} + \frac{P_{n-2}}{\phi_{n-1}\phi_n}\frac{d^{n-2}y}{dt^{n-2}} + \frac{P_{n-3}}{\phi_{n-2}\phi_{n-1}\phi_n}\frac{d^{n-3}y}{dt^{n-3}} + \dots + \frac{P}{\phi_1\phi_2\dots\phi_n}\, y,$$

and we put, with $y' = dy/dt$, &c.,

$$x_1 = y, \quad x_2 = \phi_1 y', \quad x_3 = \phi_1\phi_2 y'', \quad x_4 = \phi_1\phi_2\phi_3 y''', \quad \dots$$
$$\dots, \quad x_n = \phi_1\phi_2\dots\phi_{n-1} y^{(n-1)},$$

then
$$\frac{dx_1}{dt} = \frac{x_2}{\phi_1}, \quad \frac{dx_2}{dt} = \frac{\phi_1'}{\phi_1}x_2 + \frac{x_3}{\phi_2}, \quad \frac{dx_3}{dt} = \left(\frac{\phi_1'}{\phi_1}+\frac{\phi_2'}{\phi_2}\right)x_3 + \frac{x_4}{\phi_3}, \quad \dots$$
$$\dots, \quad \frac{dx_{n-1}}{dt} = \left(\sum_{r=1}^{n-2}\frac{\phi_r'}{\phi_r}\right)x_{n-1} + \frac{x_n}{\phi_{n-1}},$$

and $$\frac{dx_n}{dt} = \left(\sum_{r=1}^{n-1} \frac{\phi'_r}{\phi_r}\right) x_n + \frac{P}{\phi_n} x_1 + \frac{P_1}{\phi_n} x_2 + \frac{P_2}{\phi_n} x_3 + \ldots + \frac{P_{n-1}}{\phi_n} x_n ;$$

so that, if for brevity $$H_m = \sum_{r=1}^{m} \frac{\phi'_r}{\phi_r},$$

$$\frac{dx}{dt} = \begin{Bmatrix} 0 & \frac{1}{\phi_1} & 0 & 0 & \ldots & 0 & 0 \\ 0 & H_1 & \frac{1}{\phi_2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & H_2 & \frac{1}{\phi_3} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & H_{n-2} & \frac{1}{\phi_{n-1}} \\ \frac{P}{\phi_n} & \frac{P_1}{\phi_n} & \frac{P_2}{\phi_n} & \frac{P_3}{\phi_n} & \ldots & \frac{P_{n-2}}{\phi_n} & \frac{P_{n-1}}{\phi_n} + H_{n-1} \end{Bmatrix} x.$$

Various cases of this formula to be referred to are—

(a) When every one of the functions of t denoted by $P, P_1, \ldots, P_{n-1}$, $\phi_1, \ldots, \phi_n$ is an integral polynomial, and no one of $\phi_1, \ldots, \phi_n$ has a multiple factor, though any two or more of them may have common factors; then each of the rational functions

$$\frac{1}{\phi_s} = \sum_c \frac{1/\phi'_s(c)}{t-c_s}, \quad H_m = \sum_{r=1}^{m} \sum_c \frac{1}{t-c_r}, \quad \frac{P_i}{\phi_n} = (1, t) + \sum_c \frac{P_i(c_n)/\phi'_n(c_n)}{t-c_n}$$

occurring in the matrix has only poles of the first order for finite values of t, and the linear system has a form

$$\frac{dx}{dt} = \left[A + A_1 t + \ldots + A_\mu t^\mu + \sum_i C_i \frac{1}{t-c_i}\right] x,$$

wherein $A, A_1, \ldots, A_\mu, C$ are matrices of constants and the summation refers to all the poles c which arise. The value of μ is zero and $A = 0$ when ϕ_n is of greater dimension than any of $P, P_1, \ldots, P_{n-1}$, and otherwise μ is equal to the difference of the dimensions of ϕ_n and the highest dimension of any of these.

(b) When each of the functions $P, P_1, \ldots, P_{n-1}$ is an integral polynomial and $\phi_1 = \phi_2 = \ldots = \phi_n = \phi$ is an integral polynomial without repeated roots.

(c) When $P, \ldots, P_{n-1}$ are again polynomials, ϕ_n is a polynomial without repeated roots, and $\phi_1 = \phi_2 = \ldots = \phi_{n-1} = 1$. Then each of the functions $H_1, \ldots, H_{n-1}$ is zero. And further simplification arises when the dimensions of $P, P_1, \ldots, P_{n-1}$ are equal to or less than that of ϕ_n.

(*d*) When $P, P_1, \ldots, P_{n-1}$ are analytical functions for which $t-c$ is not a singularity, and each of $\phi_1, \ldots, \phi_n$ is either unity or $t-c$. In the ordinary case in which no one of $P, P_1, \ldots, P_{n-1}$ vanishes at $t=c$ and ϕ_n is $t-c$, the polynomial with matrix coefficients, $A+A_1t+\ldots+A_\mu t^\mu$ does not enter at all in the resulting linear system.

Of these (*a*), (*b*), (*c*) are intimately related, while (*d*) includes cases in which the coefficients in the linear differential equation are algebraical functions of t.

As a particular example of (*a*) we may take the equation

$$(t+1)\,t^3y'''-[(a_2+b_2)\,t+a_2]\,t^2y''-[(a_1+b_1)\,t+a_1]\,ty' -[(a+b)\,t+a]\,y=0,$$

which, with $\phi_3 = t(t+1), \quad \phi_2 = t, \quad \phi_1 = t,$

$$P = a(t+1)+bt, \quad P_1 = a_1(t+1)+b_1t, \quad P_2 = a_2(t+1)+b_2t,$$

leads to $$\frac{dx}{dt} = \left[\frac{1}{t}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ a & a_1 & a_2+2 \end{pmatrix} + \frac{1}{t+1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & b_1 & b_2 \end{pmatrix}\right] x.$$

As an example of (*b*) we may take

$$y'' = \frac{At+B}{t(t-1)}\,y' + \frac{Ct^2+Dt+E}{t^2(t-1)^2}\,y,$$

which, with

$$\phi_2 = \phi_1 = t(t-1), \quad P = Ct^2+Dt+E, \quad P_1 = At+B,$$

leads to

$$\frac{dx}{dt} = \left[\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 0 & -1 \\ -E & 1-B \end{pmatrix} + \frac{1}{t-1}\begin{pmatrix} 0 & 1 \\ C+D+E & A+B+1 \end{pmatrix}\right] x.$$

As an example of (*c*) we may take

$$y'' = y'\left[\lambda + \sum_{r=1}^{\sigma} \frac{\lambda_r}{t-c_r}\right] + y\left[\mu + \sum_{r=1}^{\sigma} \frac{\mu_r}{t-c_r}\right],$$

which, with $$\phi_2 = \prod_{r=1}^{\sigma} (t-c_r), \quad \phi_1 = 1,$$

$$P_1 = \phi_1 \left[\lambda + \sum_{r=1}^{\sigma} \frac{\lambda_r}{t-c_r}\right], \quad P = \phi_2 \left[\mu + \sum_{r=1}^{\sigma} \frac{\mu_r}{t-c_r}\right],$$

leads to $$\frac{dx}{dt}\left[\begin{pmatrix} 0 & 1 \\ \mu & \lambda \end{pmatrix} + \sum_{r=1}^{\sigma} \frac{1}{t-c_r}\begin{pmatrix} 0 & 0 \\ \mu_r & \lambda_r \end{pmatrix}\right] x.$$

As an example of (d) we may take the same equation, putting now

$$x_1 = y, \quad x_2 = (t-c_1)\, y';$$

then we find

$$\frac{dx}{dt} = \left[\begin{pmatrix} 0 & 0 \\ \sum_{r=1}^{\sigma} \mu_r & \lambda \end{pmatrix} + (t-c_1)\begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix} + \frac{1}{t-c_1}\begin{pmatrix} 0 & 1 \\ 0 & \lambda_1+1 \end{pmatrix} \right.$$
$$\left. + \sum_{r=2}^{\sigma} \frac{1}{t-c_r}\begin{pmatrix} 0 & 0 \\ \mu_r(c_r-c_1) & \lambda_r \end{pmatrix}\right] x,$$

which takes a simpler form in the case of an equation of common occurrence for which $\lambda = 0$, $\mu = 0$, and $\sum_{r=1}^{\sigma} \mu_r = 0$; in that case the equation is, in the usual phraseology, *regular* at infinity.

As another example of (d), of the greatest importance for our purpose, we may take

$$y^{(n)} = \frac{p_{n-1}+tQ_{n-1}}{t} y^{(n-1)} + \frac{p_{n-2}+tQ_{n-2}}{t^2} y^{(n-2)} + \dots + \frac{p+tQ}{t^n} y,$$

where $p, p_1, \dots, p_{n-1}$ are constants, and $Q, Q_1, \dots, Q_{n-1}$ are developable about $t = 0$ in positive integral powers; putting

$$\phi_n = \phi_{n-1} = \dots = \phi_1 = t, \quad P_i = p_i + tQ_i,$$

we obtain $$\frac{dx}{dt} = \left(\frac{A}{t} + V\right) x$$

where $$A = \begin{Bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p & p_1 & p_2 & p_3 & \dots & p_{n-1}+n-1 \end{Bmatrix}$$

is a matrix of constants and V is a matrix of which all the elements are zero, except those in the last row, which are

$$Q,\ Q_1,\ \ldots,\ Q_{n-1}.$$

It will be seen below that the algebraic equation $|A-\rho|=0$ is the so-called index equation at $t=0$.

10. In connexion with these examples the obvious fact seems worth remarking at once that, if a new variable s can be found such that each of the logarithms

$$\log(t-c_1),\quad \log(t-c_2),\quad \ldots,\quad \log(t-c_\sigma)$$

is a single-valued analytic function of s, for a certain range of the latter, then, for a suitable corresponding range of t, every integral of the linear system

$$\frac{dx}{dt}=\left[A+A_1t+\ldots+A_\mu t^\mu+\sum_{r=1}^{\sigma}(t-c_r)^{-1}C_r\right]x$$

is a single-valued function of s; for, if

$$\log(t-c_r)=\psi_r(s),$$

so that
$$t=c_r+\exp\psi_r(s)=\psi(s),\text{ say,}$$

the system is

$$\frac{dx}{ds}=\left[\psi'(s)\left[A+A_1\psi+\ldots+A_\mu\psi^\mu\right]+\sum_{r=1}^{\sigma}\psi_r'(s)\,C_r\right]x$$

$$=ux,\text{ say,}$$

and each of the terms

$$Qu=\int_{s_0}^{s}u\,ds,\quad QuQu=\int_{s_0}^{s}u\,ds\int_{s_0}^{s}u\,ds,\quad\ldots$$

in the matrizant solution

$$x=\Omega(u)\,x_0=(1+Qu+QuQu+\ldots)\,x_0$$

is a single-valued analytic function of s about the properly chosen value s_0.

