On Sphero-Cyclides. *By* HENRY M. JEFFERY, F.R.S.

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1. These spherical quartics are the lines of intersection of spheres both with cyclides and quadrics. M. Laguerre, who first pointed out their genesis (*infra*, §4), designated them Anallagmatic Spherical Curves, because they are unaltered, when inverted from any of its four poles of inversion. (Chasles, "Rapport sur les progrès de la Géometrie," p. 315.) Under the name of sphero-quartics, their properties have been studied by Dr. Casey ("Cyclides and Sphero-Quartics," *Phil. Trans.*, 1871, pp. 585—721). But, since they are only a species of spherical binodal quartics, and do not include all the intersections of quartic surfaces with spheres, their name is here altered. They might be also called spherical quadro-quadrics or sphero-quadrics (§6).

2. Sphero-cyclides, being binodal, are curves of the eighth class, and have two double, and four single, foci: the former are the two single foci of the dirigent or focal sphero-conics, from which they are generated. If the two double foci coincide in a quadruple focus, the cyclide is known as a Sphero-Cartesian (Casey, p. 677). These curves may have an additional node or cusp, whereby the class is reduced to the sixth or fifth respectively.

3. Sphero-cyclides have two double cyclic arcs, which are the single cyclic arcs of the complementary or polar conics, of which cyclic arcs the single foci of the focal conics or the double foci of the sphero-cyclides are the spherical centres or quadrantal poles. Sphero-Cartesians have each a quadruple cyclic arc, whose spherical centre is the quadruple focus, or centre of the dirigent circle, from which it is generated. This conjugate property of double arcs and double cyclic arcs is common to all spherical curves (*Quarterly Math. Journal*, Vol. xv., p. 140).

4. A sphero-cyclide may be generated in four different ways, as the envelope of a variable small circle, whose centre moves on a dirigent or focal sphero-conic (F), and which cuts a fixed small circle (J) orthogonally. (Laguerre, *Bulletin de la Société Philomathique*, 1867; Casey, §41, Cor.)

These dirigent conics (F) are doubly confocal; and the fixed circles (J) are mutually orthotomic, and all the eight figures are interdependent. The centres of the four (J) circles are the vertices of the quadrangle, in which any J and F pair intersect, and, taken three and three together, are the angular points of triangles, which are self-conjugate with respect both to the (F) conics and (J) circles. Each triad of centres has the fourth for the orthocentre of the triangle constituted by them; and each of these four triangles is self-conjugate in respect to one of the four circles and its corresponding focal conic.

The confocal conics are thus also interdependent. The twelve points in which the sides of the quadrilateral circumscribed about any pair J and F intersect, lie by tetrads on the three remaining (F) focal conics.

The line of nodes in the sphero-cyclide is the polar of the centre of any (J) circle with respect to the corresponding (F) focal conic.

5. The three anallagmatic congeners, the cyclide, the sphero-cyclide, and the bicircular quartic, constitute a geometrical trilogy, as exhibited by Professor Casey in his two classical memoirs.

Dr. Hart has shown analytically how the bicircular is generated from each of the four (F) conics (*Proceedings*, Vol. xi., pp. 143—151), and has promised this Society the corresponding memoir on the Five Focal Quadrics of a Cyclide (Vol. xii., p. 109), the MS. of which he has allowed me to see and copy. It is hoped he will shortly publish it.

Following his steps, I have investigated by spherical coordinates the generation of the sphero-cyclide from each of its four focal sphero-conics, and thereby hope to complete the series of the trilogy.

The singular forms of the curve will be considered, and a method given for finding its points of undulation, and therefrom its points of inflexion generally.

6. The equation to the sphero-cyclide is derived from those to the cyclide and quadric, by transformation of coordinates.

Let OAB be an octant of a sphere, whose centre is any origin of coordinates for the cyclide, and whose radius is unity.

Take O for the origin of spherical coordinates in Gudermann's system.

$BP = \theta$, $AM = \phi$: $OM = X$, $ON = Y$, $OP = R$.

Cartesian coordinates are thus transformed to spherical:

$$x = \sin\theta\cos\phi = \tan X\cos R,$$
$$y = \sin\theta\sin\phi = \cos R,$$
$$z = \cos\theta = \tan Y\cos R.$$

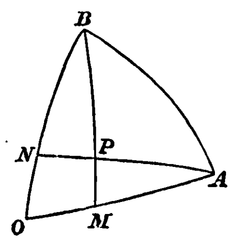

Fig. 1.

(tan X, tan Y are usually written X, Y, for brevity).

The equation to the cyclide in Cartesian coordinates is

$$a(x^2+y^2+z^2)^2+b(x^2+y^2+z^2)(u_1+u_0)+v_2+v_1+v_0=0,$$

where v_2; v_1, u_1; v_0, u_0 denote quadric and linear functions of x, y, z, and constants.

Let the same symbols v_2', ... denote the same functions of $\tan X$, 1, $\tan Y$.

$$a\sec^2 R+b\sec R\,(u_1'+u_0'\sec R)+v_2'+v_1'\sec R+v_0'\sec^2 R=0.$$

After dropping the accents, we obtain the transformed equation to the sphero-cyclide

$$\{(a+bu_0+v_0)\sec^2 R+v_2\}^2=\sec^2 R\,(bu_1+v_1)^2.$$

The curve is binodal, and is touched by the imaginary great circle ($\sec^2 R=1+\tan^2 X+\tan^2 Y=0$) in four points $\sec^2 R=0$, with $v_2=0$; it has for two nodes the two points $bu_1+v_1=0$, with $(a+bu_0+v_0)\sec^2 R+v_2=0$.

The equation to the quadric is derived from that to the cyclide, when $a=0$, $b=0$; for the corresponding sphero-cyclide,

$$\{v_0(1+X^2+Y^2)+v_2\}^2=(1+X^2+Y^2)\,v_1^2,$$

If $v_1=0$, or the quadric is referred to its centre as origin, the sphero-cyclide becomes two coincident sphero-conics.

Cor.—If $v_2\equiv c(x^2+z^2)+dy^2$, or the cyclide has the imaginary circle at infinity as a cuspidal edge, it is called by Dr. Casey a Sphero-Cartesian. Its equation in spherical coordinates becomes

$$\{(v_0+c)(1+X^2+Y^2)+d-c\}^2=v_1^2\,(1+X^2+Y^2).$$

It is thus recognised to be the intersection of a sphere and a quadric of revolution (Casey, §235).

The following theorems are preliminary; and it is necessary to premise that, in two-point coordinates, x, y denote $\tan x$, $\tan y$, and in the three-point system α, β, γ; p, q, r denote the sines of those arcs; a, b, c also represent the sines of the arcs of the triangle of reference:—

7. Two spherical small circles are mutually orthotomic, if $\cos\rho_1\cos\rho_2=\cos D$; where the symbols denote their spherical radii and the mutual distance of their centres. For in that case the centres and either point of intersection constitute a right angle.

