ruler and compass the length of the circumference is also the limit of an infinite series of operations and therefore unattainable. But change the process by using the integraph, and what was before a limit and just out of reach, becomes attainable and we get a line equal to the circumference.

THE MATHEMATICAL HANDBOOK OF AHMES.

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The Handbook of Ahmes was written about 1700 B. C.—more than a thousand years before the beginning of the classic period of Greek mathematics. It stands as an isolated peak in the history of mathematics and practically marks the beginning of this history. It seems to have been written as a compendium of useful and curious mathematical facts for the learned Egyptian priests living at about the time when the Israelites were slaves in their country.

The book is replete with facts of the greatest interest, not only to the students of mathematics, but also to those who are interested in the history of the development of the human intellect. Even the title of the book is naive. It is as follows: "Directions for obtaining a knowledge of all dark things * * * of all secrets which are involved in the objects." This title gives evidence of the ancient belief in the power and comprehensiveness of mathematical knowledge, and is comparable with the much more recent saying of Isidorus, bishop of Seville, who expressed his admiration of number in his encyclopedia in the following words: "Take away number from all things and everything goes to destruction."

The five parts of the Handbook of Ahmes are devoted respectively to the following subjects: Arithmetic, stereometry, geometry, calculation of pyramids, collection of practical examples. The first part begins with a table in which the forty-eight fractions having two for a numerator and the odd numbers from 5 to 99 as denominators are expressed as sums of different fractions having unity for their common numerator. The table i[illegible] curious that we reproduce it here, omitting only the verific[illegible] which were given with each fraction.

This table is of great historical importance. It ap[illegible]hat no

$\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$; $\frac{2}{7}=\frac{1}{4}+\frac{1}{28}$; $\frac{2}{9}=\frac{1}{6}+\frac{1}{18}$; $\frac{2}{11}=\frac{1}{6}+\frac{1}{66}$;

$\frac{2}{13}=\frac{1}{8}+\frac{1}{52}+\frac{1}{104}$; $\frac{2}{15}=\frac{1}{10}+\frac{1}{30}$; $\frac{2}{17}=\frac{1}{12}+\frac{1}{51}+\frac{1}{68}$;

$\frac{2}{19}=\frac{1}{12}+\frac{1}{76}+\frac{1}{114}$; $\frac{2}{21}=\frac{1}{14}+\frac{1}{42}$; $\frac{2}{23}=\frac{1}{12}+\frac{1}{276}$;

$\frac{2}{25}=\frac{1}{15}+\frac{1}{75}$; $\frac{2}{27}=\frac{1}{18}+\frac{1}{54}$; $\frac{2}{29}=\frac{1}{24}+\frac{1}{58}+\frac{1}{174}+\frac{1}{232}$;

$\frac{2}{31}=\frac{1}{20}+\frac{1}{124}+\frac{1}{155}$; $\frac{2}{33}=\frac{1}{22}+\frac{1}{66}$; $\frac{2}{35}=\frac{1}{30}+\frac{1}{42}$;

$\frac{2}{37}=\frac{1}{24}+\frac{1}{111}+\frac{1}{296}$; $\frac{2}{39}=\frac{1}{26}+\frac{1}{78}$; $\frac{2}{41}=\frac{1}{24}+\frac{1}{246}+\frac{1}{328}$;

$\frac{2}{43}=\frac{1}{42}+\frac{1}{86}+\frac{1}{129}+\frac{1}{301}$; $\frac{2}{45}=\frac{1}{30}+\frac{1}{90}$;

$\frac{2}{47}=\frac{1}{30}+\frac{1}{141}+\frac{1}{470}$; $\frac{2}{49}=\frac{1}{28}+\frac{1}{196}$; $\frac{2}{51}=\frac{1}{34}+\frac{1}{102}$;

