

THE COLLINEARITIES OF THE POINTS OF INFLEXION OF A NON-SINGULAR PLANE CUBIC AND THEIR PRETANGENTIALS.

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1. If a point P be taken on a cubic, it is in general possible to draw four tangents from it to touch the curve elsewhere. From each of the points of contact four tangents may likewise be drawn, and so on indefinitely. We may speak of the first four points as Pre-tangentials of P of the first order, P being their common Post-tangential. More generally, by a Pretangential of P of the m^{th} order is meant a point whose m^{th} Post-tangential taken consecutively is P .

When P is an inflexion the peculiarity arises that it may be considered as a Pretangential or Post-tangential of the m^{th} order to itself. For the same reason the Pretangentials of the m^{th} order for an inflexion include all those of lower order, while the inflexion has no Post-tangentials distinct from itself. By a proper Pretangential of P of the m^{th} order is meant a point Q whose m^{th} Post-tangential is the first to coincide with P .

It is then easily proved that the line joining two Pretangentials of the same or different inflexions cuts the cubic again in a point which is a Pretangential of an inflexion.

These have already been discussed in the simplest case of the Pretangentials of the first order by EMCH ¹⁾ and GIRAUD ²⁾.

I propose to consider these in greater detail and to furnish a general method for their discussion by synthetic geometry.

The Inflexions.

2. It is a familiar theorem that the join of two inflexions cuts the cubic again in an inflexion.

¹⁾ *The Configurations of the Points of Inflexion of a Plane Cubic and Harmonic Polars* [Rendiconti del Circolo Matematico di Palermo, t. XXIII (1° sem. 1907), pp. 251-254].

²⁾ *Complemento ad una Nota del sig. EMCH* [Rendiconti del Circolo Matematico di Palermo, t. XXIV (2° sem. 1907), pp. 44-45].

The nine inflexions could then be obtained from three non-collinear inflexions, 1, 2, 3, as follows:

Form the collinearities

1	2	<i>iii</i>
2	3	<i>i</i>
3	1	<i>ii</i>

so that *i*, *ii*, *iii*, are three new inflexions. They are *not* collinear.

Form the collinearities

<i>i</i>	<i>ii</i>	<i>III</i>
<i>ii</i>	<i>iii</i>	<i>I</i>
<i>iii</i>	<i>i</i>	<i>II</i>

and we obtain three new inflexions likewise non-collinear. From I, II, III, we come back to 1, 2, 3 for then

<i>I</i>	<i>II</i>	3
<i>II</i>	<i>III</i>	1
<i>III</i>	<i>I</i>	2.

In addition there are the collinearities

1	<i>i</i>	<i>I</i>
2	<i>ii</i>	<i>II</i>
3	<i>iii</i>	<i>III</i>

and the 12 «axes of inflexions» are then completed ³⁾.

Pretangentials of the First Order for the Inflexions.

3. From each inflexion three tangents distinct from the inflexional tangent may be drawn, and their three points of contact lie on a line, the harmonic polar of the inflexion.

Any chord joining an inflexion to a Pretangential of another inflexion must pass through a Pretangential of a third inflexion. Let I_1, I_2, \dots, I_9 be the 9 inflexions, P a Pretangential of I_k , and form the collinearity IPP' . Their tangentials must be collinear, therefore $II_k X$, say. Hence X is an inflexion and P' is a Pretangential of X . (Of course when P belongs to I we obtain IPP).

In exactly the same way it follows that the join of two Pretangentials passes either through a third Pretangential or through an inflexion. The joins formed by the inflexions and the Pretangentials must therefore furnish a closed system which we proceed to investigate.

4. There are, in the first place, the 12 axes of inflexions, the 9 harmonic polars,

³⁾ Vide WHITE, *Inflexional Lines, Triplets, and Triangles Associated with the Plane Cubic Curve* [Bulletin of the American Mathematical Society, Vol. IV (1898), pp. 258-260].

and the 27 tangents from the inflexions. If we join an inflexion to the 24 Pretangentials of the other inflexions we obtain 12 lines. Hence there are 108 lines each through an inflexion and two Pretangentials. There are also 72 lines, each through three Pretangentials. For there are 12 axes of inflexions. Let I_1, I_2, I_3 be an axis with Pretangentials $P_1, P_2, P_3; Q_1, Q_2, Q_3; R_1, R_2, R_3$ respectively. Then P_1 joined to the tetrad I_2, Q_1, Q_2, Q_3 furnishes, in some order, the tetrad R_1, I_3, R_2, R_3 , say. Two only of the lines through P_1 pass through other two Pretangentials. The three points P thus furnish 6 such collinearities for the axis I_1, I_2, I_3 . Hence, in all, there are 72 such lines.

5. Let P_i belong to I_i , and form the collinearities

$$\begin{array}{ccc} P_1 & I_1 & P_1 \\ P_1 & I_2 & P_2 \\ \cdot & \cdot & \cdot \\ P_1 & I_9 & P_9. \end{array}$$

The 9 points P are all distinct, and by examining their Post-tangentials and using § 2 we see that they belong to the different inflexions.