8. To find the condition that two small circles intersect orthogonally. First, let their equations in three-point coordinates be

$$d\alpha+e\beta+f\gamma=g,\quad l\alpha+m\beta+n\gamma=h.$$

Then $\cos\rho_1\sqrt{\Sigma(d^2-2ef\cos A)} = g$; $\cos\rho_2\sqrt{\Sigma(l^2-2mn\cos A)} = h$;

$$\cos D\sqrt{\Sigma(d^2-2ef\cos A)}\,.\,\sqrt{\Sigma(l^2-2mn\cos A)}$$
$$= d(l-m\cos C-n\cos B)+\dots$$

The required condition is

$$gh = dl+em+fn-(en+fm)\cos A-(fl+dn)\cos B-(dm+el)\cos C.$$

Next, let their equations in two-point (Gudermann's) coördinates be

$$1+ax+by = g\sqrt{(1+x^2+y^2)},\quad 1+cx+dy = h\sqrt{(1+x^2+y^2)}.$$

For mutual orthotomy, it is necessary that

$$1+ac+bd = gh.$$

9. If ABC be a spherical triangle, and O its orthocentre, then the four small circles which have A, B, C, O for their centres are mutually orthotomic, if

$$\cos a\cos\delta_1 = \cos b\cos\delta_2 = \cos c\cos\delta_3 = \sqrt{(\cos a\cos b\cos c)},$$
$$\cos\delta_4\sqrt{\Sigma(\tan^2 A+2\tan B\tan C\cos a)}$$
$$= \tan A\tan B\tan C\sqrt{(\cos a\cos b\cos c)}.$$

The radii are denoted by δ_1, δ_2, δ_3, δ_4.

The propositions in the first line are evident from § 7.

In like manner,

$$\cos a\cos\delta_4 = \cos BO\cos\delta_3 = \cos CO\cos\delta_2 = \sqrt{(\cos a\cos BO\cos CO)}.$$

The proposition in the second line is established by knowing the distance (δ) between two points from the formula

$$\sin^2 b\sin^2 c\sin^2 A\cos\delta = \Sigma(\alpha\alpha_1\sin^2 a)+\Sigma[\sin b\sin c\cos a(\beta\gamma_1+\beta_1\gamma)].$$

At the orthocentre

$$\alpha\cos A = \beta\cos B = \gamma\cos C.$$

Hence $$\cos AO\sqrt{\Sigma(\tan^2 A+2\tan B\tan C\cos a)}$$
$$= \tan A+\tan B\cos c+\tan C\cos b = \tan A\tan B\tan C\cos b\cos c.$$

By symmetry,

$$\cos a\cos AO = \cos b\cos BO = \cos c\cos CO.$$

Hence $\cos\delta_4 = \sec\delta_1\cos AO = \sec\delta_2\cos BO = \sec\delta_3\cos CO$,
and the circles are mutually orthotomic. Since

$$\Sigma(\tan^2 A+2\tan B\tan C\cos a) = \mu^2\tan^2 A\tan^2 B\tan^2 C,$$

where $\mu^2 = 2\cos a \cos b \cos c - 1 + (6V)^2 \operatorname{cosec}^2 A \operatorname{cosec}^2 B \operatorname{cosec}^2 C$,

$$\mu \cos AO = \cos b \cos c,$$

and $$\mu \cos \delta_4 = \sqrt{(\cos a \cos b \cos c)}.$$

The analogue to this theorem for Plane Geometry is given by Dr. Casey (*Sequel to Euclid*, p. 108).

Note.—By $6V$ will be hereinafter denoted six times the volume of a certain tetrahedron constituted by three radii of the sphere, and the connectors of their extremities, so that the fundamental relation is

$$6V = bc \sin A = \sqrt{\Sigma} \, (a^2 \alpha^2 + 2bc\,\beta\gamma \cos a).$$

10. To find the discriminant of the binary quartic

$$(fx^2 + 2gxy + hy^2)^2 = (ux^2 + 2w'xy + vy^2)(sx + ty)^2.$$

Let A, B be invariants of single quadrics

$$A = fh - g^2, \qquad B = uv - w'^2.$$

C, D, E are invariants of systems of two quadrics

$$C = s^2 v - 2stw' + t^2 u, \qquad D = s^2 h - 2stg + t^2 f,$$

$$E = uh - 2w'g + vf,$$

also $F = u\,(sh - tg)^2 - 2w'\,(sh - tg)(sg - tf) + v\,(sg - tf)^2 = DE - AC$.

The function F occurs in investigating I_4 by symbolical methods,

$$\left(u \frac{d^2}{dy^2} - 2w' \frac{d^2}{dy\,dx} + v \frac{d^2}{dx^2}\right)\left(s \frac{d}{dy} - t \frac{d}{dx}\right)^2 (fx^2 + 2gxy + hy^2)$$

$$= \underline{|4} \; (F + \tfrac{1}{2}AC).$$

If I_4, I_6 denote the quartic and sextic invariants of the given quartic, they can be expressed in terms of the subordinate invariants:

$$3I_4 = 4\,(A + \tfrac{1}{4}C)^2 - 3DE,$$

$$27I_6 = 8\,(A + \tfrac{1}{4}C)^3 - 9DE\,(A + \tfrac{1}{4}C) + \tfrac{27}{4}BD^2,$$

$$(I_4)^3 - 27\,(I_6)^2 = E^2\left[(A + \tfrac{1}{4}C)^2 - DE\right] - 4B\,(A + \tfrac{1}{4}C)^3 + \tfrac{9}{2}BDE\,(A + \tfrac{1}{4}C)$$
$$- \tfrac{27}{16}B^2D^2.$$

This factorial form will be employed to prove that sphero-cyclides have two double and four single foci.

