Partial Operational Completeness of the Complex Numeric Representational System

Working Paper — Problem 4 of the CNRS Programme

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Abstract

Problem 4 of the Complex Numeric Representational System (CNRS) programme asks whether the system is *complete* in three distinct senses: coverage of $\mathbb{C}$, metric completeness in the $(-2+i)$-adic topology, and operational closure under all intended arithmetic operations. The first sense (Q1, coverage) is essentially resolved by the Thurston tiling theorem. The second (Q2, metric) requires p-adic specialist input and remains open. The third (Q3, operational) is proved here in the *partial* sense defined precisely below.

With Problem 3 now complete [9], we can state and prove *partial operational completeness* of the CNRS system across its three layers:

- **Q3a (Layer 1, CNRS-A)**: CNRS-A is closed as a *ring* under addition, subtraction, and multiplication. All three operations are computable by explicit finite automata. The addition transducer has exactly 14 states and 350 transitions. Restricted division (by base powers and Gaussian units) is also closed. General division remains open.
- **Q3b (Layer 2)**: Logarithm and exponentiation are single-valued and operationally closed in the extended element representation.
- **Q3c (Layer 3, CNRS-H)**: Differentiation and integration are operationally closed as exact fixed digit-shift operators.

What remains open in Problem 4: metric completeness (Q2); general division in CNRS-A; differentiation directly on CNRS-A digit strings (blocked on the e-base CNS theorem, the central open problem of the programme).

Contents

1 Introduction and Context

1.1 The four problems of the CNRS programme

The Complex Numeric Representational System (CNRS) programme [5] constructs a positional numeral system in which complex numbers appear as single values and integration and differentiation are the structural primitive operations. The programme is organised around four problems:

	Problem	Content
P1	Base definition	Rigorous $(-\beta)$-expansion for all $\beta > 1$; admissibility; the uniform conjecture
P2	Single-value encoding	Three-layer architecture: CNRS-A (Layer 1, arithmetic), Layer 2 (branch/logarithm), CNRS-H (Layer 3, calculus)
P3	Arithmetic closure	Addition and multiplication of CNRS-A digit strings computable by finite automata
P4	Completeness	Coverage of $\mathbb{C}$; topological and metric completeness; operational closure

The present paper addresses Problem 4. With P3 now complete [9], the operational completeness sub-questions (Q3a–Q3c) can be stated and proved.

1.2 Three senses of completeness, decomposed by layer

Definition 1.1 (Three senses of completeness for CNRS)**.** *(Q1)* ***Coverage completeness (Q1).*** *Every $c \in \mathbb{C}$ has a representation as a bi-infinite digit string in base $-2+i$ with digits in $\{0,1,2,3,4\}$.*

(Q2) ***Metric completeness (Q2).*** *The $(-2+i)$-adic metric $d(c,c') = |\operatorname{Val}(c-c')|_{-2+i}$ defines a topology on CNRS-A digit strings; is the completion of this metric space isomorphic (as a topological ring) to $\mathbb{C}$?*

(Q3) ***Operational completeness (Q3).*** *The system is closed under its intended arithmetic operations, decomposed by layer:*

- *(Q3a)* Layer 1 (CNRS-A)*: ring closure under addition, subtraction, and multiplication; restricted division by base powers and units.*
- *(Q3b)* Layer 2*: closure under logarithm and exponentiation via the extended element representation.*

(Q3c) Layer 3 (CNRS-H): *closure under differentiation and integration as digit-shift operations on place values* $z_0^n/n!$.

1.3 Status before this paper

Sub-question	Status (pre-P3 completion)	Blocker
Q1 (coverage)	Essentially resolved	None; Thurston (1989)
Q2 (metric)	Open	Requires p-adic specialist
Q3a (Layer 1)	Partially open	P3 (addition, multiplication)
Q3b (Layer 2)	Resolved	P2 capstone [7]
Q3c (Layer 3)	Resolved	P2 capstone [7]

This paper proves the Q3a results that become available from P3 completion, and records Q3b and Q3c for completeness.

2 The CNRS-A System: Definitions

We recall the necessary definitions from [7, 9, 8].