$\frac{2}{53}=\frac{1}{30}+\frac{1}{318}+\frac{1}{795}$; $\frac{2}{55}=\frac{1}{30}+\frac{1}{330}$; $\frac{2}{57}=\frac{1}{38}+\frac{1}{114}$;

$\frac{2}{59}=\frac{1}{36}+\frac{1}{236}+\frac{1}{531}$; $\frac{2}{61}=\frac{1}{40}+\frac{1}{244}+\frac{1}{488}+\frac{1}{610}$;

$\frac{2}{63}=\frac{1}{42}+\frac{1}{126}$; $\frac{2}{65}=\frac{1}{39}+\frac{1}{195}$; $\frac{2}{67}=\frac{1}{40}+\frac{1}{335}+\frac{1}{736}$;

$\frac{2}{69}=\frac{1}{46}+\frac{1}{138}$; $\frac{2}{71}=\frac{1}{40}+\frac{1}{568}+\frac{1}{710}$;

$\frac{2}{73}=\frac{1}{60}+\frac{1}{219}+\frac{1}{292}+\frac{1}{365}$; $\frac{2}{75}=\frac{1}{50}+\frac{1}{150}$;

$\frac{2}{77}=\frac{1}{44}+\frac{1}{308}$; $\frac{2}{79}=\frac{1}{60}+\frac{1}{237}+\frac{1}{316}+\frac{1}{790}$; $\frac{2}{81}=\frac{1}{54}+\frac{1}{162}$;

$\frac{2}{83}=\frac{1}{60}+\frac{1}{332}+\frac{1}{415}+\frac{1}{498}$; $\frac{2}{85}=\frac{1}{51}+\frac{1}{255}$; $\frac{2}{87}=\frac{1}{58}+\frac{1}{174}$;

$\frac{2}{89}=\frac{1}{60}+\frac{1}{356}+\frac{1}{534}+\frac{1}{890}$; $\frac{2}{91}=\frac{1}{70}+\frac{1}{130}$;

$\frac{2}{93}=\frac{1}{62}+\frac{1}{186}$; $\frac{2}{95}=\frac{1}{60}+\frac{1}{380}+\frac{1}{570}$;

$\frac{2}{97}=\frac{1}{56}+\frac{1}{679}+\frac{1}{776}$; $\frac{2}{99}=\frac{1}{66}+\frac{1}{198}$

general rule was followed in its construction, although such rules could have easily be given. One such rule* is contained in the classic work of Leonardo of Pisa, written in 1202, A. D. Probably the table is a collection of results obtained by many different scholars handed down from generation to generation.

The Egyptians had a special symbol for $\frac{2}{3}$ but all the other fractions employed by them had unity for their numerator.* Even if they desired to employ only fractions with unity as a numerator, this table cannot have been of much real value, for it is easier to represent $\frac{2}{13}$ by $\frac{1}{13}+\frac{1}{13}$ than by $\frac{1}{8}+\frac{1}{52}+\frac{1}{104}$. The table is a good example of the fact that cumbersome methods are frequently employed before the easier methods are discovered. The fact that the Greeks employed such fractions along with the general fractions gives evidence of the difficulty of replacing the useless by the useful in the development of knowledge.

While this table bears definite evidence of the immaturity of the Egyptian intellect it also bears evidence of great strides in intellectual development. The appreciation of such truths, which require some continuity of thought to verify, shows that the Egyptians at this early date were very far in advance of many uncivilized nations at the present time. This fact will become clearer when the other parts of this marvelous work are exhibited.

The second section of the arithmetical part consists of only six closely related examples. They illustrate how the numbers 1, 3, 6, 7, 8, 9, respectively may be divided into ten equal parts. As is the case with most of the examples throughout the book, Ahmes gives only the answers and verifications. For instance, he says $\frac{9}{10}=\frac{2}{3}+\frac{1}{5}+\frac{1}{30}$ because 10 times $\frac{2}{3}+\frac{1}{5}+\frac{1}{30}=9$. In order to multiply these fractions by 10 he always doubles them, then doubles these results and thus obtains their four-fold. He finally doubles these results and thus obtains their eight-fold. To multiply by 10 he simply adds the double to the eight-fold. That is, he employs the principle that any natural number is the sum of different powers of 2.