11. To transform from a sphero-conic to a sphero-cyclide.

I. If O be any origin of coordinates, p the perpendicular arc drawn from it on any tangent arc of the conic, its equation in two-line coordinates is

$$(u\cos^2\theta+2w'\cos\theta\sin\theta+v\sin^2\theta)\cot^2 p$$
$$+2(u'\sin\theta+v'\cos\theta)\cot p+w=0.$$

Let r denote a corresponding arc of the sphero-cyclide, and δ a constant. The formulæ of quadric transformation may take either of the forms (Casey, § 24),

$$\cos p=\cos(p-r)\cos\delta,\quad \cot p=\frac{\tan r}{\sec r\sec\delta-1}\ \ldots\ldots\ldots(1).$$

The transformed equation denotes the sphero-cyclide in two-point coordinates,

$$ux^2+2w'xy+vy^2+2(u'y+v'x)(\sec r\sec\delta-1)+w(\sec r\sec\delta-1)^2,$$
$$(\sec^2 r=1+\tan^2 r=1+x^2+y^2).$$

Formulæ of inversion are derived from (1),

$$\sec\delta=\cos r+\sin r\tan p.$$

Denote by r_1, r_2 the vector arcs of two conjugate points P_1, P_2,

$$\cos r_1\cos r_2=\sec^2\delta\cos^2 p-\sin^2 p,\quad \cos r_1+\cos r_2=2\sec\delta\cos^2 p\,;$$

$$\tan\frac{r_1}{2}\tan\frac{r_2}{2}=\frac{\sec\delta-1}{\sec\delta+1}=\tan^2\frac{\delta}{2}\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

O is therefore a centre of inversion, such that the curve and its equation are unaltered, when $\cot\frac{R}{2}\tan^2\frac{\delta}{2}$ is substituted for $\tan\frac{r}{2}$.

The centre of inversion O and the radius δ are arbitrary; but, when they are once fixed, the other centres A, B, C, and the other constants δ_1, δ_2, δ_3 are mutually related by the coorthotomic conditions of § 9.

The two conjugate points P_1, P_2 are the points of intersection in two consecutive positions of the generating circle, which cuts orthogonally the four fixed (J) circles, whose centres are A, B, C, O.

II. Dr. Casey has also assigned a remarkably elegant mode of transformation for three-point coordinates. (Casey, § 40.)

If U, V, W denote in three-point coordinates the fixed coorthotomic circles J_1, J_2, J_3, and if the dirigent focal sphero-conic (F) be defined by the tangential equation

$$(a, b, c, f, g, h \between p, q, r)^2=0,$$

then the sphero-cyclide, thence generated, has the identical form of equation $(a, b, c, f, g, h \between U, V, W)^2 = 0.$

The triangle of reference is constituted by the centres of the (J) circles. It should be premised that, if the coordinates denote a point not on a circle U,

$$U = \cos AP - \cos \delta_1.$$

This follows from the equation to a circle, whose centre is (l, m),

$$U \equiv \frac{lx+my+1}{\sqrt{(l^2+m^2+1)}\,\sqrt{(x^2+y^2+1)}} - \cos \delta_1 = 0.$$

It denotes the distance of that point from the plane of the small circle.

To prove that $p : q : r :: U : V : W$. Let F be a dirigent focal conic, O the corresponding centre of inversion; so that, by § 11 (I.),

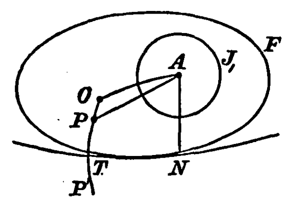

FIG. 2.

$$\cos OT = \cos \delta \cos PT = \cos \delta \cos P'T.$$

Let this be written

$$\cos P = \cos \delta \cos (P-R) \;\ldots\ldots\; (1).$$

Let A be the centre, and δ_1 the radius of (J_1), one of the other three centres of inversion; $AN = p$: TO, AN, when produced, form an angle θ.

$$\cos AO = \sin P \sin p + \cos P \cos p \cos \theta,$$

$$\cos AP = \sin p \sin (P-R) + \cos p \cos (P-R) \cos \theta.$$

Eliminate θ by the aid of (1),

$$\cos AO \sec P - \cos AP \sec (P-R) = \sin p \sin R \sec P \sec (P-R).$$

Hence $U = \cos AP - \cos \delta_1 = \cos AP - \cos AO \sec \delta$, from orthotomy (§ 9),

$$= \cos AP - \cos AO \cos (P-R) \sec P$$

$$= -\sin p \sin R \sec P,$$

V, W are like multiples of $\sin q$, $\sin r$; so that, after dropping the word sin, as stated in § 6, the formulæ for quadric transformation are

$$p : q : r = U : V : W.$$

I. *On the General Form of Sphero-Cyclides.*

12. To determine the equations to the four fixed co-orthotomic

small circles J, J_1, J_2, J_3, and the corresponding doubly confocal dirigent and focal sphero-conics (F), by means of which the sphero-cyclide is generated in four different ways.

Let O, A, B, C, the orthocentres of the spherical triangles ABC, BOC, COA, AOB, be the centres of four fixed coorthotomic circles J, J_1, J_2, J_3. The same centres, taken by triads, are the angular points of triangles, which are self-conjugate with respect to the four (J) circles, and to the four corresponding (F) dirigent sphero-conics.

Let ABC be first taken as the triangle of reference; then the equation in spherics to the circle (J), with respect to which it is self-conjugate, is

$$(J) \quad \alpha^2 \cos A \tan a + \beta^2 \cos B \tan b + \gamma^2 \cos C \tan c = 0,$$

or, by the aid of the fundamental relation

$$(6V)^2 = \Sigma\,(a^2\alpha^2 + 2bc\,\beta\gamma \cos a),$$

$$(J) \quad \alpha \tan a + \beta \tan b + \gamma \tan c = 6V\surd(\sec a \sec b \sec c).$$

In like manner (J_1), one of the other three circles, may be denoted in four different forms of the same equation:

$$(J_1) \quad \alpha^2 \cos A \tan a + \beta^2 \cos B \tan b + \gamma^2 \cos C \tan c$$
$$-2\alpha \cos A\,(\alpha \tan a + \beta \tan b + \gamma \tan c) + \alpha^2 \cos^2 A \tan a \tan b \tan c = 0,$$

$$(J_1) \quad \alpha \tan a + \beta \tan b + \gamma \tan c - 6V\surd(\sec a \sec b \sec c)$$
$$+ \alpha \tan a \tan b \tan c \cos A = 0,$$

$$(J_1) \quad a\alpha + b\beta \cos c + c\gamma \cos b = 6V \sec a\,\surd(\cos a \cos b \cos c),$$

$$(J_1) \quad \alpha' = \sec a\,\surd(\cos a \cos b \cos c),$$

if α', β', γ' denote the coordinates of a point in (J_1), with respect to the polar triangle of ABC.

This last form determines independently the radius of the (J_1) circle given in § 9,

$$\cos a \cos \delta_1 = \surd(\cos a \cos b \cos c).$$

In like forms the equations to the two remaining circles J_2, J_3 may be written. From the forms of all four equations to these circles, it is recognised that, of their twenty-four points of intersection, twelve lie on the perpendiculars OA, OB, OC, BC, CA, AB. These arcs are therefore their radical axes; and O, A, B, C are the radical centres of the four triads of circles.

The equation to some one dirigent conic (F), that corresponding to the (J) circle, is assumed to be

$$la^2\alpha^2+mb^2\beta^2+nc^2\gamma^2=0,$$

or, in three-line coordinates,

$$\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0.$$

The equations to the confocal conics will be expressed in terms of (F) in § 15.