Definition 2.1 (CNRS-A digit strings). *Let* $z_0 = -2+i \in \mathbb{Z}[i]$ *and* $\mathcal{D} = \{0, 1, 2, 3, 4\}$. *A* CNRS-A digit string *is a bi-infinite sequence* $(d_n)_{n\in\mathbb{Z}}$ *with* $d_n \in \mathcal{D}$ *and only finitely many non-zero terms with positive index (i.e. the string has a finite integer part). The* value map *is*

$$\mathrm{Val}((d_n)) = \sum_{n\in\mathbb{Z}} d_n(-2+i)^n \in \mathbb{C}.$$

A digit string is finite *if* $d_n = 0$ *for all* $n < n_0$ *and all* $n > n_1$ *for some* $n_0, n_1 \in \mathbb{Z}$. *Finite strings represent Gaussian integers and their base-*z_0 *translates.*

Definition 2.2 (Canonical number system property). *Base* $z_0 = -2+i$ *is a* canonical number system *(CNS) base: every* $\alpha \in \mathbb{Z}[i]$ *has a unique finite representation* (d_n) *with* $d_n \in \mathcal{D}$. *By Kátai–Szabó (1975) [1] and the (F) property [4], this is equivalent to saying that every element of* $\mathbb{Z}[i]$ *has a finite* $(-2+i)$*-expansion.*

Definition 2.3 ((F) property). *Base* $z_0 = -2+i$ *has the* finiteness property (F)*: every element of* $\mathbb{Z}[i]$ *has a finite* $(-2+i)$*-expansion, equivalently, the carry sequence in the addition algorithm terminates in finitely many steps from any starting carry. The carry automaton is acyclic from every non-zero carry state.*

3 Coverage Completeness (Q1): Essentially Resolved

Q1: Coverage completeness (Thurston 1989, recalled)

Every $c \in \mathbb{C}$ has a representation as a bi-infinite CNRS-A digit string. Finite representations cover all $\alpha \in \mathbb{Z}[i]$. For arbitrary $c \in \mathbb{C}$, the representation is bi-infinite (finitely many positive-index digits, infinitely many negative-index digits) and is unique for Lebesgue-almost all c (with non-uniqueness on the tiling boundary, a set of Lebesgue measure zero).

Remark 3.1. *The precise statement is that the attractor of the iterated function system $\{f_d(z) = (z - d)/z_0 : d \in \mathcal{D}\}$ tiles $\mathbb{C}$ by $\mathbb{Z}[i]$-translates, giving the representation [2]. The Thurston coverage chain (IFS attractor → complete residue system → bi-infinite coverage of $\mathbb{C}$) is fully spelled out in the Layers 1+2 standalone paper [8], which also establishes the Lemma 2.4 (Thurston coverage chain) used there. This paper imports Q1 from those results; it does not re-prove it. The non-uniqueness boundary caveat should be kept prominent: coverage completeness holds almost everywhere, not universally.*

4 Partial Operational Completeness: Layer 1 (Q3a)

4.1 Addition

Theorem 4.1 (Addition is operationally complete in CNRS-A). *Given any two CNRS-A digit strings $A = (a_n)$ and $B = (b_n)$ representing $\alpha, \beta \in \mathbb{C}$, there exists a CNRS-A digit string $C = (c_n)$ representing $\alpha + \beta$, computable by a finite transducer operating left-to-right on the input streams.*

The transducer has exactly 14 *states and* 350 *transitions. Its transition function is given by the carry recurrence*

$$c_k = (a_k + b_k + \kappa_k) \bmod (-2 + i), \tag{1}$$

$$\kappa_{k+1} = \frac{a_k + b_k + \kappa_k - c_k}{-2 + i}, \tag{2}$$

where $c_k \in \mathcal{D}$, κ_{k+1} is the new carry, and the reachable carry set is the 14-element set

$$\mathcal{K} = \{0,\ 1,\ -1,\ i,\ -i,\ 1{+}i,\ -1-i,\ -2,\ 2{+}i,\ -2-i,\ -2-2i,\ 2{+}2i,\ -3-i,\ -3-2i\}.$$

Proof. The (F) property (Definition 2.3) guarantees that the carry sequence eventually reaches $\kappa = 0$ and stays there. The exact carry set $|\mathcal{K}| = 14$ is established in [9] by exhaustive breadth-first search: every carry reachable from any input pair $(a_k, b_k) \in \mathcal{D}^2$ and any carry $\kappa \in \mathcal{K}$ is itself in $\mathcal{K}$, and every state in $\mathcal{K}$ reaches carry 0 within at most 5 steps (maximum drain chain length). The bounding rectangle $|\operatorname{Re}(\kappa)| \leq 3$, $|\operatorname{Im}(\kappa)| \leq 2$ is tight (attained by $-3-i$ and $-2-2i$ respectively). Correctness: $\operatorname{Val}(C) = \operatorname{Val}(A) + \operatorname{Val}(B)$ follows from the carry recurrence and the ring-homomorphism property of Val [9]. □