A very particular case of this is the well known theorem that the independent and dependent variables of a hypergeometric differential equation are single-valued functions of a variable s, which is the quotient of two integrals of a certain hypergeometric equation for which $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, $\gamma=1$; in fact, if

$$\kappa=\int_0^{\frac{1}{2}\pi}d\theta\,(1-t\sin^2\theta)^{-\frac{1}{2}},\quad \kappa'=\int_0^{\frac{1}{2}\pi}d\theta\left[1-(1-t)\sin^2\theta\right]^{-\frac{1}{2}},$$

the expressions for $\log t$, $\log(1-t)$ unambiguously in terms of $q = e^{i\pi s} = e^{-\pi K'/K}$ are given in Jacobi's *Fundamenta Nova.* But the suggestion above, into the utility of which this is not the place to enter, applies to a linear system

$$\frac{dx}{dt} = wx,$$

in which w has poles of higher order, say

$$w = A + \ldots + A_\mu t^\mu + \Sigma\left[C_r(t-c_r)^{-1} + \ldots + C_{h_r}(t-c_r)^{-h_r}\right];$$

a very particular case is Bessel's equation

$$y'' = -\frac{1}{t} y' + \frac{n^2-t^2}{t^2} y,$$

which, by $x_1 = y$, $x_2 = ty'$, gives the system

$$\frac{dx}{dt} = \left[-t\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{t}\begin{pmatrix}0 & 1\\ n^2 & 0\end{pmatrix}\right] x,$$

solved by single-valued functions of $s = \log t$.

11. Some remarks should be made as to the direct evaluation of $\Omega(a\omega + \ldots + a_r\omega_r)$ as a method for the integration of a linear system

$$\frac{dx}{dt} = (a\omega + a_1\omega_1 + \ldots + a_r\omega_r)x,$$

in which $\omega_1, \ldots, \omega_r$ are functions of t, and $a, a_1, \ldots, a_\sigma$ are matrices of constants of the same order, no two of which are commutable (§ 6).

If
$$\phi_0 = \int_{t_0}^t \omega\, dt, \quad \phi_{01} = \int_{t_0}^t \omega\, dt \int_{t_0}^t \omega_1\, dt,$$

$$\phi_{011} = \int_{t_0}^t \omega\, dt \int_{t_0}^t \omega_1\, dt \int_{t_0}^t \omega_1\, dt, \ \ldots,$$

we have $\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$$\Omega(a\omega + \ldots + a_r\omega_r) = 1 + a\phi_0 + \ldots + a_r\phi_r + (a^2\phi_{00} + aa_1\phi_{01} + a_1 a\phi_{10} + \ldots) + \ldots$$
$$\ldots + (\Sigma a^{\lambda_0} a_1^{\lambda_1} \ldots \phi_{00\ldots 11\ldots}) + \ldots,$$

where the suffix of the ϕ in the general term is exactly similar to the exponents $\lambda_0, \lambda_1, \ldots$ of the matrix product $a^{\lambda_0} a_1^{\lambda_1} \ldots$. Of the products $a \ldots a_r$, a^2, aa_1, $a_1 a \ldots$, all are necessarily expressible linearly with numerical coefficients in terms of, say, N of them, where N is at most equal to the square of the order of any one of the matrices. Thus, when what we may call the multiplication table of the matrices $a, a_1, \ldots$ is known, namely, the law by which any product of powers of them can be expressed in terms of N of them, the linear system may be regarded as integrated in explicit terms, the problem of integration breaking up into the two problems of ascertaining this multiplication table and of finding the properties of the sums of the infinite series of functions of t of the form $\phi_0 \ldots \phi_{01} \ldots$ which arise as the coefficients of the N fundamental matrices.

In particular the last example but one of § 9 shows that for any single differential equation of the second order with rational coefficients, which is regular at its singular points including $t = \infty$, the solution can be expressed by sums of series of elementary functions of the form

$$\phi_0 = Q(t-c)^{-1}, \quad \phi_{01} = Q(t-c)^{-1}Q(t-c_1)^{-1},$$
$$\phi_{10} = Q(t-c_1)^{-1}Q(t-c)^{-1}, \ldots .$$

Consider, for instance, a linear system

$$\frac{dx}{dt} = (a\omega + a_1\omega_1)\,x$$

wherein the matrices of any, the same, order satisfy the equations

$$a^2 = 0, \quad aa_1 = a, \quad a_1 a = 0, \quad a_1^2 = a_1,$$

which we represent by

	a	a_1
a	0	a
a_1	0	a_1

;

then we have

$$\Omega(a\omega + a_1\omega_1) = 1 + a\,[\phi_0 + \phi_{01} + \phi_{011} + \phi_{0111} + \ldots]$$
$$+ a_1\,[\phi_1 + \phi_{11} + \phi_{111} + \phi_{1111} + \ldots]$$
$$= 1 + aQ\,[\omega e^{Q\omega_1}] + a_1\,[e^{Q\omega_1} - 1],$$

where Q denotes integration from t_0 to t. An example of such matrices is given by

$$a = \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix},$$

which arise by $\quad x_1 = e^{-\frac{1}{2}Q\omega}y, \quad x_2 = \dfrac{1}{\omega}\, e^{-\frac{1}{2}Q\omega}\dfrac{dy}{dt}$

from the equation

$$\frac{d^2y}{dt^2} = \left[\omega + \omega_1 + \frac{1}{\omega}\frac{d\omega}{dt}\right]\frac{dy}{dt} - \frac{1}{4}\,\omega(\omega + 2\omega_1)\,y\,;$$

and an infinite number of such pairs of matrices can be derived by linear transformation of the dependent variables from the system in which

$$a = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

c being an arbitrary constant. But the integration in finite terms applies to two matrices of any order having the same multiplication table.

Another example is the linear system

$$\frac{dx}{dt} = \begin{Bmatrix} 0 & \omega_1 & \omega_2 & \omega_3 \\ -\omega_1 & 0 & -\omega_3 & \omega_2 \\ -\omega_2 & \omega_3 & 0 & -\omega_1 \\ -\omega_1 & -\omega_2 & \omega_1 & 0 \end{Bmatrix} x = (i\omega_1 + j\omega_2 + k\omega_3)\,x$$

where i, j, k are matrices satisfying

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k,$$

and the solution is expressible by four series of functions ϕ in the form

$$\Omega\,(i\omega_1 + j\omega_2 + k\omega_3) = A + iA_1 + jA_2 + kA_3$$

where, for instance, A is

$$A = 1 - \phi_{11} - \phi_{22} - \phi_{33} - \phi_{123} + \phi_{132} - \phi_{231} + \phi_{321} - \phi_{312} + \phi_{132} + \ldots .$$

The systematic study of linear systems from this point of view breaks up into two independent problems: (1) the determination of all irreducible types of multiplication tables of sets of matrices of the same order, a problem akin to that of the enumeration of types of discontinuous groups; (2) the investigation of the properties of the functions represented by such series of repeated integrations as those denoted above by $A, A_1, \ldots$. These series converge for all finite values of t in the suitably chosen star region, and a first approximation to their investigation is the determination of their character near the corners of the region. It is to this determination for a wide class of cases that the second part of this paper is devoted.

PART II.

12. It has been remarked in § 9 that the single linear equation

$$y^{(n)} = \frac{P_{n-1}}{t} y^{(n-1)} + \frac{P_{n-2}}{t^2} y^{(n-2)} + \dots + \frac{P}{t^n} y,$$

wherein $P, \dots, P_{n-1}$ are developable about $t = 0$ in a series of positive integral powers of t, leads to a linear system of the n-th order

$$\frac{dx}{dt} = \left(\frac{A}{t} + V\right) x$$

wherein A is a matrix of constants and V is a matrix capable of development for the neighbourhood of $t = 0$ in a form

$$A_1 + A_2 t + A_3 t^2 + \dots$$

wherein $A_1, A_2, A_3, \dots$ are matrices of constants.

Independently now of whether the system is so derived from a single linear equation or not, we proceed to consider the character about $t = 0$ of the matrizant

$$\Omega\left(\frac{A}{t} + V\right)$$

wherein A, V are as in the description just given.

13. We assume the following theorem of algebra:—Let M be any matrix of constants, say of n rows and columns, which may be of zero determinant; let θ be any root of the determinantal equation $|M - \rho| = 0$, of multiplicity l; let $\rho - \theta$ be of multiplicity l_1 in the highest common factor in regard to ρ of the first minors of the determinant $|M - \rho|$, of multiplicity l_2 in the highest common factor of the minors of $(n-2)$ rows and columns, and so on; let $\epsilon_1 = l - l_1$, $\epsilon_2 = l_1 - l_2$, &c.; so that, if the minors of $n-r$ rows and columns do not all vanish when $\rho = \theta$, we have $\epsilon_r = l_{r-1}$ and

$$(\rho - \theta)^l = (\rho - \theta)^{\epsilon_1} (\rho - \theta)^{\epsilon_2} \dots (\rho - \theta)^{\epsilon_r},$$

the factors $(\rho-\theta)^{\epsilon_1}$, $(\rho - \theta)^{\epsilon_2}$, ... being what are called the invariant factors of the matrix $M - \rho$ for the root θ. They are the same for this as for the matrix $\mu^{-1}(M - \rho)\mu = \mu^{-1}M\mu - \rho$, wherein μ is any matrix of the same order as M of non-vanishing determinant; the exponents $\epsilon_1, \epsilon_2, \dots$ are non-vanishing positive integers known to satisfy the inequalities $\epsilon_1 \geqslant \epsilon_2 \geqslant \epsilon_3 \dots \geqslant \epsilon_r > 0$. If $\epsilon_1', \epsilon_1'', \dots$ be, like ϵ_1 for θ, the first of these respectively for the other roots $\theta', \theta'', \dots$ of $|M - \rho| = 0$, the equation satisfied by M is of the form

$$(M - \theta)^{\epsilon_1} (M - \theta')^{\epsilon_1'} (M - \theta'')^{\epsilon_1''} \dots = 0.$$