13. To find the equation to the sphero-cyclide, by considering it as the envelope of a circle, whose centre moves on the focal conic (F), and which cuts the circle (J) orthogonally.

Let (λ, μ, ν) be the centre of the variable circle; its equation is (§ 9)

$$(6V)^2\cos r = a\lambda\,(a\alpha+b\beta\cos c+c\gamma\cos b)+\ldots,$$

or, if it be referred to the polar triangle of ABC,

$$6V\cos r = a\lambda\alpha'+b\mu\beta'+c\nu\gamma'.$$

For the fixed (J) circle (§ 12),

$$\alpha\tan a+\beta\tan b+\gamma\tan c = 6V\sqrt{(\sec a\sec b\sec c)}.$$

From the condition of orthotomy (§ 8)

$$(1)\quad \lambda\tan a+\mu\tan b+\nu\tan c = 6V\cos r\sqrt{(\sec a\sec b\sec c)}$$
$$= (a\lambda\alpha'+b\mu\beta'+c\nu\gamma')\sqrt{(\sec a\sec b\sec c)}.$$

The centre moves on the dirigent (F),

$$(2)\quad la^2\lambda^2+mb^2\mu^2+nc^2\nu^2=0.$$

The equation to the sphero-cyclide, as the envelope of (1), subject to the condition (2), is

$$\frac{1}{la^2}\{a\alpha'\sqrt{(\sec a\sec b\sec c)}-\tan a\}^2$$
$$+\frac{1}{mb^2}\{b\beta'\sqrt{(\sec a\sec b\sec c)}-\tan b\}^2$$
$$+\frac{1}{nc^2}\{c\gamma'\sqrt{(\sec a\sec b\sec c)}-\tan c\}^2=0.$$

If we revert to the primitive triangle of reference ABC, it is written

$$\Sigma\,\frac{1}{l}\,\{a\alpha+b\beta\cos c+c\gamma\cos b-6V\sec a\sqrt{(\cos a\cos b\cos c)}\}^2=0.$$

14. If U, V, W denote, as in § 11, the J_1, J_2, J_3 circles, the equation to the sphero-cyclide takes the form

$$\frac{U^2}{l}+\frac{V^2}{m}+\frac{W^2}{n}=0.$$

If this form be compared with the tangential equation of (F),

$$\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0,$$

as in § 11, $$p:q:r::U:V:W.$$

The equation to the sphero-cyclide was anticipated from Dr. Casey's theorem.

15. To express the three focal sphero-conics F_1, F_2, F_3 in terms of their confocal F.

Being given $$F\equiv\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0,$$

if ABC be the triangle of reference, to determine the coefficients, if OBC be the new triangle, in the assumed equation

$$F_1\equiv\frac{P^2}{l_1}+\frac{q^2}{n_1}+\frac{r^2}{m_1}=0.$$

P denotes the perpendicular from O ($\alpha\cos A=\beta\cos B=\gamma\cos C$) on any tangent arc, so that

$$P^2\Sigma(\tan^2 A+2\tan B\tan C\cos a)=(p\tan A+q\tan B+r\tan C)^2$$
$$=\tan A\tan B\tan C\,(p^2\tan A\cos a+q^2\tan B\cos b+r^2\tan C\cos c)$$
$$-\Sigma(p^2\sin^2 A-2qr\cos A\sin B\sin C)\sec A\sec B\sec C.$$

Make this substitution, and denote by μ, as in § 9, the ratio

$$\Sigma(\tan^2 A+2\tan B\tan C\cos a)=\mu^2\tan^2 A\tan^2 B\tan^2 C,$$

$$F_1\equiv\frac{1}{l_1\mu^2}\cot A\cot B\cot C\,(p^2\tan A\cos a+q^2\tan B\cos b+r^2\tan C\cos c)$$
$$-\frac{6V}{l_1\mu^2}\cot A\cot B\cot C+\frac{q^2}{n_1}+\frac{r^2}{m_1}.$$

Since F and F_1 are doubly confocal conics, they must be identical, if the constant term be omitted. Write θ for $\frac{1}{\mu^2}\cot A\cot B\cot C$, and equate coefficients.

$$\frac{1}{l} = \frac{\theta}{l_1} \tan A \cos a, \quad \frac{1}{m} = \frac{\theta}{l_1} \tan B \cos b + \frac{1}{n_1};$$

$$\frac{1}{n} = \frac{\theta}{l_1} \tan C \cos c + \frac{1}{m_1}.$$

Substitute this value for θ in F_1,

$$F_1 \equiv \frac{p^2}{l} + \frac{q^2}{m} + \frac{r^2}{n} - \frac{6V}{l} \cot A \sec a = 0.$$

F_2, F_3 have similar forms.

16. The following identity connects the several (J) forms of § 12:

$$\tan A \cos a \{a\alpha + b\beta \cos c + c\gamma \cos b - 6V \sec a \sqrt{(\cos a \cos b \cos c)}\}^2$$
$$+ \tan B \cos b \{a\alpha \cos c + b\beta + c\gamma \cos a - 6V \sec b \sqrt{(\cos a \cos b \cos c)}\}^2$$
$$+ \tan C \cos c \{a\alpha \cos b + b\beta \cos a + c\gamma - 6V \sec c \sqrt{(\cos a \cos b \cos c)}\}^2$$
$$= \cos^2 a \cos^2 b \cos^2 c \tan A \tan B \tan C$$
$$\times \{\alpha \tan a + \beta \tan b + \gamma \tan c - 6V \sqrt{(\sec a \sec b \sec c}\}^2.$$

The proof depends upon identities of the type

$$\tan A + \tan B \cos c + \tan C \cos b = \cos b \cos c \tan A \tan B \tan C.$$

Hence it may be shown that the equation to the sphero-cyclide may be obtained from any other pair of circles (J_1) and dirigent conics (F_1).

When referred to tangential coordinates, OBC being the triangle considered,

$$(F_1) \qquad \frac{p^2}{l_1} + \frac{q^2}{n_1} + \frac{r^2}{m_1} = 0.$$

By Prof. Casey's theorem, cited in § 11, II., the equation is deduced to the sphero-cyclide

$$\frac{J_1^2}{l_1} + \frac{J_2^2}{n_1} + \frac{J_3^2}{m_1} = 0.$$

But, from this article,

$$\frac{J^2}{t} = \tan A \cos a J_1^2 + \tan B \cos b J_2^2 + \tan C \cos c J_3^2,$$

if
$$\frac{1}{t} = \cos^2 a \cos^2 b \cos^2 c \tan A \tan B \tan C.$$

If this value of J^2 be substituted, and the result compared with the former equation $$\frac{J_1^2}{l}+\frac{J_2^2}{m}+\frac{J_3^2}{n}=0,$$

it is seen that

$$\frac{1}{l}=\frac{t}{l_1}\tan A\cos a,\qquad \frac{1}{m}=\frac{t}{l_1}\tan B\cos b+\frac{1}{n_1},$$
$$\frac{1}{n}=\frac{t}{l_1}\tan C\cos c+\frac{1}{m_1}.$$

These relations are those given in § 15.

