Exact Valuation Transport in a Split 2-Torsion 2-Adic Regime

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Abstract

Let

$$E : y^2 = x^3 - Ax$$

be an elliptic curve over $\mathbb{Q}_2$ satisfying

$$A \equiv 1 \pmod 8.$$

We isolate a localized valuation regime in which the duplication map satisfies an exact transport identity. Translation by a nontrivial 2-torsion point acts locally by valuation inversion and transports formal group points into a resonant shell near the torsion divisor. Inside this shell, ultrametric valuation separation eliminates cancellation and forces the exact identity

$$v_2(x(2P)) = -\delta(P),$$

where

$$\delta(P) = v_2(x(P)^2 - A).$$

After this exact transport step, all subsequent iterates return to the formal group and satisfy strict inductive valuation decay.

1 Introduction

Let v_2 denote the normalized 2-adic valuation on $\mathbb{Q}_2$.

We study local duplication dynamics on elliptic curves

$$E : y^2 = x^3 - Ax$$

over $\mathbb{Q}_2$ under the assumption

$$A \equiv 1 \pmod 8.$$

Duplication dynamics in formal groups are typically asymptotic. The purpose of this note is to isolate a localized regime in which duplication becomes exact.

The mechanism arises from translation by a nontrivial 2-torsion point. In local coordinates, torsion translation acts by valuation inversion and transports formal group points into a resonant shell near the torsion divisor. Inside this shell, ultrametric valuation separation forces a rigid valuation identity without any asymptotic remainder.

The resulting dynamics have the form

$$E_1(\mathbb{Q}_2) \xrightarrow{+T} \mathcal{R}_A \xrightarrow{[2]} E_1(\mathbb{Q}_2).$$

More precisely,

$$v_2(x(P)) < 0 \xrightarrow{+T} \delta(P) \gg 0 \xrightarrow{[2]} v_2(x(2P)) \ll 0.$$

2 Local Setup

Throughout the paper,

$$E : y^2 = x^3 - Ax$$

is defined over $\mathbb{Q}_2$ with

$$A \equiv 1 \pmod 8.$$

Because

$$A \equiv 1 \pmod 8,$$

Hensel's lemma yields

$$\alpha = \sqrt{A} \in \mathbb{Q}_2.$$

Hence

$$T = (\alpha, 0) \in E[2]$$

is a nontrivial 2-torsion point.

Lemma 2.1 (Parity of Formal Valuations)**.** *Suppose*

$$P = (x, y) \in E(\mathbb{Q}_2)$$

satisfies

$$v_2(x(P)) < 0.$$

Then

$$v_2(x(P)) = -2n$$

for some integer $n \geq 1$.

Proof. Write

$$v_2(x(P)) = -k < 0.$$

Then

$$v_2(x^3) = -3k, \qquad v_2(Ax) = -k.$$

Since

$$-3k < -k,$$

the cubic term dominates in

$$y^2 = x^3 - Ax.$$

Hence

$$v_2(y^2) = 2v_2(y) = -3k.$$

Because $2v_2(y)$ is even, $-3k$ must be even. Since 3 is odd, k is even. Therefore

$$k = 2n$$

for some integer $n \geq 1$. □

Lemma 2.2 (Formal Group Criterion)**.** *Let*

$$P = (x, y) \in E(\mathbb{Q}_2).$$

Then

$$P \in E_1(\mathbb{Q}_2) \iff v_2(x(P)) < 0.$$

Proof. Let

$$t(P) = -\frac{x(P)}{y(P)}$$

be the standard formal parameter at the origin.