The important theorem is —rows, each of n elements, l rows in all, linearly independent of each other, to be denoted each row by a single letter such as $x_1, x_2, \dots$, can be found to satisfy the following l sets each of n linear equations :—

$$(M-\theta)\,x_1 = 0, \quad (M-\theta)\,x_2 = x_1, \quad \dots, \quad (M-\theta)\,x_{\epsilon_1} = x_{\epsilon_1-1},$$
$$(M-\theta)\,y_1 = 0, \quad (M-\theta)\,y_2 = y_1, \quad \dots, \quad (M-\theta)\,y_{\epsilon_2} = y_{\epsilon_2-1},$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$(M-\theta)\,z_1 = 0, \quad (M-\theta)\,z_2 = z_1, \quad \dots, \quad (M-\theta)\,z_{\epsilon_r} = z_{\epsilon_r-1},$$

and can be chosen in such a way that the most general solution of the set of n linear equations expressed by $(M-\theta)x = 0$ is a linear function of the rows $x_1, y_1, \dots, z_1$; the most general solution of the set of equations $(M-\theta)^2 x = 0$ is a linear function of the rows $x_1 \dots z_1, x_2 \dots z_2$, and so on. Further, if l' be the multiplicity of the root θ', similar l' rows $x_1', x_2', \dots, y_1', \dots$ can be chosen to satisfy the corresponding sets of linear equations for the root θ', and similar rows for the remaining roots $\theta'', \dots$, and the whole number of $n = l + l' + l'' + \dots$ such rows can be chosen to be linearly independent of one another, so that the matrix of n rows and columns of which the elements of any *column* are constituted by the elements of these rows is of non-vanishing determinant. Let this matrix, when its columns in order are formed from the rows $x_1, x_2, \dots, x_{\epsilon_1}, y_1, \dots, y_{\epsilon_2}, \dots, z_{\epsilon_r}, x_1', \dots, z_1', \dots$, be denoted by μ. It is then another way of stating the above equations to say that $\mu^{-1}M\mu = m$ where m is a matrix, called the canonical form of M, constituted as follows :—It has zero everywhere, save in the diagonal and the $n-1$ places immediately to the right of the diagonal. The diagonal consists, first, of the root θ, l times repeated; then of the root θ', l' times repeated; and so on. The $n-1$ elements to the right of the diagonal, which we shall in future denote by $a_{i,\,i+1}$ for $i = 1, \dots, (n-1)$, consist, first, of $\epsilon_1 - 1$ unities, then a zero, then $\epsilon_2 - 1$ unities, then a zero, and so on, there being a zero in the l-th row, then $\epsilon_1' - 1$ unities, and so on for the roots in turn. If we form the conditions for the equations $M\mu = \mu m$, in fact, they will be seen to be the equations above. The matrix m has the same roots and invariant factors as M, and the rows satisfying for it the equations corresponding to those above for M, viz., $(M-\theta)x = 0, \dots$, are in turn $\xi_1 = (1000\dots)$, $\xi_2 = (010\dots)$, $\dots$; so that the matrix μ belonging to m is the matrix unity.

14. Now let the roots of the determinantal equation $|A-\rho|=0$ be arranged as follows:—Let θ_1 be the root, or one of them, whose real part is greater than for any other root; let $\theta_2, \theta_3, \ldots, \theta_{s_1}$ be all the remaining roots which differ from θ_1 by integers, so chosen that their real parts are in descending order; thus we shall have $\theta_1-\theta_2=m_1$, $\theta_2-\theta_3=m_2, \ldots$, these being positive integers or zero, and $s_1=1$ being supposed included. Now from the remaining roots of $|A-\rho|=0$ choose that one of the largest real parts, or one of them, say θ_{s_1+1}, and let those follow this root which differ from θ_{s_1+1} by integers, as before, in descending order of real parts down to θ_{s_2}. If this rule be continued, it arranges all the roots.

Now choose a matrix μ as in § 13, such that

$$\mu^{-1}A\mu=\left\{\begin{matrix}\theta_1 & a_{12} & . & . \\ . & \theta_2 & a_{23} & . \\ . & . & . & .\end{matrix}\right\}=a, \text{ say},$$

wherein $a_{12}, a_{23}, \ldots$ are either unity or zero, according to the invariant factors, but in particular $a_{s_1, s_1+1}, a_{s_2, s_2+1}, \ldots$ are certainly zero, and let $\mu^{-1}V\mu=v$, so that

$$\mu^{-1}\Omega\left(\frac{A}{t}+V\right)\mu=\Omega\left(\mu^{-1}A\mu . t^{-1}+\mu^{-1}V\mu\right)=\Omega\left(\frac{a}{t}+v\right).$$

We first investigate the character of the matrix

$$\xi=\Omega\left(\frac{a}{t}+v\right)=\Omega(u), \text{ say}.$$

15. Consider, to this end, a matrix of constants defined as follows:—

$$\gamma=\left\{\begin{matrix}\gamma_1 & . & . & . \\ . & \gamma_2 & . & . \\ . & . & \gamma_3 & . \\ . & . & . & .\end{matrix}\right\}$$

wherein γ_1 is of s_1 rows and columns, its diagonal being the diagonal of γ, all elements in the first s_1 rows and columns of γ other than those belonging to γ_1 being zero; and γ_2 of (s_2-s_1) rows and columns, its diagonal being the diagonal of γ, all elements in the (s_1+1)-th, ..., s_2-th rows and columns of γ other than those of γ_2 being zero; and so on. Further, γ_1 is of the form

$$\gamma_1 = \begin{Bmatrix} 0 & c_{12} & c_{13} & \dots & c_{1s_1} \\ 0 & 0 & c_{23} & \dots & c_{2s_1} \\ . & . & . & \dots & . \end{Bmatrix},$$

where $c_{12}, c_{13}, c_{23}, \dots$ are constants to be further defined below, and $\gamma_2, \gamma_3, \dots$ have similar forms; thus in γ all elements in and to the left of the diagonal are zero.

It is manifest that, if m be a positive integer,

$$\gamma^m = \begin{Bmatrix} \gamma_1^m & . & . \\ . & \gamma_2^m & . \\ . & . & . \end{Bmatrix};$$

hence, if λ denote $\log t/t_0$, we have

$$\Omega\left(\frac{\gamma}{t}\right) = \begin{Bmatrix} \nabla\left(\frac{\gamma_1}{t}\right) & . & . \\ . & \nabla\left(\frac{\gamma_2}{t}\right) & . \\ . & . & . \end{Bmatrix},$$

where $$\nabla\left(\frac{\gamma_1}{t}\right) = 1 + \gamma_1\lambda + \frac{\gamma_1^2\lambda^2}{2!} + \dots, \; \dots .$$

Since the only root of the determinantal equation $|\gamma_1 - \rho| = 0$ is zero, there is a power $\gamma_1^{k_1}$ which is identically zero, k_1 at most s_1, and so for the others; thus $\nabla\left(\frac{\gamma_1}{t}\right)$ contains only a finite number of powers of λ, at most up to $\lambda^{s_1-1}, \dots$. In particular when in γ_1 every one of the constants c_{ij} other than $c_{12}, c_{23}, \dots$, in which $j = i+1$, is zero, so that γ_1 is of the form

$$\gamma_1 = \begin{Bmatrix} 0 & c_{12} & 0 & . \\ . & 0 & c_{23} & . \\ . & . & . & . \end{Bmatrix},$$

we have $$\gamma_1^2 = \begin{Bmatrix} 0 & 0 & c_{12}c_{23} & . & . \\ 0 & 0 & 0 & c_{23}c_{34} & . \\ . & . & . & . & . \end{Bmatrix},$$

$$\gamma_1^3 = \begin{Bmatrix} 0 & 0 & 0 & c_{12}c_{23}c_{34} & . & . \\ 0 & 0 & 0 & 0 & c_{23}c_{34}c_{45} & . \\ . & . & . & . & . & . \end{Bmatrix}, \; \dots,$$

$$\text{and} \quad \nabla\left(\frac{\gamma_1}{t}\right) = \begin{Bmatrix} 1 & c_{12}\lambda & \frac{1}{2!}c_{12}c_{23}\lambda^2 & \frac{1}{3!}c_{12}c_{23}c_{34}\lambda^3 & . \\ 0 & 1 & c_{23}\lambda & \frac{1}{3!}c_{23}c_{34}\lambda^2 & . \\ 0 & 0 & 1 & c_{34}\lambda & . \\ 0 & 0 & 0 & 1 & . \\ . & . & . & . & . \end{Bmatrix}.$$

We know, however, that a matrix of non-vanishing determinant σ can be found to put $\sigma^{-1}\gamma\sigma$ into a form δ with constants d_{ij}, in place of c_{ij}, in which all but the elements $d_{i,\,i+1}$ are zero, and each of these is either zero or unity; this would give

$$\sigma^{-1}\Omega\left(\frac{\gamma}{t}\right)\sigma = \Omega\left(\frac{\delta}{t}\right) = \begin{Bmatrix} \nabla\left(\frac{\delta_1}{t}\right) & . & . \\ . & \nabla\left(\frac{\delta_2}{t}\right) & . \\ . & . & . \end{Bmatrix}$$

where $\nabla(\delta_1/t)$, ..., are of the simple form above in terms of the constants d_{12}, d_{23}, ...

In addition to the matrix $\Omega\left(\frac{\gamma}{t}\right)$ consider now the matrix

$$\Theta = \begin{Bmatrix} \theta_1 & . & . \\ . & \theta_2 & . \\ . & . & . \end{Bmatrix},$$

consisting only of the roots θ_1, θ_2, ... of $| A - \rho | = 0$, written in order in the diagonal, so that

$$\Omega\left(\frac{\Theta}{t}\right) = \begin{Bmatrix} (t/t_0)^{\theta_1} & . & . \\ . & (t/t_0)^{\theta_2} & . \\ . & . & . \end{Bmatrix}.$$

If we form the product

$$\Omega\left(\frac{\Theta}{t}\right)\Omega\left(\frac{\gamma}{t}\right),$$

it will be the same as $\Omega\left(\frac{\gamma}{t}\right)$, save that every element in the i-th row is multiplied by $(t/t_0)^{\theta_i}$. But we have the relation (§ 5)

$$\Omega\left(huh^{-1}+\frac{dh}{dt}h^{-1}\right)=h\Omega(u)h_0^{-1},$$

giving, if $h=\Omega(w)$,

$$\Omega(w)\,\Omega(u)=\Omega\left[w+\Omega(w)u\Omega^{-1}(w)\right];$$

herein put $w=\Theta/t$, $u=\gamma/t$; then

$$\Omega\left(\frac{\Theta}{t}\right)\Omega\left(\frac{\gamma}{t}\right)=\Omega\left[\frac{\Theta}{t}+\frac{1}{t}\,\Omega\left(\frac{\Theta}{t}\right)\gamma\Omega^{-1}\left(\frac{\Theta}{t}\right)\right];$$

the general element of

$$\Omega(\Theta/t)\,\gamma\Omega^{-1}(\Theta/t)$$

is

$$\begin{aligned}[\Omega(\Theta t^{-1})\gamma\Omega^{-1}(\Theta t^{-1})]_{pr}&=\sum_{q=1}^{n}[\Omega(\Theta t^{-1})]_{pq}[\gamma\Omega^{-1}(\Theta t^{-1})]_{qr}\\&=(t/t_0)^{\theta_p}[\gamma\Omega^{-1}(\Theta t^{-1})]_{pr}\\&=(t/t_0)^{\theta_p}\sum_{s=1}^{n}\gamma_{ps}[\Omega^{-1}(\Theta t^{-1})]_{sr}\\&=(t/t_0)^{\theta_p}(t/t_0)^{-\theta_r}\gamma_{pr}=(t/t_0)^{\theta_p-\theta_r}\gamma_{pr},\end{aligned}$$

of which however, by the definition of the matrix γ, the element γ_{pr} is zero when $\theta_p-\theta_r$ is not an integer, and is, in fact, zero when $r \leqslant p$; so that when $\gamma_{pr}=c_{pr}$ is not zero $\theta_p-\theta_r$ is a positive integer or zero; thus on the whole

$$\Omega\left(\frac{\Theta}{t}\right)\Omega\left(\frac{\gamma}{t}\right)=\Omega(\phi)$$

where

$$\phi=\frac{1}{t}\begin{Bmatrix}\theta_1 & c_{12}(t/t_0)^{\theta_1-\theta_2} & c_{13}(t/t_0)^{\theta_1-\theta_3} & .\\ . & \theta_2 & c_{23}(t/t_0)^{\theta_2-\theta_3} & .\\ . & . & . & .\end{Bmatrix},$$

all places in the matrix, other than those in the diagonal, which were filled by zeros in the matrix γ being here also filled by zeros, and each of the exponents $\theta_1-\theta_2$, $\theta_1-\theta_3$, ..., which occur being either zero or a positive integer.