17. If a spherical quadrilateral be circumscribed about a circle of inversion and its corresponding dirigent conic, the other three confocal dirigent conics pass through the three quartets of opposite intersections. (Casey on "Cyclides," § 124.)

The tangential equations to such a (J) circle and (F) conic (§ 12)

are $(J)\quad p^2\tan A\cos a+q^2\tan B\cos b+r^2\tan C\cos c=0,$

$$(F)\qquad \frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}=0.$$

The triangle of reference is constituted by the three vertices of the quadrangle of intersection, as before. Whence

$$\frac{p^2}{l}:\frac{q^2}{m}:\frac{r^2}{n}::m\tan B\cos b-n\tan C\cos c$$
$$:n\tan C\cos c-l\tan A\cos a$$
$$:l\tan A\cos a-m\tan B\cos b.$$

The spherical quadrilateral, thus constituted, is defined by the linear

equations $$a\alpha p\pm b\beta q\pm c\gamma r=0.$$

Two points of intersection, as well as their two antipodal points, lie on the arc BC, which passes through two vertices of the quadrangle

of intersection, $$\alpha=0,\quad b^2\beta^2q^2=c^2\gamma^2r^2.$$

Since the line-equation to F_1 (by § 15) is

$$(F_1)\quad \frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}-\frac{1}{6Vl\tan A\cos a}\Sigma\,(a^2p^2-2bcqr\cos A)=0,$$

the transformed point-equation is

$$(F_1)$$

$$a^2\cos A\,(\mu\nu-\cos^2 A)-\frac{b\beta^2}{cn}\cos B\cos C\sin A\,(n\tan C\cos c-l\cos a\tan A)$$
$$-\frac{c\gamma^2}{bm}\cos B\cos C\sin A\,(m\tan B\cos b-l\cos a\tan A)$$
$$+2a\beta\cos B\,(\sin^2 A-\sin^2 C)+2a\gamma\cos C\,(\sin^2 A-\sin^2 B)=0,$$

if $$(1-\mu)\,b^2m=(1-\nu)\,c^2n=6Vl\tan A\cos a.$$

When $a=0$

$$mb^2\beta^2\,(n\tan C\cos c-l\cos a\tan A)=nc^2\gamma^2\,(l\cos a\tan A-m\cos b\tan B),$$

or $$b^2\beta^2q^2=c^2\gamma^2r^2.$$

The arc BC therefore meets the conic (F_1) in the preceding points.

Similarly, the other two conics (F_2), (F_3) may be shown to pass through the other intersections of the quadrilateral.

18. To find the equation to the sphero-cyclide in two-point coordinates, the origin being the centre of a dirigent focal conic.

The coordinates in Gudermann's system represent tangents of arcs, so that x, y; a, b represent

$$\tan x,\ \tan y;\quad \tan a,\ \tan b;\quad \sec^2 r=1+x^2+y^2.$$

The equation to a (J) circle, whose centre is (f, g) is

$$(J)\quad \cos\delta\sqrt{(1+f^2+g^2)}\sqrt{(1+x^2+y^2)}=1+fx+gy,$$

or $$1+fx+gy=\cos\delta\sec R\sec r=t\sec r,$$

where t denotes the secant of the tangent arc drawn from the origin.

The equation to the generating circle, whose centre is (a, β) is

$$1+ax+\beta y=T\sec r.$$

The condition of orthotomy (§ 8) gives the relation

$$1+af+\beta g=tT.$$

For the generating circle, when T is eliminated,

$$t\,(1+ax+\beta y)=(1+af+\beta g)\sec r\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

For the dirigent conic

$$(F)\quad \frac{a^2}{a^2}+\frac{\beta^2}{b^2}=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The circle (1) varies, subject to the condition (2); hence

$$(f\sec r-tx)\,d\alpha+(g\sec r-ty)\,d\beta=0,$$

$$\frac{\alpha\,d\alpha}{a^2}+\frac{\beta\,d\beta}{b^2}=0.$$

The required equation to the sphero-cyclide is

$$(t-\sec r)^2=a^2(f\sec r-tx)^2+b^2(g\sec r-ty)^2.$$

When rationalised,

$$\{(a^2f^2+b^2g^2-1)\sec^2 r+t^2(a^2x^2+b^2y^2-1)\}^2=4t^2\sec^2 r\,(a^2fx+b^2gy-1)^2.$$

If this be combined with the imaginary great circle $\sec^2 r=0$,

$$(a^2x^2+b^2y^2-1)^2=0 \quad\text{or}\quad \{(a^2-b^2)\,x^2-(1+b^2)+b^2(1+x^2+y^2)\}^2=0.$$

The sphero-cyclide has two double cyclic arcs, which are the single cyclic arcs of the polar or complementary conic of (F) the focal conic. The line through the nodes is the polar of (f, g) with respect to this polar conic $(a^2x^2+b^2y^2=1)$.

The formulæ of quadric transformation from the tangential equation of (F) $(a^2\xi^2+b^2\eta^2=1)$ are seen to be

$$\xi:\eta:1=f\sec r-tx:g\sec r-ty:t-\sec r,$$

where $$t=\cos\delta\,\sqrt{(1+f^2+g^2)}.$$

19. If the four (F) conics are given, to determine the four corresponding (J) circles.

The equation to the sphero-cyclide may take other three forms of the above type,

$$(t_1-\sec r)^2=a_1^2(f_1\sec r-t_1x)^2+b_1^2(g_1\sec r-t_1y)^2.$$

For the confocal conics,

$$\frac{1+a^2}{1+b^2}=\frac{1+a_1^2}{1+b_1^2}=\frac{1+a_2^2}{1+b_2^2}=\frac{1+a_3^2}{1+b_3^2}=1+\gamma^2;\quad [\gamma=\tan(OS)];$$

O, S being a common centre and focus of the confocal conics.