Suppose

$$v_2(x(P)) = -2n < 0.$$

By the previous lemma,

$$v_2(y(P)) = -3n.$$

Hence

$$v_2(t(P)) = v_2(x) - v_2(y) = -2n - (-3n) = n > 0.$$

Therefore

$$P \in E_1(\mathbb{Q}_2).$$

Conversely, if

$$v_2(t(P)) > 0,$$

then the formal expansion

$$x = t^{-2} + O(t^{-1})$$

implies

$$v_2(x(P)) < 0.$$

□

Definition 2.3 (Torsion Proximity Invariant)**.** For

$$P = (x, y) \in E(\mathbb{Q}_2),$$

define

$$\delta(P) = v_2(x(P)^2 - A).$$

Definition 2.4 (Resonant Shell)**.** Define

$$\mathcal{R}_A = \{P \in E(\mathbb{Q}_2) : \delta(P) \geq 3\}.$$

Lemma 2.5. *If*

$$P \in \mathcal{R}_A,$$

then

$$v_2(x(P)) = 0.$$

Proof. Suppose first that

$$v_2(x(P)) > 0.$$

Then

$$v_2(x(P)^2) > 0,$$

so

$$v_2(x(P)^2 - A) = 0,$$

contrary to

$$\delta(P) \geq 3.$$

Now suppose

$$v_2(x(P)) < 0.$$

Then

$$v_2(x(P)^2) < 0,$$

and therefore

$$v_2(x(P)^2 - A) = 2v_2(x(P)) < 0,$$

again contradicting

$$\delta(P) \geq 3.$$

Hence

$$v_2(x(P)) = 0.$$

□

3 Torsion-Induced Valuation Inversion

Lemma 3.1 (Branch Localization). *Let*

$$P \in \mathcal{R}_A.$$

Then exactly one of

$$v_2(x - \alpha), \qquad v_2(x + \alpha)$$

is strictly greater than 1*, while the other equals* 1*.*

Consequently,

$$\delta(P) = \max\{v_2(x - \alpha), v_2(x + \alpha)\} + 1.$$

Proof. Since

$$(x + \alpha) - (x - \alpha) = 2\alpha$$

and

$$v_2(\alpha) = 0,$$

we obtain

$$v_2(2\alpha) = 1.$$

Therefore

$$\min\{v_2(x - \alpha), v_2(x + \alpha)\} \leq 1.$$

On the other hand,

$$v_2(x - \alpha) + v_2(x + \alpha) = \delta(P) \geq 3.$$

Since valuations are integral, one valuation equals 1 and the other is at least 2. □

Theorem 3.2 (Valuation Inversion). *Let*

$$T = (\alpha, 0) \in E[2].$$

For every

$$P \in E_1(\mathbb{Q}_2),$$

we have

$$x(P + T) - \alpha = \frac{2A}{x(P) - \alpha}.$$

Consequently,

$$v_2(x(P + T) - \alpha) = 1 - v_2(x(P) - \alpha).$$

Proof. Let
$$P = (x, y).$$
The addition formula gives
$$x(P+T) = \lambda^2 - x - \alpha, \qquad \lambda = \frac{y}{x-\alpha}.$$

Using
$$y^2 = x(x-\alpha)(x+\alpha),$$
we obtain
$$\lambda^2 = \frac{x(x+\alpha)}{x-\alpha}.$$
Hence
$$x(P+T) - \alpha = \frac{x(x+\alpha)}{x-\alpha} - x - 2\alpha.$$
Combining terms gives
$$x(P+T) - \alpha = \frac{2\alpha^2}{x-\alpha} = \frac{2A}{x-\alpha}.$$

Taking valuations yields
$$v_2(x(P+T) - \alpha) = v_2(2A) - v_2(x(P) - \alpha).$$
Since
$$v_2(A) = 0,$$
we have
$$v_2(2A) = 1.$$
Finally,
$$v_2(x(P)) < 0 < v_2(\alpha)$$
implies
$$v_2(x(P) - \alpha) = v_2(x(P)).$$
Therefore
$$v_2(x(P+T) - \alpha) = 1 - v_2(x(P) - \alpha).$$
□