The join of any two of the points P passes through an inflexion. Thus from the collinearities

$$\begin{array}{ccc} P_m & P_i & I_m \\ P_n & P_i & I_n \\ X & I_i & I_k, \text{ say,} \end{array}$$

which may be read in rows or columns, it follows that X must be an inflexion. Moreover, the inflexion through which P_m, P_n passes is sufficiently indicated. It also follows that the join of any inflexion to a point P passes through another point P' of the system.

The nine inflexions and the nine points P thus furnish a closed system of collinearities, 36 of these joining an inflexion to two Pretangentials.

6. Let Q_i be a Pretangential distinct from the 9 P 's. From it we can form the system Q_1, Q_2, \dots, Q_9 with similar properties, by joining Q_i to the inflexions. Similarly the remaining 9 form a system R_1, R_2, \dots, R_9 . Now join P_i to the points Q . We obtain 9 collinearities furnishing 9 points which can be neither I 's, P 's, nor Q 's. They are therefore the 9 points R . There are 81 such collinearities, including the 9 harmonic polars.

7. The points as so arranged give rise to minor closed systems of collinearities, some examples of which may be noted.

Thus, if three of the 27 points be taken which have collinear inflexions, the collinearities give rise at most to 12 points, viz. the three inflexions and the corresponding pretangentials. But if the three points taken are three P 's only 6 points are found, the three inflexions and the three points P given.

If we start with three pretangentials which are not all of the same denomination

and which do not possess collinear inflexions, the 27 pretangentials and the 9 inflexions may be determined therefrom by collinearities.

In particular, suppose they are all of unlike denomination, say, P_1, Q_2, R_3 belonging to non-collinear inflexions I_1, I_2, I_3 . The sides of $\Delta P_1 Q_2 R_3$ will cut the cubic in three new non-collinear points R_{111}, Q_{11}, P_i . The sides of P_i, Q_{11}, R_{111} will cut the curve again in P_{II}, Q_{III}, R_{III} . The sides of P_{II}, Q_{III}, R_{III} cut again in the points P_1, Q_2, R_3 . All the points so found belong to different inflexions (§ 2).

To obtain the inflexions take

$$\begin{array}{ccc} P_1, & Q_2, & R_{111} \\ P_1, & R_3, & Q_{11}; \\ \text{therefore } I_1, & P_i, & P_{II}, \text{ etc.} \end{array}$$

Having thus found the 9 inflexions, we may determine the 9 points of each denomination in a variety of ways which it seems needless to discuss.

Pretangentials proper of the Second Order.

8. Before proceeding further, we note the interesting theorem that.

If $P_{(m)}$ denote a Pretangential proper of an inflexion of the m^{th} order, then the join of two points $P_{(m)}, P_{(m+n)}$ passes through a point $P'_{(m+n)}$.

For example, join I to $P_{(2)}$ and form the collinearity $I, P_{(2)}, X$. Take the Post-tangentials and we obtain the collinearity I, P_2, \dots and the third point must therefore be P_2 say. Hence X is a point $P'_{(2)}$.

In exactly the same way, by taking post-tangentials, it may be proved that the collinearity beginning with $P_{(k)}, P_{(k+l)}$ has for third point $P'_{(k+l)}$.

Also it will appear that the three points belong either to three collinear inflexions or to the same inflexion.

9. We may now discuss the collinearities which include the Post-tangentials of the second Order.

Let the Pretangentials of I_1 be P_1, Q_1, R_1 . Let the Pretangentials of P_1 be $P_{11}, P'_{11}, \pi_{11}, \pi'_{11}$.

Also let I_1, P_{11}, P'_{11} form a collinearity; therefore so do I_1, π_{11}, π'_{11} .

Join P'_{11} to the nine inflexions forming the collinearities

$$\begin{array}{ccc} P'_{11}, & I_1, & P_{11} \\ P'_{11}, & I_2, & P_{21} \\ \dots & \dots & \dots \\ P'_{11}, & I_9, & P_{91}. \end{array}$$

The points in the third column are of the second order. They form a set of nine points, the joins of pairs of which pass through the 9 points P_1, P_2, \dots, P_9 .

Thus

$$\begin{array}{ccc} P'_{11}, & P'_{11}, & P_1, \\ I_2, & I_3, & I_k \quad (\text{say}), \\ \text{therefore } P_{21}, & P_{31}, & P_k. \end{array}$$

Thus the joins of a point P_{k1} to the nine points $P_1 \dots P_9$ furnish the nine points P'_{11}, \dots, P'_{91} .

10. Join the point P_{11} to the inflexions, when we obtain the points $P'_{11}, P'_{21}, \dots, P'_{91}$ say; and they possess similar properties, i.e. The join of any two of them passes through one of the points P_1, \dots, P_9 . They are, in fact, Pretangentials of the 9 points P_1, \dots, P_9 .