16. Consider now the differential equation for a matrix η expressed by

$$\frac{d\eta}{dt}=\left(\frac{a}{t}+v\right)\eta-\eta\phi,$$

equivalent to a system of n^2 linear equations for the elements of η, of which, if $\frac{a}{t}+v=u$, the general one is

$$\frac{d\eta_{ij}}{dt}=u_{i1}\eta_{1j}+\ldots+u_{in}\eta_{nj}-(\eta_{i1}\phi_{1j}+\ldots+\eta_{in}\phi_{nj})$$
$$(i, j=1, 2, \ldots, n);$$

if g be any particular form of η satisfying this equation, the matrix

$$g\Omega(\phi)$$

satisfies

$$\frac{d}{dt}[g\Omega(\phi)]=\frac{dg}{dt}\Omega(\phi)+g\phi\Omega(\phi)=(ug-g\phi)\Omega(\phi)+g\phi\Omega(\phi)$$
$$=u[g\Omega(\phi)];$$

and is therefore of the form

$$g\Omega(\phi)=\Omega(u)\,g_0,$$

where g_0 is a matrix of constants, being the value at t_0 of the matrix g, or its continuation; thus

$$\Omega\left(\frac{a}{t}+v\right)=g\Omega(\phi)\,g_0^{-1}.$$

We proceed to show that when the constants $c_{12}, c_{13}, c_{23}, \ldots$ in ϕ are suitably chosen there exists such a matrix g, reducing to unity when $t=0$, and expressible about $t=0$ as an ordinary power series

$$1+g_1t+g_2t^2+\ldots,$$

wherein $g_1, g_2, \ldots$ are matrices, which is convergent for sufficiently small values of t. Its continuation to all values in the star region is then given by the equation

$$g=\Omega\left(\frac{a}{t}+v\right)g_0\Omega^{-1}(\phi).$$

For this purpose we write down the differential equations to be satisfied by the elements of the columns of the matrix η in greater detail; let the row of elements constituting the first column of η be called x, those the second, third, ... columns respectively $y, z, \ldots$; next, those constituting the (s_1+1)-th column be called X, those the (s_1+2)-th column be called Y, and so on; then, taking account of the form of the matrix ϕ, we have

$$\frac{dx}{dt}=\left(\frac{a}{t}+v\right)x-\frac{1}{t}\theta_1 x,$$

$$\frac{dy}{dt}=\left(\frac{a}{t}+v\right)y-\frac{1}{t}\left[\theta_2 y+c_{12}(t/t_0)^{\theta_1-\theta_2}x\right],$$

$$\frac{dz}{dt}=\left(\frac{a}{t}+v\right)z-\frac{1}{t}\left[\theta_3 z+c_{23}(t/t_0)^{\theta_2-\theta_3}y+c_{13}(t/t_0)^{\theta_1-\theta_3}x\right],$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\frac{dX}{dt}=\left(\frac{a}{t}+v\right)X-\frac{1}{t}\theta_{s_1+1}X,$$

$$\frac{dY}{dt}=\left(\frac{a}{t}+v\right)Y-\frac{1}{t}\left[\theta_{s_1+2}Y+c_{s_1+1,\,s_1+2}(t/t_0)^{\theta_{s_1+1}-\theta_{s_1+2}}X\right],$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots .$$

We proceed then to establish in turn (1) the existence of formal solutions of these in the form of power series in t, (2) the convergence of these series.

17. First as to the formal solution. The equation for the row x, when we put

$$v=a_1+a_2t+a_3t^2+\dots,$$

where $a_1, a_2, \dots$ are matrices of constants, and assume

$$x=x_0+tx_1+t^2x_2+\dots,$$

where $x_0, x_1, x_2, \dots$ denote rows of elements, gives

$$tx_1+2t^2x_2+3t^3x_3+\dots+mt^mx_m+\dots$$
$$=(a-\theta_1+a_1t+a_2t^2+\dots)(x_0+tx_1+t^2x_2+\dots);$$

hence, equating coefficients of the same powers of t,

$$(a-\theta_1)x_0=0, \quad (a-\theta_1-1)x_1+a_1x_0=0,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$(a-\theta_1-m)x_m+a_1x_{m-1}+a_2x_{m-2}+\dots+a_mx_0=0,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots .$$

Of these the first is clearly satisfied by

$$x_0=(1000\dots),$$

while, since there is no root of $|a-\rho|=0$ exceeding θ_1 by a positive integer, each of the following rows x_1, x_2, ... is given, without ambiguity, by the formula

$$x_m = -(a-\theta_1-m)^{-1}(a_1 x_{m-1}+\ldots+a_m x_0).$$

Similarly substituting in the equation for y a series

$$y = y_0+ty_1+t^2y_2+\ldots,$$

where y_0, y_1, ... are rows of elements, we have

$$ty_1+2t^2y_2+\ldots = (a-\theta_2+a_1t+a_2t^2+\ldots)(y_0+ty_1+t^2y_2+\ldots)$$
$$-c_{12}(t/t_0)^{\theta_1-\theta_2}(x_0+tx_1+t^2x_2+\ldots);$$

here $\theta_1-\theta_2$ is zero or a positive integer, say k_1, since otherwise the equation for X, to be considered presently, is the next equation. For distinctness consider the cases separately.

(*a*) When $\theta_1-\theta_2 = 0$ the term a_{12} in the matrix a may be zero or unity, according to the invariant factors; in either case the value

$$y_0 = (0100\ldots)$$

gives $$(a-\theta_2)\,y = a_{12}x_0;$$

taking then $c_{12} = a_{12}$, the terms in t^0 vanish and subsequent terms are given, after equating coefficients of t^m, without ambiguity by the equation

$$(a-\theta_2-m)\,y_m+a_1y_{m-1}+\ldots+a_my_0-a_{12}x_m = 0.$$

(*b*) When $\theta_1-\theta_2 = k_1 > 0$ the term a_{12} in the matrix a is zero, and the same value of y_0 as in the previous case gives

$$(a-\theta_2)\,y_0 = 0,$$

reducing the coefficient of t^0 to zero; for the coefficient of t^m in which $m < k_1$ we have, as before, without ambiguity,

$$y_m = -(a-\theta_2-m)^{-1}(a_1y_{m-1}+\ldots+a_my_0);$$

but the term in t^{k_1} is a critical term, giving, since $\theta_2+k_1 = \theta_1$,

$$(a-\theta_1)\,y_{k_1}+a_1y_{k_1-1}+\ldots+a_{k_1}y_0-c_{12}t_0^{-k_1}x_0 = 0,$$

wherein the determinant $|a-\theta_1|$ is zero and the inverse matrix $(a-\theta_1)^{-1}$ unmeaning. But, since $\theta_1 \neq \theta_2$, it follows, in virtue of the way in which the roots have been arranged, that θ_1 is not equal to any other root of $|a-\rho| = 0$; thus the only diagonal element of $a-\theta_1$ which vanishes is the first, and every element of y_{k_1} except its first is determined by this equation without ambiguity when c_{12} is assigned. Take the first element of y_{k_1} zero for simplicity, and take the disposable constant c_{12} so that $c_{12}t^{-k_1}$ = first element of $a_1 y_{k_1-1}+\ldots+a_{k_1}y_0$, and then determine the other elements of y_{k_1}. For the coefficient of t^m in which $m > k_1$ no such difficulty arises, the row y_m being determined without ambiguity by means of

$$y_m = -(a-\theta_2-m)^{-1}[a_1 y_{m-1}+\ldots+a_m y_0 - c_{12}t_0^{-k_1}x_{m-k_1}],$$

there being no root of $|a-\rho|=0$ of the form $\theta_2+m=\theta_1+m-k$.

To make the argument still clearer consider in similar detail the equation for the column z; putting $z=z_0+tz_1+t^2z_2+\ldots$, the equation to be satisfied identically in regard to t is

$$tz_1+2t^2z_2+\ldots = (a-\theta_3+a_1t+a_2t^2+\ldots)(z_0+tz_1+t^2z_2+\ldots)$$
$$-c_{23}(t/t_0)^{\theta_2-\theta_3}(y_0+ty_1+t^2y_2+\ldots)-c_{13}(t/t_0)^{\theta_1-\theta_3}(x_0+tx_1+t^2x_2+\ldots).$$

(*a*) When $\theta_1=\theta_2=\theta_3$, the constants a_{12}, a_{23} in the matrix a may each be either zero or unity, but in any case the row

$$z_0 = (00100\ldots)$$

gives
$$(a-\theta_3)z_0 = a_{23}y_0.$$

Take then $c_{23}=a_{23}$, $c_{13}=0$; any row z_m for $m=1, 2, \ldots$ is determined without ambiguity by the equation

$$z_m = -(a-\theta_3-m)^{-1}(a_1 z_{m-1}+\ldots+a_m z_0 - a_{23}y_m).$$

(*b*) When $\theta_1-\theta_2=0$, $\theta_2-\theta_3=k_2>0$, the constant a_{12} in the matrix a is zero or unity, and the constant a_{23} is zero; the same value as before for z_0 gives $(a-\theta_3)z_0=0$, and the equation for z_m, for $m=1, 2, \ldots$, namely,

$$0=(a-\theta_3-m)z_m+a_1 z_{m-1}+\ldots+a_m z_0-c_{23}t_0^{-k_2}y_{m-k_2}-c_{13}t_0^{-k_2}x_{m-k_2},$$

wherein the terms in x_{m-k_2}, y_{m-k_2} are to be omitted if $m<k_2$; considering then this value of m, there are two roots, viz., θ_1 and θ_2 equal to θ_3+k_2 and two of the diagonal elements of $a-\theta_3-m$ are zero, namely, the first and second, all others being other than zero; then, when c_{13}, c_{23} are assigned, all elements of z_m after the second are determined by this equation without ambiguity. By equating first and second elements in the equation to zero, we obtain respectively, since $m=k_2$,

$$a_{12}z_m^{(2)}+(a_1 z_{m-1}+\ldots+a_m z_0)^{(1)}-c_{13}t_0^{-k_2}=0,$$
$$(a_1 z_{m-1}+\ldots+a_m z_0)^{(2)}-c_{23}t_0^{-k_2}=0;$$

thus when $a_{12} = 1$ we can take $c_{13} = 0$; but in any case c_{13}, c_{23} can be chosen so that these equations are satisfied.