Corollary 4.2 (Subtraction)**.** *Subtraction is operationally complete. Since -1 has a finite $(-2+i)$-expansion by the (F) property (specifically, $-1 = (4, 4, 1)$ in the greedy expansion [9]), negation $-\alpha = (-1)\cdot\alpha$ reduces to one-argument multiplication (Theorem 4.4).*

4.2 Multiplication

Lemma 4.3 (Cauchy convolution law)**.** *If $A = \sum_n d_n z_0^n$ and $B = \sum_m e_m z_0^m$ are two CNRS-A digit strings, the naive product coefficients are*

$$(A \cdot B)_k = \sum_{n+m=k} d_n \, e_m \quad \in \mathbb{Z}, \tag{3}$$

and $\operatorname{Val}(A \cdot B) = \operatorname{Val}(A) \cdot \operatorname{Val}(B)$: the value map is a ring homomorphism [9]. The naive coefficients may lie outside $\mathcal{D}$; carry normalisation (a 14-state Phase 2 transducer) reduces them to $\mathcal{D}$.

Theorem 4.4 (One-argument multiplication: single-pass transducer)**.** *Let $B = (e_0, \ldots, e_{J-1})$ be a fixed CNRS-A digit string of length J representing multiplier $c \in \mathbb{Z}[i]$. Multiplication of any CNRS-A string A by B is computable by a single-pass finite transducer with $|\mathcal{K}_c| \cdot 5^{J-1}$ states, where $\mathcal{K}_c$ is the multiplier-specific multiplication carry set for c.*

The multiplication carry set $\mathcal{K}_c$ depends on c, not just on the base. For $c = 2$ $(J = 1)$: $|\mathcal{K}_2| = 14$, giving 14 states and 70 transitions (the minimal non-trivial case). For $c = 3$ $(J = 1)$: $|\mathcal{K}_3| = 32$. Full table in [10].

Proof. The transducer state encodes: (i) the current carry $\kappa \in \mathcal{K}_c$ ($|\mathcal{K}_c|$ choices, finite by the (F) property), and (ii) the last $J-1$ digits of A seen so far (5^{J-1} choices), sufficient to compute the contribution of A's current digit to all convolution positions up to lag $J-1$. The carry influence of any digit of A terminates within a bounded number of positions (by the (F) property), ensuring the state space is finite and the automaton halts. Termination of carry normalisation follows from the (F) property. See [9, 10] for full detail and verified examples. □

Theorem 4.5 (Two-argument sequential multiplication: single-pass)**.** *If $B = (e_0, \ldots, e_{N_B-1})$ is fully known before processing begins, multiplication of any CNRS-A string A by*

B is computable in a single pass over A, with state space $|\mathcal{K}_B| \cdot 5^{N_B-1}$ (where B is treated as the fixed multiplier).

Proof. Knowing B entirely before processing A reduces this to the one-argument case (Theorem 4.4) with $J = N_B$. □

Theorem 4.6 (Two-argument online multiplication: two passes necessary)**.** *When both A and B are presented as simultaneous input streams of unbounded length, no finite-state single-pass online transducer can compute $A \cdot B$. Two passes always suffice.*

Proof. The impossibility applies specifically to *finite-state, one-pass, online transduction of two simultaneous unbounded input streams.* This is not a statement that multiplication is generally non-computable in one pass; it is specific to the online setting with no prior knowledge of either input.

By a pigeonhole argument on the state space [9]: suppose a single-pass transducer with S states exists. For any $N > S$, choose inputs A, A' of length N differing only in their first digit. By pigeonhole, the transducer is in the same state after reading S positions from both streams. But the Cauchy convolution coefficient (3) at position N involves the pair (d_0, e_N), which depends on the first digit of A — a dependency that cannot be recovered from a state reached via two indistinguishable prefixes. Contradiction.

Two passes always suffice: Phase 1 computes naive Cauchy convolution coefficients; Phase 2 applies the 14-state carry normalisation transducer. □

Corollary 4.7 (Division by base and by units)**.** *Division by the base $z_0 = -2 + i$ is a single digit shift (right by one position). Division by units $\{\pm 1, \pm i\}$ of $\mathbb{Z}[i]$ reduces to one-argument multiplication by the unit inverse, which has a length-1 CNRS-A representation. Both are computable by a 14-state transducer.*

4.3 Master theorem: ring closure of CNRS-A

Theorem 4.8 (CNRS-A is ring-closed under addition, subtraction, and multiplication)**.** *Given any two CNRS-A digit strings A and B:*

1. *$A + B$ is a CNRS-A digit string, computed by the 14-state addition transducer (Theorem 4.1).*
2. *$A - B$ is a CNRS-A digit string, computed by negation followed by addition (Corollary 4.2).*
3. *$A \cdot B$ is a CNRS-A digit string, computed by the two-phase multiplication algorithm (Lemma 4.3, Theorems 4.4–4.6).*

In each case the output is an explicit CNRS-A digit string and the computation is realised by a finite automaton. CNRS-A is therefore operationally closed as a ring under these three operations. Division is not a ring operation; its status is stated separately in Corollary 4.7 (closed for base powers and units) and Open Problem 2 (open for general Gaussian integers).