There is no evidence that any multiplication table was in use among the Egyptians. On the contrary, the examples that have

* Cf. Harzer, Jahresberich der Deutschen Mathematiker-Vereinigung, vol. 14 (1905), p. 315.

* In his History of Mathematics, Ball says that the Egyptians used also the fraction ¾ ; 3d edition, p. 4. This statement is incorrect.

come down to us indicate that multiplication was accomplished by successive addition. The earliest evidence of a multiplication table is found in the work of Nicomachus who lived about 100 A. D., and wrote the first classic arithmetic. Even this table may have served another purpose.

As in the preceding section so we find in this section evidences of immaturity of thought along side with the display of judgment. An instance of the latter is furnished by the omission of the numbers 2, 4, and 5. Ahmes seems to have realized that $\frac{2}{10}$ and $\frac{2}{10}$ and respectively equal to $\frac{1}{5}$ and $\frac{1}{2}$, and hence the division of 2 and 5 into 10 equal parts need not be considered. Moreover, $\frac{4}{10} = \frac{2}{5}$ and hence it is not necessary to divide 4 into 10 equal parts as $\frac{2}{5}$ has been resolved into unit-fractions in the preceding table. An instance of immaturity is furnished by the fact that 1 is divided into ten equal parts, and the result $\frac{1}{10} = \frac{1}{10}$ is proved by showing that the double of $\frac{1}{10}$ is $\frac{1}{5}$ and the eight-fold of $\frac{1}{10}$ is $\frac{2}{3} + \frac{1}{10} + \frac{1}{30}$ From the fact that $\frac{1}{5} + \frac{2}{3} + \frac{1}{10} + \frac{1}{30} = 1$ it therefore follows that $\frac{1}{10} = \frac{1}{10}$.

The third section of the arithmetical part consists of eighteen examples. Fifteen of these give the fractions obtained by adding to a given fraction its one-half and its one-fourth, or its two-third and one-third. In the latter case the fractions are simply doubled by the operation, but Ahmes goes through the details in each of the six examples where a fraction is increased by its $\frac{2}{3}$ and $\frac{1}{3}$. His methods appear very cumbersome. For instance, when he increases $\frac{1}{4}$ by its $\frac{2}{3}$ and its $\frac{1}{3}$, he observes that $\frac{2}{3}$ of $\frac{1}{4} = \frac{1}{6}$ and $\frac{1}{3}$ of $\frac{1}{4} = \frac{1}{12}$. Instead of adding $\frac{1}{4} + \frac{1}{6} + \frac{1}{12}$ he reduces them to the common denominator 18 and notes that $\frac{1}{4} = \frac{4\frac{1}{2}}{18}$, $\frac{1}{6} = \frac{3}{18}$, and $\frac{1}{12} = \frac{1\frac{1}{2}}{18}$. Hence, the result is $\frac{9}{18} = \frac{1}{2}$

In the last three examples of this section it is required to find the difference between a given fraction and unity, or between a given fraction and $\frac{2}{3}$. The first one of these reads as follows: "You are told to complete $\frac{2}{3} + \frac{1}{51}$ to 1." This is done by observing that $\frac{2}{3} = \frac{10}{15}$, and hence the required fraction is $\frac{4}{15}$. To reduce this to Egyptian fractions Ahmes determines the number which must be multiplied into 15 to give 4. He first multiplies 15 by $\frac{1}{15}$ and thus obtains 1. He then multiplies by $\frac{1}{10}$ and obtains $1\frac{1}{2}$. Finally he multiplies by $\frac{1}{5}$ and obtains 3. By adding the first and last result together he obtains the required number 4. Hence $\frac{1}{5} + \frac{1}{15}$ must be added to $\frac{2}{3} + \frac{1}{15}$ to obtain 1.