By equating coefficients in the identical forms, when developed,

$$\left.\begin{aligned}\frac{1}{t}(a^2f^2+b^2g^2-1+a^2t^2)&=\frac{1}{t_1}(a_1^2f_1^2+b_1^2g_1^2-1+a_1^2t_1^2)\\ \frac{1}{t}(a^2f^2+b^2g^2-1+b^2t^2)&=\frac{1}{t_1}(a_1^2f_1^2+b_1^2g_1^2-1+b_1^2t_1^2)\\ \frac{1}{t}(a^2f^2+b^2g^2-1-t^2)&=\frac{1}{t_1}(a_1^2f_1^2+b_1^2g_1^2-1-t_1^2)\end{aligned}\right\}\quad\ldots\ldots(1),$$

$$a^2f = a_1^2f_1 = a_2^2f_2 = a_3^2f_3 = \lambda \text{ suppose,}$$
$$b^2g = b_1^2g_1 = b_2^2g_2 = b_3^2g_3 = \mu.$$

The line through the nodes ($a^2fx+b^2gy = 1$) is fixed and seen to be the polar of the centre of each (J) circle with respect to the polar conic of its corresponding (F) conic.

$$(a^2+1)\, t = (a_1^2+1)\, t_1 = \dots = \nu \text{ suppose,}$$
$$(b^2+1)\, t = (b_1^2+1)\, t_1 = \dots = \frac{\nu}{1+\gamma^2}.$$

(1) may be written

$$\frac{1}{t}\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}-1\right)+a^2t = \frac{1}{t_1}\left(\frac{\lambda^2}{a_1^2}+\frac{\mu^2}{b_1^2}-1\right)+a_1^2t_1.$$

This is simplified to the condition of orthotomy,

$$\frac{\lambda^2}{a^2a_1^2}+\frac{\mu^2}{b^2b_1^2}+1 = tt_1, \quad \text{or} \quad ff_1+gg_1+1 = tt_1.$$

By symmetry, $\qquad ff_2+gg_2+1 = tt_2.$

By subtraction,

$$\frac{\lambda f}{a_1^2a_2^2}+\frac{\mu g}{b_1^2b_2^2} = 1, \quad \text{or} \quad \frac{\lambda^2}{a^2a_1^2a_2^2}+\frac{\mu^2}{b^2b_1^2b_2^2} = 1.$$

By symmetry, $\qquad \dfrac{\lambda^2}{a^2a_1^2a_3^2}+\dfrac{\mu^2}{b^2b_1^2b_3^2} = 1.$

Hence $\qquad \lambda^2\gamma^2 = a^2a_1^2a_2^2a_3^2, \quad \dfrac{\mu^2\gamma^2}{1+\gamma^2} = -b^2b_1^2b_2^2b_3^2.$

Since $\qquad \dfrac{\lambda^2}{a^2a_1^2}+\dfrac{\mu^2}{b^2b_1^2}+1 = tt_1 = \dfrac{\nu^2}{(a^2+1)(a_1^2+1)},$

$$\nu^2(1+\gamma^2) = (a^2+1)(a_1^2+1)(a_2^2+1)(a_3^2+1).$$

Thus the radii (δ) and the centres (f, g) of the (J) circles have been determined in terms of the axes of the (F) sphero-conics.

Cor.—If the centres of the (J) circles are collinear, the foci of the sphero-cyclide are also collinear. In this case, three, and not four, pairs of (F) focal conics and (J) circles are necessary; if $g = g_1 = g_2 = g_3 = 0$, $\lambda^2 = a^2a_1^2a_2^2$, $\nu^2 = (a^2+1)(a_1^2+1)(a_2^2+1)$.

20. To determine the two double and four single foci of the sphero-cyclide, and its equivalent class-octavic.

Its equation is taken from § 19, when m is written for $a^2f^2+b^2g^2-1$,

$$\{m(x^2+y^2+1)+t^2(a^2x^2+b^2y^2-1)\}^2 = 4t^2(x^2+y^2+1)(a^2fx+b^2gy-1)^2.$$

The equivalent class-equation is found by combining it with an arbitrary tangent $(x\xi+y\eta-1)$; the resulting binary quartic must have two equal roots, in which case $(I_4)^3-27\,(I_6)^2=0$, as in § 10.

The resulting binary quartic is

$$\{[m+a^2t^2+(m-t^2)\,\xi^2]\,x^2+2\,(m-t^2)\,\xi\eta\,xy+[m+b^2t^2+(m-t^2)\,\eta^2]\,y^2\}^2$$
$$=4t^2\,[(\xi^2+1)\,x^2+2\xi\eta xy+(\eta^2+1)\,y^2]\,[(a^2f-\xi)\,x+(b^2g-\eta)\,y]^2.$$

Then the system of invariants, given in § 10, has the following values:

$$A=(m+a^2t^2)(m+b^2t^2)+(m-t^2)\,[(m+a^2t^2)\,\eta^2+(m+b^2t^2)\,\xi^2]$$
$$=t^4\,(a^2+1)\,(b^2+1)+(t^4-mt^2)\,(a^2\xi^2+b^2\eta^2-1)$$
$$+[t^4+mt^2\,(a^2+b^2-1)+m^2]\,(\xi^2+\eta^2+1),$$
$$B=\xi^2+\eta^2+1.$$

In determining the foci, B and all other terms, which involve $(\xi^2+\eta^2+1)$ as a factor, are neglected, since the foci are obtained by the intersections of common tangents to the quartic and this imaginary great circle.

$$\frac{C}{4t^2}=(a^2f-\xi)^2+(b^2g-\eta)^2+(a^2f\eta-b^2g\xi)^2$$
$$=(a^4f^2+b^4g^2+1)\,(\xi^2+\eta^2+1)-(a^2f\xi+b^2g\eta-1)^2,$$
$$\frac{D}{4t^2}=(m+b^2t^2)(a^2f-\xi)^2+(m+a^2t^2)(b^2g-\eta)^2+(m-t^2)(a^2f\eta-b^2g\xi)^2$$
$$=t^2\,[(b^2+1)(a^2f-\xi)^2+(a^2+1)(b^2g-\eta)^2]+(t^2-m)(a^2f\xi+b^2g\eta+1)^2$$
$$-(t^2-m)(a^4f^2+b^4g^2+1)(\xi^2+\eta^2+1),$$
$$\frac{E}{4t^2}=(\xi^2+1)\,[m+b^2t^2+(m-t^2)\,\eta^2]-2\xi^2\eta^2\,(m-t^2)$$
$$+(\eta^2+1)\,[m+a^2t^2+(m-t^2)\,\xi^2]$$
$$=-t^2\,(a^2\xi^2+b^2\eta^2-1)+[2m+(a^2+b^2-1)\,t^2]\,(\xi^2+\eta^2+1).$$

If these values be substituted in the expression for $(I_4)^3-27\,(I_6)^2$ in § 10, the equivalent class octavic may be obtained.

The foci are found by rejecting B, and therefore the last three terms. For the double foci,

$$E^2=t^4\,(a^2\xi^2+b^2\eta^2-1)^2.$$

They are therefore the single foci of (F) the focal conic, from which the sphero-cyclide was generated.