4.4 Summary table: partial operational completeness of CNRS-A (Q3a)

Partial operational completeness of CNRS-A

Operation	Closed?	Automaton / authority
Addition	**Yes**	14-state, 350-transition transducer; Thm 4.1, [9]
Subtraction	**Yes**	Negation + addition; Cor 4.2, [9]
Mult. (one-arg, J-digit multiplier c)	**Yes**	Single-pass; $\lvert\mathcal{K}_c\rvert \cdot 5^{J-1}$ states; Thm 4.4, [9, 10]
Mult. (two-arg, B known first)	**Yes**	Single-pass; $\lvert\mathcal{K}_B\rvert \cdot 5^{N_B-1}$ states; Thm 4.5
Mult. (two-arg, online)	**Yes**	Two passes; Phase 2 has 14 states; Thm 4.6, [9]
Division by base z_0	**Yes**	Single digit shift; Cor 4.7
Division by units $(\pm 1, \pm i)$	**Yes**	14-state transducer; Cor 4.7
General division	**Open**	Open Problem 2

5 Operational Completeness: Layers 2 and 3 (Q3b, Q3c)

5.1 Layer 3: CNRS-H calculus layer (Q3c)

The CNRS-H system (Layer 3 of Problem 2 [7]) uses place values $\Pi_n = z_0^n/n!$ rather than z_0^n. Its operational completeness under differentiation and integration is a structural consequence of the EGF digit convention and does not depend on the P3 arithmetic closure results. Full development is in [7].

Q3c: CNRS-H closed under differentiation and integration

In CNRS-H, with the EGF digit convention ($d_n = g^{(n)}(0)$, raw n-th derivative at 0):

1. **Differentiation.** The left digit-shift operator $\mathcal{D}_H : (d_n)_{n\geq 0} \mapsto (d_{n+1})_{n\geq 0}$ (dropping the leading digit) exactly realises $d/d\rho$:

$$\mathrm{Val}_H\big(\mathcal{D}_H(f),\, \rho\big) = \frac{d}{d\rho}\,\mathrm{Val}_H(f, \rho).$$

   This is the EGF digit-shift identity; the $n!$ denominators in $\Pi_n = z_0^n/n!$ make simple index-dropping exact [7].
2. **Integration.** The right digit-shift operator (prepending a zero digit) realises indefinite integration; the prepended digit encodes the constant of integration.
3. **Round-trip.** $(d/d\rho) \circ \int d\rho = \mathrm{id}$ holds exactly at the digit level.
4. **Product rule.** $d/d\rho[f \cdot g] = f' \cdot g + f \cdot g'$ holds at the digit level via the binomial convolution law.

All four are realised by fixed digit-shift operators. CNRS-H is operationally closed under differentiation and integration.

Remark 5.1 (CNRS-A vs CNRS-H: distinct layers, distinct jobs)**.** *CNRS-A (Layer 1) and CNRS-H (Layer 3) use the same base $z_0 = -2 + i$ but different place-value conventions (z_0^n vs $z_0^n/n!$) and therefore different digit conventions. The operational closure results are independent: CNRS-A closure is arithmetic (finite transducers); CNRS-H closure is analytic (fixed digit-shift operators).*

5.2 Layer 2: logarithm and exponentiation (Q3b)

Q3b: Layer 2 closed under logarithm and exponentiation

In Layer 2 [7], the extended element $\tilde{z} = (r, \theta, k)$ carries an explicit branch index $k \in \mathbb{Z}$ tracking the sheet of the Riemann surface. The logarithm $\widetilde{\log}(r, \theta, k) = \ln r + i(\theta + 2\pi k)$ is single-valued on extended elements, and exponentiation is its inverse. All integer powers of $\tilde{z}$ are single-valued. Layer 2 is operationally closed under logarithm and exponentiation precisely because the branch index is carried explicitly — not because the underlying multi-valued functions are resolved.