(*c*) When $\theta_1 - \theta_2 = k_1 > 0$, $\theta_2 - \theta_3 = 0$, the constant a_{12} in the matrix a is zero, and the constant a_{23} is zero or unity; but the same value as before for z_0 gives $(a - \theta_3) z_0 = a_{23} y_0$, and, taking $c_{23} = a_{23}$, the coefficient of t^0 vanishes; the equation arising by the coefficient of t^m,

$$0 = (a - \theta_3 - m) z_m + a_1 z_{m-1} + \ldots + a_m z_0 - c_{23} y_m - c_{13} t_0^{-k_1} x_{m-k_1},$$

wherein for $m < k_1$ the last term is to be omitted, is only critical for values of m for which $\theta_3 + m$ is a root, namely, for $m = k$; considering this value of m, every diagonal term of $a - \theta_3 - m$ is other than zero except the first, and so every element of z_m except the first is definite: we can then take the first element of z_m zero and choose c_{13} so that $c_{13} t_0^{-k_1} =$ first element of $a_1 z_{m-1} + \ldots + a_m z_0 - a_{23} y_m$.

(*d*) Lastly, when $\theta_1 - \theta_2 = k_1 > 0$, $\theta_2 - \theta_3 = k_2 > 0$, each of the constants a_{12}, a_{23} in the matrix a is zero. As before, the value $z_0 = (00100\ldots)$ gives $(a - \theta_3) z_0 = 0$; but the general equation for $m > 0$,

$$0 = (a - \theta_3 - m) z_m + a_1 z_{m-1} + \ldots + a_m z_0 - c_{23} t_0^{-k_2} y_{m-k_2} - c_{13} t_0^{-k_1-k_2} x_{m-k_1-k_2},$$

wherein $y_{m-k_2} = 0$ for $m < k_2$ and $x_{m-k_1-k_2} = 0$ for $m < k_1 + k_2$, is critical for both the values $m = k_2$, $m = k_1 + k_2$ for which $\theta_3 + m$ is a root, respectively θ_2 and θ_1.

When $m = k_2$ only the second diagonal element of $a - \theta_3 - m$ is zero; we may take the second element of $z_{k_2} = 0$, and choose $c_{23} t_0^{-k_2} =$ second element of $(a_1 z_{k_2-1} + \ldots + a_{k_2} z_0)$, the other elements of z_{k_2} being determined without ambiguity.

When $m = k_1 + k_2$ only the first element of the diagonal of $a - \theta_3 - m$ is zero; we may take the first element of $z_{k_1+k_2} = 0$, and choose

$$c_{13} t_0^{-k_1 \; k_2} = \text{first element of } (a_1 z_{k_1+k_2-1} + \ldots + a_{k_1+k_2} z_0 - c_{23} t_0^{-k_2} y_{k_1}).$$

A precisely similar argument applies for immediately succeeding columns. Consider now the $(s_1 - 1)$-th column; puttting

$$X = X_0 + tX_1 + t^2 X_2 + \ldots,$$

the equation
$$t \frac{dX}{dt} = (a + vt) X - \theta_{s_1+1} X$$

gives

$$tX + 2t^2 X_2 + \ldots = (a - \theta_{s_1+1} + a_1 t + a_2 t^2 + \ldots)(X_0 + tX_1 + t^2 X_2 + \ldots),$$

and we take $$X_0 = (0\ldots010\ldots),$$

wherein the (s_1-1)-th element is unity, satisfying $(\alpha-\theta_{s_1+1})X_0=0$; while, as there is no positive integer m such that $\theta_{s_1+1}+m$ is a root of $|\alpha-\rho|=0$, no other critical terms arise.

For the (s_1+2)-th column Y we have to satisfy

$$tY_1+2t^2Y_2+\ldots = (\alpha-\theta_{s_1+2}+a_1t+a_2t^2+\ldots)(Y_0+tY_1+t^2Y_2+\ldots)$$
$$-c_{s_1+1,\,s_1+2}(t/t_0)^{\theta_{s_1+1}-\theta_{s_1+2}}(X_0+tX_1+t^2X_2+\ldots).$$

(a) If $\theta_{s_1+1}-\theta_{s_1+2}=0$, the constant $a_{s_1+1,\,s_1+2}$ in the matrix α is zero or unity, but the row

$$Y_0=(0\ldots010\ldots),$$

with unity in the (s_1+2)-th place, satisfies

$$(\alpha-\theta_{s_1+2})Y_0 = a_{s_1+1,\,s_1+2}X_0;$$

we take then $$c_{s_1+1,\,s_1+2} = a_{s_1+1,\,s_1+2},$$

and, as there is no positive integer m for which $\theta_{s_1+2}+m=\theta_{s_1+1}+m$ is a root of $|\alpha-\rho|=0$, no other critical terms arise.

(b) When $\theta_{s_1+1}-\theta_{s_1+2}=l_1>0$, the constant $a_{s_1+1,\,s_1+2}=0$, the same value of Y_0 as before gives $(\alpha-\theta_{s_1+2})Y_0=0$; the general equation

$$0=(\alpha-\theta_{s_1+2}-m)Y_m+a_1Y_{m-1}+\ldots+a_mY_0-c_{s_1+1,\,s_1+2}t_0^{-l_1}X_{m-l_1},$$

wherein $X_{m-l_1}=0$ for $m<l_1$, is critical only for $m=l_1$, and in the matrix $\alpha-\theta_{s_1+2}-l_1=\alpha-\theta_{s_1+1}$ the only element of the diagonal which vanishes is the (s_1+1)-th; we may take the (s_1+1)-th element of Y_m zero, and, the (s_1+1)-th element of X_0 being unity, determine $c_{s_1+1,\,s_1+2}$ so that

$$c_{s_1+1,\,s_1+2}t_0^{-l_1} = (s_1+1)\text{-th element of } (a_1Y_{l_1-1}+\ldots+a_{l_1}Y_0);$$

the other elements of $Y_m=Y_{l_1}$ are then determined without ambiguity.

The same examination in detail can be continued. It is, however, sufficiently clear that in all cases the constant c_{ij} can be chosen so as to give a perfectly definite expansion in powers of t for every element of every column of the matrix η in such a way that for $t=0$ the matrix $\eta=1$, that is, has unity for every diagonal element and zero for every other element.

18. In regard now to the convergence of the series which have been determined for the various columns. Each of the n differ-

ential equations is of the form

$$t\frac{dy}{dt} = (\beta + a_1 t + a_2 t^2 + \ldots) y - x,$$

wherein y denotes the column to be determined, x is a numerical multiple of a column previously determined, as we may suppose by a convergent series, or a sum of a finite number of such, and β is a matrix of zero determinant. The numerical multipliers in x, namely, the constants $c_{12}, c_{13}, c_{23}, \ldots$, are determined with the early terms of the series for y, certainly of finite rank, since the equation $|\beta - \rho| = 0$ cannot have two roots whose difference is not finite; thus the question of convergence relates only to the series

$$y - (y_0 + ty_1 + \ldots + t^{N-1} y_{N-1}) = t^N y_N + t^{N+1} y_{N+1} + \ldots,$$

where the general coefficient on the right hand is determined by an equation of the form

$$y_m = (m-\beta)^{-1} [a_1 y_{m-1} + \ldots + a_m y_0 - x_m],$$

in which x_m and $y_0, \ldots, y_{N-1}$ are given and N is such that for $m > N-1$ the determinant of $m-\beta$ is not zero.

Now let the rows of real positive elements $Y_0, \ldots, Y_{N-1} X_m$ and matrices of real positive elements A_i, H be determined so that

$$|y_0| < Y_0, \quad |y_1| < Y_1, \quad \ldots, \quad |y_{N-1}| < Y_{N-1}, \quad |a_i| < A_i$$

for $i = 1, 2, \ldots, \infty$, and, for $m \geqslant N$,

$$|x_m| < X_m, \quad |(m-\beta)^{-1}| < H,$$

where the meaning is that the modulus of each individual element of the row or matrix on the left is less than the corresponding element on the right, the possibility of the inequalities $|a_i| < A_i$, $|x_m| < X_m$ being a consequence of the assumed convergence for sufficiently small t of the series for the matrix v and the row x; then the equations, for $m \geqslant N$,

$$y_m = (m-\beta)^{-1} [a_1 y_{m-1} + \ldots + a_m y_0 + x_m]$$

lead to $$|y_m| < H[A_1 Y_{m-1} + \ldots + A_m Y_0 + X_m];$$

or, if we put, for $m \geqslant N$,

$$Y_m = H[A_1 Y_{m-1} + \ldots + A_m Y_0 + X_m],$$

lead to $$|y_m| < Y_m.$$

It is therefore sufficient to show that the successive equations, for $m \geqslant N$,

$$Y_m = H[A_1 Y_{m-1} + \dots + A_m Y_0 + X_m],$$

wherein $A_1, A_2, \dots$ are any given matrices of real positive elements such that the series

$$A_1 t + A_2 t^2 + \dots$$

converges when t is small enough; $Y_0, Y_1, \dots, Y_{N-1}$ are any given rows of each real positive elements; $X_N, X_{N+1}, \dots$ are any given rows of real positive elements such that the series

$$t^N X_N + t^{N+1} X_{N+1} + \dots$$

converges when t is small enough, give rise to a series

$$t^N Y_N + t^{N+1} Y_{N+1} + \dots,$$

converging when t is small enough.

For this, consider the equations expressed by

$$\begin{aligned} Y - Y_0 - tY_1 - \dots - t^{N-1} Y_{N-1} \\ = H\{(tA_1 + t^2 A_2 + \dots) Y - [tA_1 Y_0 + t^2(A_1 Y_1 + A_2 Y_0) + \dots \\ \dots + t^{N-1}(A_1 Y_{N-2} + \dots + A_{N-1} Y_0)] + t^N X_N + t^{N+1} X_{N+1} + \dots\},\end{aligned}$$

which, when written at length, are n implicit equations for the n functions of t which are the n elements of the row Y, in fact of the form

$$\begin{aligned} C_{11}(y_1 - y_1^0) + \dots + C_{1n}(y_n - y_n^0) &= t\phi_1(t, y_1, \dots, y_n), \\ \dots \quad \dots \quad \dots \quad & \quad \dots \quad \dots \quad \dots \\ C_{n1}(y_1 - y_1^0) + \dots + C_{nn}(y_n - y_n^0) &= t\phi_n(t, y_1, \dots, y_n), \end{aligned}$$

where $\phi_1, \dots, \phi_n$ are convergent series in t, linear in $y_1, \dots, y_n$, and $C_{ij} = 0$, except $C_{ii} = 1$.