Corollary 3.3 (Resonant Injection)**.** *Suppose*
$$v_2(x(P)) = -2n < 0.$$
Then
$$P + T \in \mathcal{R}_A$$
and
$$\delta(P+T) = 2n + 2.$$
Proof. By Valuation Inversion,
$$v_2(x(P+T) - \alpha) = 2n + 1.$$
By Branch Localization,
$$v_2(x(P+T) + \alpha) = 1.$$
Hence
$$\delta(P+T) = (2n+1) + 1 = 2n + 2.$$
□

4 Exact Valuation Transport

Proposition 4.1 (Resonant Rigidity)**.** *If*

$$P \in \mathcal{R}_A,$$

then

$$v_2(x(P)^2 + A) = 1.$$

Proof. Since

$$P \in \mathcal{R}_A,$$

we have

$$v_2(x(P)) = 0.$$

Therefore

$$v_2(x(P)^2) = 0.$$

Because

$$A \equiv 1 \pmod 8,$$

every 2-adic unit square satisfies

$$x(P)^2 \equiv 1 \pmod 8.$$

Hence

$$x(P)^2 + A \equiv 2 \pmod 8.$$

Therefore

$$v_2(x(P)^2 + A) = 1.$$

□

Theorem 4.2 (Exact Transport Identity)**.** *Let*

$$P \in \mathcal{R}_A.$$

Then

$$v_2(x(2P)) = -\delta(P).$$

Proof. The duplication formula gives

$$x(2P) = \frac{(x^2 + A)^2}{4x(x^2 - A)}.$$

By Resonant Rigidity,

$$v_2((x^2 + A)^2) = 2.$$

Moreover,

$$v_2(4x(x^2 - A)) = 2 + 0 + \delta(P) = 2 + \delta(P).$$

Therefore

$$v_2(x(2P)) = 2 - (2 + \delta(P)) = -\delta(P).$$

□

5 Asymptotic Escape

Lemma 5.1 (Inductive Valuation Decay). *Suppose*

$$v_2(x_n) < 0.$$

Then

$$v_2(x_{n+1}) = v_2(x_n) - 2.$$

Proof. Using the duplication formula,

$$x_{n+1} = \frac{(x_n^2 + A)^2}{4y_n^2}.$$

Since

$$v_2(x_n) < 0,$$

we have

$$v_2(x_n^2 + A) = 2v_2(x_n).$$

Hence

$$v_2((x_n^2 + A)^2) = 4v_2(x_n).$$

Similarly,

$$y_n^2 = x_n^3 - Ax_n$$

implies

$$v_2(y_n^2) = 3v_2(x_n).$$

Therefore

$$v_2(4y_n^2) = 2 + 3v_2(x_n).$$

Subtracting valuations gives

$$v_2(x_{n+1}) = 4v_2(x_n) - (2 + 3v_2(x_n)) = v_2(x_n) - 2.$$

□

Theorem 5.2 (Escape After Exact Transport). *Let*

$$P \in \mathcal{R}_A$$

and define

$$P_n = [2]^n P.$$

Then:

(i)

$$v_2(x(P_1)) = -\delta(P).$$

(ii) For every $n \geq 1$,

$$v_2(x(P_n)) = -\delta(P) - 2(n-1).$$

(iii)

$$v_2(x(P_n)) \to -\infty.$$

Proof. Part (i) is the Exact Transport Identity.

Since

$$v_2(x(P_1)) = -\delta(P) < 0,$$

all subsequent iterates lie in the formal group.

Applying the previous lemma inductively yields

$$v_2(x(P_n)) = -\delta(P) - 2(n-1).$$

Hence

$$v_2(x(P_n)) \to -\infty.$$

□

6 Conclusion

Translation by a nontrivial 2-torsion point induces a valuation inversion mechanism transporting formal group points into a resonant shell near the torsion divisor. Inside this shell, ultrametric valuation separation forces the exact identity

$$v_2(x(2P)) = -\delta(P).$$

Subsequent iterates return to the formal group and satisfy strict inductive valuation decay.

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