But, in addition, the join of any dashed letter P'_{k1} to any undashed letter P_{m1} passes through an inflexion

$$\begin{array}{ccc} \text{for } P'_{k1}, & I_k, & P_{11} \\ & P_{m1}, & I_m, & P'_{11} \\ \text{therefore } I_i, & I_l, & I_i \quad \text{say.} \end{array}$$

Thus we get the same dashed letters by joining any one of the 9 points P_{11}, \dots, P_{91} to the inflexions; and the 18 points thus found form a closed system in conjunction with the 9 inflexions and the 9 points P_1, \dots, P_9 .

11. If we next start with π_{11} we obtain other two sets of 9 points

$$\begin{array}{ccc} \pi_{11}, & \dots, & \pi_{91}; \\ \text{and} & & \pi'_{11}, \dots, \pi'_{91} \end{array}$$

possessing exactly similar properties. We also thus obtain the 36 Pretangentials of the 9 points P_1, \dots, P_9 .

What happens if we join P_{11} to the points $\pi_{11}, \dots, \pi_{91}$? P_{11}, π_{11} must pass through either Q_i or R_i , say Q_i (since it can pass through neither I_i nor P_i). Then the joins of any two points P_{k1}, π_{l1} passes through one of the 9 points Q_1, \dots, Q_9 .

Thus

$$\begin{array}{ccc} P_{11}, & \pi_{11}, & Q_i \\ & P_{k1}, & \pi_{l1}, & X \\ \text{therefore } P_m, & P_n, & I_r \end{array}$$

Hence X is Q_r .

Similarly, the join of P_{k1} and π'_{l1} passes through one of the 9 points R_1, \dots, R_9 ; as do the points P'_{k1}, π_{l1} ; while the joins P'_{k1}, π'_{l1} pass through Q_1, \dots, Q_9 .

We have thus a closed system consisting of the inflexions, their first Pretangentials, and the 36 Pretangentials of P_1, \dots, P_9 .

12. To the collinearities just obtained are to be added the closed systems formed with the aid of (i) the 36 Pretangentials of Q_1, \dots, Q_9 , and (ii) the 36 Pretangentials of R_1, \dots, R_9 .

Owing to these systems being closed, it follows that the join of any point P_{k_1} to any point Q_{l_1} passes through a point R_{m_1} . There are therefore $36^2 = 1296$ collinearities each through three Pretangentials of the second order.

Generalisation.

13. The collinearities obtained from the Pretangentials of the third order might similarly be constructed gradually with the aid of those already obtained, etc.

These are not the only groups of points on the cubic which lead to closed systems of collinearities, but they are the simplest of the kind and frequently form part of more extensive systems.

With respect to such groups of points, we may note a few theorems.

14. Let I_1, I_2, \dots, I_n be a group of points on the cubic such that the join of any two of the points passes through another of the group:

I. Let A_1 be any point on the cubic, and form the collinearities by joining A_1 to the points I

$$\begin{array}{l} A_1, I_1, B_1 \\ A_1, I_2, B_2 \\ \dots \dots \dots \\ A_1, I_n, B_n. \end{array}$$

Then the join of any two points B passes through one of n points.

Thus

$$\begin{array}{l} A_1, I_k, B_k \\ A_1, I_l, B_l \end{array}$$

Therefore α_1, I, \dots

where α_1 is the Post-tangential of A_1, \dots , etc.

II. Let the points I include their own Post-tangentials and join B_1, \dots, B_n to I_1, \dots, I_n , when only n distinct points are obtained, A_1, A_2, \dots, A_n .

Thus form

$$\begin{array}{l} B_1, I_1, A_1 \\ B_1, I_2, A_2 \\ \dots \dots \dots \\ B_1, I_n, A_n. \end{array}$$

Then

$$\begin{array}{l} B_2, I_2, X \\ A_1, I_1, B_1 \\ I_2, I_1, I_m \end{array}$$

i. e. X is the point A_m .

III. Let the points I be as in II, and let $\alpha_1, \alpha_2, \dots, \alpha_n; \beta_1, \beta_2, \dots, \beta_n$ be the post-tangentials of $A_1, \dots, A_n; B_1, \dots, B_n$. Then the two groups of Post-tangentials are similarly related, and the join of an α to a β passes through a point I .

IV. If one of the points A coincides with one of the points B , then the two groups are identical, and in such a case the Post-tangentials of the group A belong to the group of points I .

Similarly, the points A may be distinct from points B , while the points α and β are coincident.

V Let A'_i be a point distinct from $A_1, \dots, A_n; B_1, \dots, B_n$; and form the analogous groups by joining it to the points I , viz.,

$$A'_1, \dots, A'_n; \quad B'_1, \dots, B'_n.$$

Then the collinearities formed by joining a point B to a point B' pass through n points at most.

Thus

$$A'_1, \quad I_1, \quad B'_1$$

$$A'_i, \quad I_i, \quad B'_i$$

$$a_i, \quad I_i, \quad \dots$$

in which a_i, A_i, A'_i form a collinearity.

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