Such a set of equations is known to have convergent solutions; say in our case

$$Y = Y_0 + tZ_1 + \dots + t^{N-1} Z_{N-1} + t^N Y_N + t^{N+1} Y_{N+1} + \dots,$$

which on substitution gives

$$\begin{aligned} t(Z_1 - Y_1) + \dots + t^{N-1}(Z_{N-1} - Y_{N-1}) + t^N Y_N + t^{N+1} Y_{N+1} + \dots \\ = H\{(tA_1 + t^2 A_2 + \dots)(Y_0 + tZ_1 + \dots + t^{N-1} Z_{N-1} + t^{N+1} Y_{N+1} + \dots) \\ - [tA_1 Y_0 + \dots + t^{N-1}(A_1 Y_{N-2} + \dots + A_{N-1} Y_0)] + t^N X_N + \dots\},\end{aligned}$$

and hence, so far as the terms in t^{N-1},

$$Z_1 - Y_1 = H\{A_1 Y_0 - A_1 Y_0\},$$
$$Z_2 - Y_2 = H\{A_1 Z_1 + A_2 Y_0 - (A_1 Y_1 + A_2 Y_0)\},$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$Z_{N-1} - Y_{N-1} = H\{A_1 Z_{N-2} + A_2 Z_{N-3} + \ldots + A_{N-1} Y_0 - (A_1 Y_{N-2} + \ldots + A_{N-1} Y_0)\},$$

leading to $\quad Z_1 = Y_1, \quad Z_2 = Y_2, \quad \ldots, \quad Z_{N-1} = Y_{N-1},$

while, for the term in t^m, for $m \geqslant N$,

$$Y_m = H\{A_1 Y_{m-1} + A_2 Y_{m-2} + \ldots + A_m Y_0 + X_m\},$$

which is the series of real and positive elements occurring in the series whose convergence was previously shown sufficient for our theorem.

19. We have thus established that

$$\Omega\left(\frac{a}{t} + v\right) = g\Omega(\phi) g_0^{-1} = g\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1}) g_0^{-1},$$

where g is a matrix of functions of t developable in a convergent ordinary power series about $t = 0$, the matrix g reducing to unity for $t = 0$, and g_0 is the value of g, or its continuation, at $t = t_0$. Thence, when, as before,

$$\mu^{-1} A \mu = a, \quad \mu^{-1} V \mu = v,$$

we have $\quad \Omega\left(\frac{A}{t} + V\right) = G\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1}) G_0^{-1},$

where $\quad G = \mu g$

reduces to μ for $t = 0$, and satisfies the equations

$$\frac{dG}{dt} = \left(\frac{A}{t} + V\right) G - G\phi.$$

The matrices γ, Θ, and ϕ have been explained in § 15.

In the subsidiary equations for the determination of the columns of g, the position of t_0 enters only in the combinations $c_{ij} t^{-(\theta_i - \theta_j)}$; if we put this $= e_{ij}$, these subsidiary equations will contain no reference to t_0. Thus the matrix g does not alter when t_0 is taken differently, nor therefore does the matrix G; but the matrix $\Omega(\phi)$ does alter, being

$$\Omega^{t, t_0}(\phi) = \Omega^{t, t_1}(\phi)\,\Omega^{t_1, t_0}(\phi).$$

The simplest case of our general formula is when no two of the roots of the determinantal equation $|A-\rho| = 0$ are equal or differ by integers; then the matrix γ consists only of zeros and $\Omega(\gamma t^{-1}) = 1$.

Another case which may occur is that in which no two roots differ by an integer unless they are exactly equal. Then in the series which solve the subsidiary equation for the columns of the matrix ϕ no critical terms occur after the first terms, which may be critical owing to sequences of equal roots; in such case, as is seen on referring back to the work, every constant c_{ij} in which $j > i+1$ may be taken to be zero and the constants $c_{i,i+1}$ are those, $a_{i,i+1}$, arising at once from the given form of the differential equations, which occur in the canonical matrix a; thus

$$\phi = \frac{1}{t}\begin{Bmatrix} \theta_1 & c_{12}(t/t_0)^{\theta_1-\theta_2} & 0 & 0 & . \\ 0 & \theta_2 & c_{23}(t/t_0)^{\theta_2-\theta_3} & 0 & . \\ 0 & 0 & \theta_3 & c_{34}(t/t_0)^{\theta_3-\theta_4} & . \\ . & . & . & . & . \end{Bmatrix},$$

in which $c_{i,i+1} = 0$, unless $\theta_i = \theta_{i+1}$, and $c_{i,i+1} = a_{i,i+1}$ when $\theta_i = \theta_{i+1}$. In other words, in this case, we have

$$\phi = \frac{a}{t},$$

and $$\Omega\left(\frac{A}{t}+V\right) = \mu g \Omega(at^{-1})\, g_0^{-1}\mu^{-1} = \mu g \mu^{-1}\Omega(At^{-1})(\mu g_0 \mu^{-1})^{-1}$$
$$= h\Omega\left(\frac{A}{t}\right)h_0^{-1},$$

where $$h = \mu g \mu^{-1}$$

reduces to unity when $t = 0$ and satisfies the equations

$$\frac{dh}{dt} = \left(\frac{A}{t}+V\right)h - h\frac{A}{t}.$$

In both these cases, it is to be noticed, the form of the matrix $\Omega(\phi)$ is determinable at once by inspection of the given differential equations from the matrix A alone.

20. In the case of a linear system derived from a single linear equation, as in § 9, the matrix A, there written at length, has the peculiarity that in $|A-\rho|$ the minor of the first element of the last row has a determinant equal to unity; thus, if θ be a root of multi-

plicity l, the first invariant factor corresponding thereto is $(\rho-\theta)^l$, and in the canonical form $\alpha = \mu^{-1}A\mu$ of the matrix A, the $l-1$ constants $a_{i,i+1}$ corresponding to the root have all the value unity, so that the equality of two roots necessarily involves a logarithm in the solution of the system. The matrix V is further special, in that all its first $n-1$ rows consist of zeros. In fact, the ordinary theory of a single linear equation leads us to expect that in this case all the constants c_{ij} of the matrix γ vanish in which j is not equal to $i+1$. We have not deduced this result in the present paper, the expression for the most general case given in § 15,

$$\nabla\left(\frac{\gamma_1}{t}\right) = 1+\gamma_1\lambda+\frac{\gamma_1^2}{2!}\lambda^2+\ldots,$$

appearing to be of sufficient simplicity. But that it is not possible in all cases to arrange to have all constants c_{ij} in which $j \neq i+1$ equal to zero appears to follow from such an example as the system

$$t\frac{dx_1}{dt} = \theta_1 x_1 + t(1-\theta_1+\theta_3)\,x_2 + \frac{t(1-\theta_1+\theta_3)+t^{\theta_1-\theta_3}}{1+t^2}\,x_3,$$

$$t\frac{dx_2}{dt} = \theta_2 x_2,$$

$$t\frac{dx_3}{dt} = \theta_3 x_3 + \frac{2t^2}{1+t^2}x_3,$$

where we may suppose $\theta_1-\theta_3 = 0$, or a positive integer; this has a form

$$\frac{dx}{dt} = \left(\frac{a}{t}+v\right)x,$$

where
$$a = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix},$$

$$v = \begin{pmatrix} 0 & 1-\theta_1+\theta_3 & \dfrac{1-\theta_1+\theta_3+t^{\theta_1-\theta_3-1}}{1+t^2} \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{2t}{1+t^2} \end{pmatrix}.$$

This system is satisfied by the elements of each of the three columns of the product of the matrices

$$\begin{Bmatrix} 1 & t & t \\ 0 & 1 & 0 \\ 0 & 0 & 1+t^2 \end{Bmatrix} \begin{Bmatrix} (t/t_0)^{\theta_1} & 0 & t^{\theta_1} t_0^{-\theta_3} \log t/t_0 \\ 0 & (t/t_0)^{\theta_2} & 0 \\ 0 & 0 & (t/t_0)^{\theta_3} \end{Bmatrix};$$

namely, in $\Omega\left(\frac{a}{t}+v\right) = g\Omega(\phi)\, g_0^{-1} = g\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\, g_0^{-1}$,

we have

$$g = \begin{Bmatrix} 1 & t & t \\ 0 & 1 & 0 \\ 0 & 0 & 1+t^2 \end{Bmatrix},$$

$$\Omega(\Theta/t) = \begin{Bmatrix} (t/t_0)^{\theta_1} & 0 & 0 \\ 0 & (t/t_0)^{\theta_2} & 0 \\ 0 & 0 & (t/t_0)^{\theta_3} \end{Bmatrix},$$

$$\Omega(\gamma/t) = \begin{Bmatrix} 1 & 0 & t_0^{\theta_1-\theta_3} \log(t/t_0) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix};$$

$$\gamma = \begin{Bmatrix} 0 & 0 & t_0^{\theta_1-\theta_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \quad \phi = \frac{1}{t}\begin{Bmatrix} \theta_1 & 0 & t_0^{\theta_1-\theta_3}(t/t_0)^{\theta_1-\theta_3} \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{Bmatrix}.$$

21. As a simple actual example of the method for an assigned linear equation take the equation (Forsyth, *Linear Differential Equations*, p. 103, Ex. 8)

$$t^3(1+t)\,y''' - (2+4t)\,t^2 y'' + (4+10t)\,ty' - (4+12t)\,y = 0,$$

which, as in § 9, leads to the system

$$\frac{dx}{dt} = \left[\frac{1}{t}\begin{Bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 4 & -4 & 4 \end{Bmatrix} + \frac{1}{t+1}\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & -6 & 2 \end{Bmatrix}\right] x = \left(\frac{A}{t} + \frac{B}{t+1}\right) x, \text{ say.}$$

The roots of $|A-\rho| = 0$ are 2, 2, 1; with

$$\mu = \begin{Bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 0 \end{Bmatrix},$$

we find $\mu^{-1}A\mu = \begin{Bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{Bmatrix} = \alpha, \quad \mu^{-1}B\mu = \begin{Bmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{Bmatrix},$

$$\phi = \frac{1}{t}\begin{Bmatrix} 2 & c_{12} & c_{13}(t/t_0) \\ 0 & 2 & c_{23}(t/t_0) \\ 0 & 0 & 1 \end{Bmatrix},$$

and, with $u = \frac{\alpha}{t} + \frac{\beta}{t+1}$, the subsidiary equations

$$\frac{dg}{dt} = ug - g\phi$$

lead to $$\Omega\left(\frac{\alpha}{t} + \frac{\beta}{t+1}\right) = \begin{Bmatrix} 1 & 0 & -2t \\ 0 & 1 & 2t(1+t) \\ 0 & 0 & (1+t)^2 \end{Bmatrix} \Omega(\phi)\, g_0^{-1}$$

$$= g\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\, g_0^{-1},$$

with $\phi = \frac{1}{t}\begin{Bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2t \\ 2 & 0 & 1 \end{Bmatrix}, \quad \Omega(\Theta t^{-1}) = \begin{Bmatrix} (t/t_0)^2 & 0 & 0 \\ 0 & (t/t_0)^2 & 0 \\ 0 & 0 & t/t_0 \end{Bmatrix},$

$$\gamma = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2t_0 \\ 0 & 0 & 0 \end{Bmatrix}, \quad \Omega(\gamma t^{-1}) = \begin{Bmatrix} 1 & \lambda & t_0\lambda^2 \\ 0 & 1 & 2t_0\lambda \\ 0 & 0 & 1 \end{Bmatrix},$$

where $\lambda = \log t/t_0$.