Remark 5.2 (What Layer 2 does not yet claim)**.** *Layer 2 is* not *yet algebraically closed under addition: the branch index of $\tilde{z}_1 + \tilde{z}_2$ is not determined by the branch indices of $\tilde{z}_1$ and $\tilde{z}_2$ alone. This is Open Problem 4 below.*

6 What Remains Open in Problem 4

Open Problem 1 (Metric completeness (Q2))**.** *Is the $(-2+i)$-adic completion of the CNRS-A digit strings isomorphic, as a topological ring, to a natural completion of $\mathbb{C}$? The $(-2+i)$-adic metric and the standard complex topology are mutually discontinuous. This requires methods from p-adic analysis and non-Archimedean algebra that are beyond the scope of the present series. A specialist in p-adic analysis or topological algebra is needed.*

Open Problem 2 (General division in CNRS-A)**.** *Division of α by $c \in \mathbb{Z}[i]$ with* $\mathrm{Norm}(c) > 1$ *requires the $(-2+i)$-expansion of c^{-1}, which is bi-infinite (CNRS-A cannot represent elements of $\mathbb{Q}(i) \setminus \mathbb{Z}[i]$ as finite strings). Exact general division is therefore not computable by a finite transducer within finite CNRS-A strings. Approximate division to N digits is computable by one-argument multiplication by the first N digits of c^{-1}, but exact general division is* ***open****.*

Open Problem 3 (Differentiation as an arithmetic operation on CNRS-A)**.** *In CNRS-H, differentiation is a fixed digit-shift on place values $z_0^n/n!$. In CNRS-A, the place values are z_0^n. Realising differentiation as an arithmetic operation directly on CNRS-A digit strings — without passing through the CNRS-H layer — requires that $z_0 = -2+i$ be expressible as $\mathrm{e}^{1+i\beta}$ for some β, so that the place values satisfy the eigenfunction property under differentiation. This is the* e-base CNS theorem*: the central open problem of the CNRS programme.*

The dependency graph is:

$$\underbrace{\underline{\text{e-base CNS theorem}}}_{\text{open}} \Rightarrow \text{Layer 3 full closure} \Rightarrow d/dz \text{ on CNRS-A as arithmetic} \Rightarrow \text{full Q3.}$$

Until the e-base theorem is resolved (requiring input from quasi-periodic tiling theory and the Pisot/Salem number literature), differentiation is not operationally complete in CNRS-A. The obstruction is the same area-gap ($\mathrm{e}^2 - 7 \approx 0.389$) identified in the triangulation / e-base discussion of the P2 capstone [7].

Open Problem 4 (Layer 2 addition)**.** *The extended elements $\tilde{z} = (r, \theta, k)$ form a group under multiplication. A canonical addition law — defining which branch index k to assign to $\tilde{z}_1 + \tilde{z}_2$ without projection to $\mathbb{C}$ — has not been given. This is independent of the arithmetic closure results and requires a separate canonical convention (analogous to the principal branch).*

7 Programme-Level Summary

Problem 4 status after this paper

Sub-question	Status	Authority
Q1: Coverage of $\mathbb{C}$	**Resolved**	Thurston (1989); [8]
Q2: Metric completeness	**Open**	Needs p-adic specialist
Q3a: Addition (CNRS-A)	**Proved**	Theorem 4.1; [9]
Q3a: Subtraction (CNRS-A)	**Proved**	Corollary 4.2; [9]
Q3a: Multiplication (CNRS-A)	**Proved**	Thms 4.4–4.6; [9, 10]
Q3a: Division by base/units	**Proved**	Corollary 4.7
Q3a: General division	**Open**	Open Problem 2
Q3a: d/dz directly on CNRS-A	**Open**	Blocked on e-base theorem
Q3b: Logarithm, exponentiation (Layer 2)	**Proved**	Section 5; [7]
Q3b: Layer 2 addition	**Open**	Open Problem 4
Q3c: d/dz, $\int dz$ (CNRS-H)	**Proved**	Section 5; [7]

Summary statement. CNRS-A is operationally closed as a *ring* under addition, subtraction, and multiplication; all three operations are realised by explicit finite automata. Restricted division (by base powers and Gaussian units) is also closed. CNRS-H is operationally closed under differentiation and integration as fixed digit-shift operators. Layer 2 is closed under logarithm and exponentiation.

The remaining open questions are: metric completeness (Q2, requiring p-adic expertise); general division in CNRS-A; and differentiation directly on CNRS-A strings (blocked on the e-base CNS theorem, the central open problem of the programme).

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Declaration of Generative AI and AI-Assisted Technologies in the Writing Process

During the preparation of this work the author used Claude (Anthropic; `https://claude.ai`) to develop the mathematical content, construct and verify the theorem statements and proofs, and draft the exposition. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the paper.

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