Section IV. is the most interesting part of the whole book. It is devoted to elementary algebra. Some of the problems are similar to those met in our present text-books on algebra, except that the unknown is called "heap," instead of x. The following examples exhibit the nature of these problems: "Heap, its seventh, its whole, it makes 19." "Heap, its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, it makes 33." It is a significant fact that the oldest mathematical work extant should include elementary algebra. The equations to which the problems give rise are all of the first degree. The Egyptians could not solve the quadratic equation as they did not even know how to extract the square root.

The fifth and last section of the arithmetical part is devoted to the division into unequal parts. The first examples state that 100 loaves are divided among 10 people. Four of these receive 50 loaves while the remaining six receive also 50 loaves. It is required to find the difference between the amount received by each of the four and each of the six. The second example requires to find five terms of an arithmetical progression such that the sum of the terms is 100 and that $\frac{1}{7}$ of the sum of the first three terms is equal to the sum of the last two terms.

The last problem is solved by assuming that the common difference is $5\frac{1}{2}$ and that the last term is 1. The terms obtained in this way are 23, $17\frac{1}{2}$, 12, $6\frac{1}{2}$, 1. As their sum is 60 instead of 100, Ahmes increases each of these terms by $\frac{2}{3}$ of their value and thus obtains the values of five numbers in arithmetical progression which satisfy the conditions of the problem. This completes the arithmetical part of the work under consideration. As this is by far the most important part of the book we proceed to give a brief summary of its contents.

In addition to the table of unit fractions, it is composed of 40 problems, while the remaining four parts together are composed of 44 problems. The most advanced of these problems relate to linear equations and to arithmetical progression. The main difference between the method of operation pursued by Ahmes and those of the present day is due to the fact that the only fractions which were allowed to appear in the results were $\frac{2}{3}$ and those having unity as a numerator, while the denominator was frequently more than 1000. This restriction seems to us very unfortunate. While it is no more arbitrary than some of those under which we labor at the present day (e. g. the restriction to the rule and the circle in plane geometry constructions), yet it seems to have been more detrimental to progress. It had the advantage that fractions could be multiplied by merely multiplying the denominators. Hence the multiplication of fractions was just as simple as the multiplication of integers.

The second and third parts of the Handbook of Ahmes deal respectively with stereometry and plane geometry. We would naturally have expected that these two subjects would have been treated in the reverse order, as the plane figures are involved in the mensuration of solids. It is possible that the arrangement was made according to what appeared the relative importance of the subjects and we are reminded of the fact that spherical trigonometry was developed earlier than the plane trigonometry.

The part on stereometry begins with the following example: "Directions to calculate a round granary, whose diameter is 9 and height 10." The problem is solved by reducing the diameter by its $\frac{1}{9}$ and then squaring the remainder for the area of the circle. This result is multiplied by $\frac{3}{2}$ of the height to obtain the volume. Several interesting facts appear in this operation. In the first place, the Egyptians regarded the circle equal to a square whose side is $\frac{8}{9}$ of the diameter of the circle. This is equivalent to considering $\pi = 3.1604$ . . . It is interesting to note in this connection that the Japanese used to consider $\pi = 3.16$.

The fact that the area of the circular base is multiplied by $\frac{3}{2}$ of the height instead of by the height presents greater difficulties, which have not been definitely solved. It may be that granaries had sloping sides and that the given base is the smaller of the two bases. In the examples in which the base is a square the area of the base is also multiplied by $\frac{3}{2}$ of the height. All the examples dealing with the mensuration of solids relate to the determination of the contents of granaries whose dimensions are given or to the determination of the dimensions when the contents are given. In the latter it is assumed that the base is a square whose side is 10 units.