The four single foci are those of the class-quartic

$$\left(A+\frac{C}{4}\right)^2-DE.$$

Cor.—The two double foci unite in a quadruple focus, when $a = b$, or (F) becomes a circle. This sphero-cyclide is called by Dr. Casey a sphero-Cartesian.

21. The sphero-cyclide has in all 28 foci, real and imaginary.

Prof. Cayley remarks that this quartic has two nodes, and, besides, touches the imaginary circle $S\,(x^2+y^2+1=0)$ in four points. The number of its class is thus $4.3-2.2$, or 8; and the number of common tangents to the quartic and the circle S would thus be 2.8 or 16; but among these are included the four lines touching along the points of contact, each twice; the number of common tangents is thus $16-2.4$, or 8.

These eight lines intersect in $28\left(=\frac{8.7}{1.2}\right)$ points, which are the foci of this quartic.

II. *On Sphero-cyclides with Collinear Foci.*

22. Their equations have been deduced in § 19, Cor., from the general form; but they are here also investigated, when a focus of the focal conic is the origin, and the constants determined in terms of the directrices.

There are only three (F) focal sphero-conics,

$$(F_1)\quad \lambda\,(x^2+y^2)-(x-a)^2=0, \qquad (F_2)\quad \mu\,(x^2+y^2)-(x-b)^2=0,$$

$$(F_3)\quad \nu\,(x^2+y^2)-(x-c)^2=0.$$

There are three corresponding coorthotomic circles of inversion,

$$(J_1)\ 1+fx=\rho_1\sec r,\quad (J_2)\ 1+gx=\rho_2\sec r,\quad (J_3)\ 1+hx=\rho_3\sec r.$$

Let the generating circle for the first pair be

$$1+ax+\beta y=\gamma\,(1+x^2+y^2)^{\frac{1}{2}}=\gamma\sec r.$$

The condition of orthotomy (§ 8) is

$$1+af=\rho_1\gamma.$$

The envelope is required of the variable circle

$$\rho_1\,(1+ax+\beta y)=(1+af)\sec r,$$

subject to the dirigent condition, that its centre moves on the conic,

$$\lambda\,(a^2+\beta^2)-(a-a)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (F_1).$$

The equation to the envelope is found, as in § 19, to be

$$\{(1+af)\sec r-\rho_1\,(1+ax)\}^2+a^2\rho_1^2y^2=\lambda\,(\sec r-\rho_1)^2.$$

But, since the dirigent conics are doubly confocal, S, H being the common foci,

$$\frac{1}{2a}(\lambda-1+a^2)=\frac{1}{2b}(\mu-1+b^2)=\frac{1}{2c}(\nu-1+c^2)=d=-\cot SH.$$

The preceding equation to the sphero-cyclide may be written

$$\{a(\rho_1^2+f^2+1)+2f-2d\}\sec^2 r$$
$$+2\rho_1\{2d-(1+af)\,x-f-a\}\sec r+2\rho_1^2(x-d)=0.$$

The quadrantal polars of the origin $(1=0)$, and of the other common focus $(x=d)$, are double cyclic arcs of the sphero-cyclide.

The line of nodes $[(1+af)\,x+f+a=2d]$ is the polar of the centre of (J) with respect to the polar conic of (F_1),

$$(ax+1)^2+a^2\left(1-\frac{1}{\lambda}\right)y^2=\lambda.$$

23. Two other forms may be written, in which ρ_2, g, b; ρ_3, h, c take the place of ρ_1, f, a. It is proposed to obtain thereby another form of the quartic, in which the coefficients shall be functions of a, b, c, the tangents of the distances of the directrices from that common focus, which is the origin of coordinates.

Equate the coefficients in the preceding and the identical quartic

$$[(1+bg)^2+b^2\rho_2^2-\mu]\sec^2 r-2\rho_2[(bg+1)+(bg+1)\,bx-\mu]\sec r$$
$$+2b\rho_2^2(x-d)=0,$$

$$\frac{1}{\sqrt{K}}=\frac{af+1}{\rho_1}=\frac{bg+1}{\rho_2},\qquad \frac{af+1-\lambda}{a\rho_1}=\frac{bg+1-\mu}{b\rho_2}=\frac{H}{\sqrt{K}},$$

$$\frac{a^2\rho_1^2+(af+1)^2-\lambda}{a\rho_1^2}=\frac{b^2\rho_2^2+(bg+1)^2-\mu}{b\rho_2^2}.$$

The symbols H, K are introduced for subsequent use.

These relations may be combined, so as to express ρ_1^2 in terms of f:

$$\frac{\rho_1}{\rho_2}=\frac{(a-b)(af+1)+b\lambda}{a\mu},$$

$$\rho_1^2(a-b)-\frac{a-b}{ab}(af+1)^2-\frac{\lambda}{a}+\frac{1}{a^2b\mu}[(a-b)(af+1)+b\lambda]^2=0.$$

After reduction,

$$\mu\rho_1^2+f^2(ab-1-2ad)-2f(a-b)+ab-1-2bd=0 \quad \text{......(1)}.$$

By symmetry,

$$\nu\rho_1^2+f^2(ac-1-2ad)-2f(a-c)+ac-1-2cd=0\ldots\ldots\ldots(2).$$

The symbols λ, μ, ν are retained, for convenience, as known functions of a, b, c, and d.

Subtract (2) from (1), and reject the factor $(b-c)$.

$$(af+1)^2-\lambda-a\rho_1^2(b+c-2d)=0,$$

or $$(af+1)^2+a^2\rho_1^2-\lambda=a\rho_1^2(a+b+c-2d).$$

Multiply (1) by c, and (2) by b; subtract, and reject the factor $(b-c)$.

$$(1+bc)\rho_1^2=(1+af)^2+\lambda f^2.$$

By eliminating ρ_1^2, there results a quadratic, which indicates two (J_1) circles,

$$f^2[-abc-a+b+c+2d(ab+ac-1)-4ad^2]$$
$$+2f(-bc+ab+ac-1-2ad)-abc-a+b+c+2bcd=0.$$

For brevity, write this quadratic

$$Af^2+2Bf+C=0.$$

The following relations connect the coefficients:

$$2Bd=A-C,\quad B^2-AC=\lambda\mu\nu.$$

Whence $$Af+B=\surd(\lambda\mu\nu),$$

$$A(af+1)=A-aB+a\surd(\lambda\mu\nu)=\lambda(b+c-2d)+a\surd(\lambda\mu\nu),$$

$$\frac{A}{a}(1-\lambda)-B=\frac{\lambda}{a}(b+c-2d-A)=\lambda+\lambda(b-2d)(c-2d)$$

$$=\lambda+\frac{\lambda}{bc}(1-\mu)(1-\nu).$$

We have the proportion above stated

$$\frac{1}{a}\left(1-\frac{\lambda}{af+1}\right)=\frac{1}{b}\left(1-\frac{\mu}{bg+1}\right)=H.$$