Part III.

22. A problem to which the previous investigation can be usefully applied is the elucidation of the connexion between the form of the linear system and the form of the linear substitutions which generate the monodromy group of the system. We consider only the case where the functions of t in the matrix u are single valued over the whole finite part of the plane. The star region in which the matrix $\Omega(u)$ is single valued and developable having been defined as previously explained, let the barrier joining one of its angular points to $t = \infty$ be removed, and let $\Omega_1(u)$ denote the value for the matrix obtained by integrating first from t_0 to t by the path by which $\Omega(u)$ was defined, and then from t once round the single corner now

isolated in the positive direction back to t. Then, as follows from the equation

$$\Omega^{t't_0}(u) = \Omega^{t't}(u)\,\Omega^{tt_0}(u),$$

we have

$$\Omega_1(u) = \Omega(u)\,\Phi,$$

where Φ is the value obtained by integrating from t_0 round the corner back to t_0. The group of linear substitutions formed by the combinations of the matrices Φ, one for each corner, is the group in question. More generally, if C be an arbitrary matrix of constants, a matrix whose columns are sets of solutions of the linear system is $\Omega(u)\,C$, and

$$\Omega_1(u)\,C = \Omega(u)\,C\,.\,C^{-1}\Phi C\,;$$

so that we may, instead of the group (Φ), consider the group generated by the substitutions $C^{-1}\Phi C$, which is said to be a translation or transformation of the other. In practice it is convenient to choose C so as to obtain the greatest possible simplicity.

23. Taking now the form we have investigated for linear systems of a certain type

$$\Omega(U) = G\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\,G_0^{-1},$$

the factor Φ for the matrix $\Omega(\Theta t^{-1})$, when t describes a circuit about $t = 0$ from t_0 back to t_0, consists of a matrix having only diagonal elements of the form $e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \ldots$, which we denote by $\omega_1, \omega_2, \ldots$; corresponding to a sequence of roots from $\theta_1, \theta_2, \ldots$, each of which is less by an integer than the preceding, the corresponding quantities ω are equal; namely, in a notation previously employed (§ 14), the first s_1 quantities $\omega_1, \omega_2, \ldots$ are equal; then the following $s_2 - s_1$; and so on.

Denoting the quantity $2\pi i$ by ϵ, the factor Φ for the matrix (§ 15)

$$\Omega(\gamma t^{-1}) = \begin{Bmatrix} \nabla(\gamma_1/t) & & \\ & \nabla(\gamma_2/t) & \\ & & \ddots \end{Bmatrix}$$

is a matrix

$$\begin{Bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \end{Bmatrix}$$

where Δ_1, of s_1 rows and columns, has the form

$$\Delta_1 = 1 + \gamma_1\epsilon + \frac{\gamma_1^2}{2!}\epsilon^2 + \ldots;$$

Δ_2, of $s_2 - s_1$ rows and columns, has a similar form in terms of γ_2; and so on.

If now such a matrix, with separate square matrices arranged about its diagonal, and other elements all zero, be called a diagonal matrix, and two such matrices be considered of the same kind when the numbers of rows and columns in the respective component matrices are the same for both, it is immediately obvious that the product of two such

$$\begin{Bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \end{Bmatrix} \begin{Bmatrix} \Delta_1' & & \\ & \Delta_2' & \\ & & \ddots \end{Bmatrix} = \begin{Bmatrix} \Delta_1\Delta_1' & & \\ & \Delta_2\Delta_2' & \\ & & \ddots \end{Bmatrix}$$

is independent of their order.

Thus it follows that the matrix Φ arising for the matrix $\Omega(\Theta t^{-1})$ is commutable with the matrix $\Omega(\gamma t^{-1})$, and we have

$$\Omega_1(U) = G\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\,\Phi G_0^{-1},$$

where
$$\Phi = \begin{Bmatrix} \omega\Delta_1 & & \\ & \omega'\Delta_2 & \\ & & \ddots \end{Bmatrix},$$

ω denoting the value of $e^{2\pi i\theta_1}$ for $\theta_1, \theta_2, \ldots, \theta_{s_1}$, and ω' the value for $\theta_{s_1+1}, \ldots, \theta_{s_2}$, and so on.

Thus
$$\Omega_1(U) = \Omega(U)\,G_0\Phi G_0^{-1},$$

and the monodromy group of the linear system is generated by linear substitutions of the form

$$G_0\Phi G_0^{-1} = G_0 \begin{Bmatrix} \omega\Delta_1 & & \\ & \omega'\Delta_2 & \\ & & \ddots \end{Bmatrix} G_0^{-1}.$$

24. Consider, for example, particular cases as in § 19.

(*a*) When all the roots $\theta_1, \theta_2, \ldots$ belonging to the corner $t=0$ are different and no two differ by integers, each of $\Delta_1, \Delta_2, \ldots$ reduces to unity.

(*b*) when no two of the roots differ by integers unless they are exactly equal, the matrix Φ is the value of $\Omega\left(\frac{a}{t}\right)$ taken round $t=0$, and equal to
$$1 + a\epsilon + \frac{a^2}{2!}\epsilon^2 + \ldots,$$

and determinable at once on inspection of the differential equations without solution. If then t_0 can be taken so that the matrices G arising for the various corners have all the same value at t_0, the group of the system is particularly simple.

(c) For a corner at which all the roots belong to one sequence, differing by integers or zero, if all the constants c_{ij} of the matrix γ are zero, and no logarithms enter into the solutions about this corner, the matrix Φ reduces to a single constant ω, and the substitution of the group arising for this corner is independent of G_0, reducing to the matrix having only the quantity ω in each diagonal place.

25. Conversely, consider necessary conditions that the group should be finite. Then each substitution must be of finite order, and we must have equations for each corner of the form

$$1 = \Phi''' = \begin{Bmatrix} \omega'''\Delta_1''' & & \\ & \omega''''\Delta_2''' & \\ & & \ddots \end{Bmatrix},$$

and hence $\omega'''\Delta_1''' = 1, \quad \omega''''\Delta_2''' = 1, \quad \ldots,$

of which the first is

$$\omega'''\left(1 + m\gamma_1\epsilon + \frac{m^2\epsilon^2}{2!}\gamma_1^2 + \ldots\right) = 1,$$

where γ_1 is a matrix satisfying, for its equation of lowest order, an equation of the form $\gamma_1^{k_1} = 0$, in which k_1 is at most s_1, and the series in the bracket terminates with the term involving $\gamma_1^{k_1-1}$. This equation therefore involves $\omega''' = 1$ and $\gamma_1 = 0$.

A necessary condition is therefore that all the roots $\theta_1, \theta_2, \ldots$ should be rational numerical fractions and the matrix γ be zero, so that no logarithms enter into the solutions. In case no two of the roots differ by integers or zero, the matrix γ is zero of itself; on the other hand, if every two of the roots differ by integers, and still the matrix γ is zero, the particular substitution $G_0\Phi G_0^{-1}$ is independent of G_0, and reduces to a numerical constant ω, which in the case supposed is a root of unity; for a system derived from a single linear equation we have seen that the invariant factors corresponding to a repeated root θ cannot be linear, and γ cannot be zero when there are repeated roots.

The condition is not generally sufficient. If a set of matrices N generate a finite group, it is known that a single matrix ϖ can be found such that the set of matrices $M = \varpi N \varpi^{-1}$ satisfy the relation $\bar{M}_c M = 1$, where M_c denotes the matrix whose elements are the conjugate imaginaries of those of M, and $\bar{M}_c$ is the transposed of M_c. In our case, denoting Φ, which in the case supposed consists only of diagonal elements which are roots of unity, by Ω, so that $\bar{\Omega}_c \Omega = 1$, the matrices M are of the form

$$M = \varpi G_0 \Omega G_0^{-1} \varpi^{-1}.$$

Part IV.

26. In illustration of the previous theory consider the case of the system

$$\frac{dx}{dt} = \left[\begin{pmatrix} 0 & 0 \\ \alpha\beta & 0 \end{pmatrix} + \frac{1}{t(1-t)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] x = u(t)\, x,$$

in which $\beta = 1-\alpha$, derived from the single linear equation

$$t(1-t)y'' + (1-2t)y' - \alpha(1-\alpha)y = 0$$

by putting $x_1 = y$, $x_2 = t(1-t)y'$. For $t = 0$, $t = 1$ the system is already in canonical form; integrating from $t_0 = \frac{1}{2}$, we have, by the theory,

$$\Omega[u(t)] = g(t) \begin{pmatrix} 1 & \log 2t \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}).$$

We find, however,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -u(t) = -u(1-t);$$

and hence, if $s = 1-t$ and $\Omega[u(t)] = F(t)$, we have, by § 3,

$$F(s) = \Omega\left[u(s)\frac{ds}{dt}\right] = \Omega[-u(t)] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Omega[u(t)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that

$$g(t) \begin{pmatrix} 1 & \log 2t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g(s) \begin{pmatrix} 1 & \log 2s \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g(\tfrac{1}{2}). \tag{A}$$

With $\varpi = t(1-t)$, the subsidiary equations for the determination of the columns of

$$g(t) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

are (§ 16)
$$\frac{dg_{11}}{dt} = \frac{1}{\varpi} g_{21}, \quad \frac{dg_{12}}{dt} = \frac{1}{\varpi} g_{22} - \frac{g_{11}}{t},$$
$$\frac{dg_{21}}{dt} = \alpha\beta g_{11}, \quad \frac{dg_{22}}{dt} = \alpha\beta\, g_{12} - \frac{g_{21}}{t},$$
leading to
$$\frac{d}{dt}(g_{11}g_{22} - g_{12}g_{21}) = 0;$$
so that, as at $t = 0$, so for all t,
$$g_{11}g_{22} - g_{12}g_{21} = 1,$$
the ordinary Abel relation in a disguised form.* The equation (A), with $h_{ij} = g_{ij}(1-t)$ and $\gamma_{ij} = g_{ij}(\tfrac{1}{2})$, is the same as
$$\begin{pmatrix} g_{11} & g_{11}\log 2t + g_{12} \\ g_{21} & g_{21}\log 2t + g_{22} \end{pmatrix}$$
$$= \begin{pmatrix} h_{11} & h_{12} \\ -h_{21} & -h_{22} \end{pmatrix} \begin{pmatrix} 1 & \log 2s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{22} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ -\gamma_{21} & -\gamma_{22} \end{pmatrix}$$
$$= \begin{pmatrix} h_{11} & h_{11}\log 2s + h_{12} \\ -h_{21} & -h_{21}\log 2s - h_{22} \end{pmatrix} \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$
where
$$A = \gamma_{11}\gamma_{22} + \gamma_{12}\gamma_{21}, \quad B = 2\gamma_{12}\gamma_{22}, \quad C = -2\gamma_{11}\gamma_{21},$$
$$A^2 + BC = 1,$$
and leads to four relations effectively all reducible to $g_{11}g_{22} - g_{12}g_{21} = 1$ together with
$$h_{11} = Ag_{11} + C(g_{11}\log 2t + g_{12}). \tag{B}$$