In the part on plane geometry there is one example in which the area of a circular field is computed. This is again done by finding the area of a square whose side is $\frac{8}{9}$ of the diameter of the field. The following example, at first sight, seems to fix a remarkable low limit to the geometrical attainments of the Egyptians. It is definite proof that Ahmes did not know how to find the area of an isosceles triangle. The problem is to find the area of an isosceles triangle whose base is 4 and whose side is 10. Ahmes simply multiplies half of the base into the side, giving 20 as the area.

In order to find the exact area from the given data it would be necessary to find the value of $\sqrt{100-4}=\sqrt{96}$. It was noted above that the Egyptians did not know how to extract the square root and hence this operation was impossible for them. Moreover, the error which Ahmes commits is not very great since his result is only about 2 per cent. too large, an error which in his day may have passed unnoticed. A similar error is made in the next problem where it is required to find the area of an isosceles trapezoid. Ahmes multiplies half the sum of the parallel sides by the other side, instead of by the altitude.

The part devoted to the calculation of pyramids has presented great difficulties. It deals with the quotient obtained by dividing one-half of a certain line in the pyramid by another line. This quotient is called Seqt, and seems to be the cosine of the angle between an edge o fthe pyramid and the diagonal of the base Hence this part is sometimes regarded as a chapter in trigonometry but the data are so meagre as to convey very little definite information. The first of these examples reads as follows: "Directions to calculate a pyramid 360 yards at the base, 250 at the edge, let me know their ratio." It is solved in the following manner: Take $\frac{1}{2}$ of 360, this gives 180; multiply 250 to find 180, this gives $\frac{1}{2}+\frac{1}{5}+\frac{1}{50}$ of a yard. Sincs a yard is 7 hand-breadths we have to multipliy 7 by $\frac{1}{2}+\frac{1}{5}+\frac{1}{50}$. Hence the Seqt is $5\frac{1}{25}$ hand-breadths.

The last part consists of a collection of twenty-three practical examples which relate to the division of loaves, wages of a herdsman, paying laborers, the feed of oxen, etc. From the type of these examples it is inferred that Ahmes had the wants of the farmer especially in mind in writing his book. Two of the examples Nos. 80 and 81, are devoted to the change from one system of measures to another—a type of problems found in our modern arithmetics. One of them seems to involve a knowledge of the formula for the sum of a geometrical progression.

While the Handbook of Ahmes raises many questions which cannot be definitely answered at the present time it gives conclusive proof of the following facts: As early as 1700 B. C., (and probably much earlier as Ahmes claimed to have modeled his book on an old work) the Egyptians had a fairly advanced knowledge of fractions, the linear equation, the arithmetic series, and probably the geometric series. They employed a formula for the area of the circle which gives a comparatively close approximation. They had made a beginning in the study of similar figures but their formula for the area of a triangle was a crude approximation.

A QUESTION.

I venture to draw your attention to "A Geometrical Fallacy" on p. 369 School Science and Mathematics for May, 1905. It seems to me the writer of the note in question has entirely misunderstood the terms "mutually equilateral" and "mutually equiangular." If, as is clearly intended in the proposition as originally stated, the n-gons are mutually equilateral and mutually equiangular, equal angles lying between equal sides in each, and read in the same order, i. e., right to left or reverse in both, then the proposition is evidently true. Further, the writer assumes two n-gons congruent to begin (line 8) and concludes by asserting that they are not in general congruent. Can you explain?

W. D. Patterson.

THE REPLY.

If, as the writer states, two polygons are mutually equilateral, mutually equiangular, and *the equal angles lie between equal sides in each*, then the polygons are congruent.

The third condition is obviously necessary, but it seems to have always been overlooked, or, at least, as in the comments of Mr. Patterson, it is assumed to be a logical consequence of the first two conditions.

That this is not true is shown in the note referred to; for though it begins with two polygons which are congruent, and which therefore satisfy the three conditions, each is afterwards altered so that the first two are satisfied and the third is not, and they no longer are congruent.

G. W. Greenwood.