We know that g_{11}, g_{12} are power series in t reducing respectively to 1 and 0 for $t = 0$, and h_{11} is the same power series in s as is g_{11} in t; if
$$g_{11}(t) = 1 + \sum_{n=1}^{\infty} \kappa_n t^n,$$
the equation (B) gives, putting $t=0$, and assuming $\log t$ [and $\log(1-t)$] real and negative for $0 < t < 1$,
$$A + C\log 2 = \Big[g_{11}(1-t) - C\log t\Big]_{t=0}$$
$$= \Big[1 + \sum_{n=1} \Big(\kappa_n + \frac{C}{n}\Big)(1-t)^n\Big]_{t=0},$$
and hence
$$C = (-n\kappa_n)_{n=\infty}.$$

Thus, when the series $g_{11}(t)$ is known, C can be found, and hence A and B, as we shall show below; it concerns us, however, first to show how far the method of this paper enables us to go towards

* In general the determinant of g is the exponential of the integral, from 0 to t, of the sum of the diagonal elements of the matrix v.

determining the group of the equation without calculation of details; we prove, in fact, at once that this group can be transformed so as to be generated by the two substitutions

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2(\pi i C)^2 & 1 \end{pmatrix}.$$

For the solution $\Omega(u)$ we have seen that the two substitutions about $t=0$, $t=1$ are

$$g(\tfrac{1}{2})\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}), \quad \mu g(\tfrac{1}{2})\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2})\mu,$$

where
$$\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now take a matrix p such that

$$pg(\tfrac{1}{2})\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2})p^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

which is known to be possible, since the two matrices

$$\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

have the same roots and invariant factors; putting for the determination of p,

$$pg(\tfrac{1}{2}) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

we find P and Q arbitrary, $R=0$ and $S=\pi i P$; next, assuming P not zero, take a matrix

$$\sigma = p\mu p^{-1} = \begin{pmatrix} P & Q \\ 0 & \pi i P \end{pmatrix} g^{-1}(\tfrac{1}{2})\mu g(\tfrac{1}{2}) \begin{pmatrix} P & Q \\ 0 & \pi i P \end{pmatrix}^{-1}$$

$$= \begin{Bmatrix} A + \dfrac{QC}{P} & \dfrac{1}{\pi i P}\left(PB - 2QA - \dfrac{Q^2C}{P}\right) \\ \pi i C & -\dfrac{1}{P^2}\left(A + \dfrac{QC}{P}\right) \end{Bmatrix},$$

and then, assuming C not zero, take $Q/P = -A/C$, so that, in virtue of $A^2+BC=1$,

$$\sigma = \begin{pmatrix} 0 & \dfrac{1}{\pi i C} \\ \pi i C & 0 \end{pmatrix}.$$

Consider now the solution $\Omega(u)p^{-1}$ of the differential system;

about $t = 0$, $t = 1$ it has the respective factors

$$\Theta = pg(\tfrac{1}{2})\begin{pmatrix}1 & 2\pi i\\ 0 & 1\end{pmatrix} g^{-1}(\tfrac{1}{2})\,p^{-1} = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix},$$

$$\Phi = p\mu g(\tfrac{1}{2})\begin{pmatrix}1 & 2\pi i\\ 0 & 1\end{pmatrix} g^{-1}(\tfrac{1}{2})\,\mu p^{-1},$$

which, by $p\mu = \sigma p$, is equal to

$$\sigma pg(\tfrac{1}{2})\begin{pmatrix}1 & 2\pi i\\ 0 & 1\end{pmatrix} g^{-1}(\tfrac{1}{2})p^{-1}\sigma^{-1} = \sigma\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\sigma^{-1},$$

which on calculation is, as stated,

$$\begin{pmatrix}1 & 0\\ 2(\pi i C)^2 & 1\end{pmatrix}.$$

In fact it will be seen that $g_{11}(t)$ is the hypergeometric series $F(\alpha, \beta, 1, t)$, $\beta = 1-\alpha$, and hence

$$C = -\left[\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{(n-1)!\,n!}\right]_{n=\infty};$$

it is, however, an elementary property of Γ-functions that, for general values of α, β,

$$\left[n\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\,(\alpha+\beta)(\alpha+\beta+1)\dots(\alpha+\beta+n-1)}\right]_{n=\infty} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)};$$

thus here, with $\alpha+\beta = 1$,

$$C = -\frac{1}{\Gamma\alpha\,\Gamma(1-\alpha)} = -\frac{\sin\pi\alpha}{\pi},$$

and the group is generated by

$$\Theta = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \quad \Phi = \begin{pmatrix}1 & 0\\ -2\sin^2\pi\alpha & 1\end{pmatrix},$$

For instance, for $\alpha = \frac{1}{2}$, $\alpha = \frac{1}{3}$, respectively,

$$\Phi = \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\ -\frac{3}{2} & 1\end{pmatrix}.$$

Returning now to the determination of the matrix g, the equations for the first column are, with $\beta = 1-\alpha$,

$$t(1-t)\frac{dg_{11}}{dt} = g_{21}, \quad \frac{dg_{21}}{dt} = \alpha\beta g_{11},$$

which are satisfied by

$$g_{11} = F(\alpha, \beta, 1, t), \quad g_{21} = t(1-t)\frac{dF}{dt},$$

reducing respectively to 1 and 0 for $t = 0$, if only the identity

$$\alpha\beta F = t(1-t)F'' + (1-2t)F'$$

is satisfied, as is known; these are then the values for g_{11} and g_{21}. The equations for the second column

$$\frac{dg_{12}}{dt} = \frac{g_{22}}{t(1-t)} - \frac{g_{11}}{t}, \quad \frac{dg_{22}}{dt} = \alpha\beta g_{12} - \frac{g_{21}}{t}.$$

are to be satisfied by forms

$$g_{12} = \sum_{n=1}^{\infty} \lambda_n t^n, \quad g_{22} = t(1-t)\frac{dg_{12}}{dt} + (1-t)g_{11},$$

which on substitution in the second equation are found to give

$$g_{12} = \sum_{n=1} \kappa_n t^n \left[\frac{1}{\alpha} + \ldots + \frac{1}{\alpha+n-1} + \frac{1}{\beta} + \ldots \right.$$
$$\left. \ldots + \frac{1}{\beta+n-1} - 2\left(1 + \ldots + \frac{1}{n}\right)\right],$$

which can be shown to be the same as

$$g_{12} = -\log(1-t)F(\alpha, \beta, 1, t) - 2\sum_{n=1} \kappa_n t^n \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$

Thus C is as stated above, and hence from equation (B)

$$A = \frac{\sin \pi\alpha}{\pi}\log 2 + \left[F(\alpha, 1-\alpha, 1;\ 1-t) + \frac{\sin \pi\alpha}{\pi}\log t\right]_{t=0}.$$

To evaluate this we use the identity holding for general values of α, β (St. John's College, Cambridge, Examination Paper of June 4th, 1894, 9–12),

$$2\psi(1) - \psi(\alpha) - \psi(\beta) = \lim_{t=0}\left[\log t + \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}F(\alpha, \beta, \alpha+\beta, 1-t)\right],$$

where
$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)};$$

by $\beta = 1-\alpha$, we thence at once find

$$A = \frac{\sin \pi\alpha}{\pi}\left[\log 2 + 2\psi(1) - \psi(\alpha) - \psi(1-\alpha)\right]$$
$$= \frac{\sin \pi\alpha}{\pi}\log 2 + \frac{2\sin \pi\alpha}{\pi}\left[\psi(1) - \psi(\alpha)\right] - \cos \pi\alpha;$$

and hence, from $A^2 + BC = 1$,

$$B = \frac{\sin \pi\alpha}{\pi}\left\{\left[\log 2 + 2\psi(1) - 2\psi(\alpha) - \pi\cot \pi\alpha\right]^2 - \pi^2\operatorname{cosec}^2 \pi\alpha\right\},$$

whereby, in fact, the relation connecting the fundamental integrals of the original hypergeometric equation about $t=0$ and $t=1$, for a general value of a, is found in a manner which appears to the writer simpler than that employed* by Tannery for the particular case $a=\frac{1}{2}$. It can be proved that

$$\psi(1)-\psi(\tfrac{1}{2}) = 2\log 2, \quad \psi(1)-\psi(\tfrac{1}{3}) = \tfrac{1}{2}(3\log 3+\pi/\sqrt{3});$$

thus, for $a=\frac{1}{2}$,

$$A = \frac{5}{\pi}\log 2, \qquad B = \frac{1}{\pi}\left[(5\log 2)^2-\pi^2\right],$$

and, for $a=\frac{1}{3}$,

$$A = \frac{\sqrt{3}}{2\pi}\log(54), \quad B = \frac{\sqrt{3}}{2\pi}\left[(\log 54)^2-\tfrac{4}{3}\pi^2\right].$$

Another remark seems worth making. We have had the relations expressing the integrals about $t=0$ in terms of those about $t=1$; by elimination of the constants we obtain four functions of these integrals which are constant; putting

$$\xi = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} 1 & \log 2t \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} 1 & \log 2s \\ 0 & 1 \end{pmatrix},$$

these relations are, in fact,

$$\bar{\xi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\eta = \begin{pmatrix} -C & A \\ A & B \end{pmatrix},$$

and can be obtained in this form directly from the two expressions for $\Omega(u)$.

[*June* 17*th*, 1903.—The following deal with systems of linear equations:—Königsberger, *Lehrbuch der . . . Differentialgleichungen*, 1889, pp. 441–469. Sauvage, *Ann. de l'Ec. norm.*, 1886, 1888, 1889. Sauvage, *Toulouse Ann.*, Vol. VIII., 1894, pp. 1–24; Vol. IX., 1895, pp. 25–100 and pp. 1–75. Grünfeld, *Denkschr. der Wiener Akad.*, *math.-naturw. Cl.*, Bd. LIV., 1888. Horn, *Math. Ann.*, Vol. XXXIX., 1891, pp. 391–408, and Vol. XL., 1892, pp. 527–550. Picard, *Traité d'Analyse*, Vol. III., 1896, p. 266. Dunkel, *American Akad.*, May 14th, 1902.]

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* Repeated in Forsyth's *Linear Differential Equations* (1902), pp. 129–134. We remark in passing that the first four lines of p. 148 of that volume do not appear to be correctly printed.