Substitute the preceding value of $af+1$, and a similar value of $bg+1$,

$$H=\left[\frac{\lambda}{bc}(1-\mu)(1-\nu)+\lambda+\surd(\lambda\mu\nu)\right][\lambda(b+c-2d)+a\surd(\lambda\mu\nu)]^{-1}$$

$$=\left[\frac{\mu}{ac}(1-\nu)(1-\lambda)+\mu+\surd(\lambda\mu\nu)\right][\mu(c+a-2d)+b\surd(\lambda\mu\nu)]^{-1}.$$

$$\text{Dividendo,} = \left\{\frac{1-\nu}{c}\left[\frac{\lambda}{b}(1-\mu)-\frac{\mu}{a}(1-\lambda)\right]+\lambda-\mu\right\}$$

$$\times\left\{(a-b)\surd(\lambda\mu\nu)+\frac{\mu}{c-a}(\nu-\lambda)-\frac{\lambda}{b-c}(\mu-\nu)\right\}^{-1}$$

$$=\{-(a-2d)(b-2d)(c-2d)-a-b-c+4d\}$$

$$\times\left\{\surd(\lambda\mu\nu)-1+\frac{1}{2d}(a-2d)(b-2d)(c-2d)-\frac{abc}{2d}\right\}^{-1},$$

which is a symmetrical function.

By the aid of this proportion, the coefficients may be expressed.

From the former proportions,

$$\frac{1}{\rho_1^2}(af+1)^2=\frac{1}{\rho_2^2}(bg+1)^2,$$

$$(af+1)^2-\lambda=a\rho_1^2(b+c-2d), \quad \text{and} \quad \frac{1}{a}\left(1-\frac{\lambda}{af+1}\right)=H.$$

$$\text{Hence} \quad K=\rho_1^2(af+1)^{-2}=\frac{1}{a}(b+c-2d)^{-1}[1-\lambda(af+1)^{-2}]$$

$$=\frac{1}{a}(b+c-2d)^{-1}\left[1-\frac{1}{\lambda}(aH-1)^2\right],$$

by symmetry,

$$=\frac{1}{b}(c+a-2d)^{-1}\left[1-\frac{1}{\mu}(bH-1)^2\right],$$

dividendo,

$$=\left[\left(\frac{b^2}{\mu}-\frac{a^2}{\lambda}\right)H^2-2\left(\frac{b}{\mu}-\frac{a}{\lambda}\right)H+\frac{1}{\mu}-\frac{1}{\lambda}\right](a-b)^{-1}(c-2d)^{-1}$$

$$=\frac{1}{\lambda\mu}(c-2d)^{-1}[-(2abd+a+b)H^2+2(ab+1)H+2d-a-b],$$

by symmetry,

$$=\frac{1}{\mu\nu}(a-2d)^{-1}[-(2bcd+b+c)H^2+2(bc+1)H+2d-b-c],$$

dividendo,

$$=\frac{a-c}{\mu}[-(2bd+1)H^2+2bH-1][\lambda(c-2d)-\nu(a-2d)]^{-1}$$

$$=\frac{1}{\mu}[(2bd+1)H^2-2bH+1]\left[1+\frac{1}{ac}(1-\lambda)(1-\nu)\right]^{-1},$$

by symmetry,

$$=\frac{1}{\nu}\left[(2cd+1)H^2-2cH+1\right]\left[1+\frac{1}{ab}(1-\lambda)(1-\mu)\right]^{-1},$$

dividendo,

$$=(-2dH^2+2H)\left\{\frac{1}{a}(1-\lambda)\left[1+\frac{1}{bc}(1-\mu)(1-\nu)\right]+b+c-2d\right\}^{-1}$$

$$=2H(-dH+1)\left\{\frac{1}{abc}(1-\lambda)(1-\mu)(1-\nu)+\frac{1}{a}(1-\lambda)+\frac{1}{b}(1-\mu)+\frac{1}{c}(1-\nu)+2d\right\}^{-1}.$$

The equation to the sphero-cyclide may therefore be thus expressed symmetrically in terms of the tangents of the distances of the directrices of the focal conics:

$$\frac{1}{2}(a+b+c-2d)\sec^2 r-\frac{1}{\sqrt{K}}(x+H)\sec r+x-d=0.$$

Cor. 1.—If $a+b+c=2d$, the satellite-conic degenerates into the cyclic arcs.

Cor. 2.—If $K=0$, the sphero-cyclide degenerates into the line of nodes twice repeated, and the imaginary great circle.

Cor. 3.—If $K=\infty$, or $(a-2d)(b-2d)(c-2d)+a+b+c-4d$, the sphero-cyclide degenerates into two coincident conics, but retains the same cyclic arcs.

III. *On Sphero-Cartesians.*

24. To generate a Sphero-Cartesian by Laguerre's method.

There are three concentric dirigent circles,

$$x^2+y^2=a_1^2\ \ldots\ldots\ (F_1),\quad x^2+y^2=a_2^2\ \ldots\ldots\ (F_2),\quad x^2+y^2=a_3^2\ \ldots\ldots\ (F_3).$$

And to these there correspond three co-orthotomic circles of inversion,

$$1+f_1x=t_1\sec r\ \ldots\ldots\ (J_1),\qquad 1+f_2x=t_2\sec r\ \ldots\ldots\ (J_2),$$

$$1+f_3x=t_3\sec r\ \ldots\ldots\ (J_3),$$

where f_1, f_2, f_3 denote the coordinates of their centres, and t_1, t_2, t_3 the secants of the touching arcs drawn from the origin.

The equation to the cyclide is found, as in § 18, or deduced from it, as the envelope of a variable circle, which cuts the (J) circles orthogonally,

$$\left(f_1^2+t_1^2-\frac{1}{a_1^2}\right)\sec^2 r+2t_1\left(f_1 x-\frac{1}{a_1^2}\right)\sec r-t_1^2\left(1+\frac{1}{a_1^2}\right)=0.$$

Similar forms involve the constants $f_2, a_2, t_2; f_3, a_3, t_3$.

By equating the coefficients of like terms in the equivalent forms,

$$f_1 a_1^2 = f_2 a_2^2 = f_3 a_3^2 = a_1 a_2 a_3,$$

$$t_1(a_1^2+1) = t_2(a_2^2+1) = t_3(a_3^2+1) = \sqrt{\{(a_1^2+1)(a_2^2+1)(a_3^2+1)\}},$$

$$\begin{aligned}(a_1^2+1)\{(f_1^2+t_1^2)a_1^2-1\} &= (a_2^2+1)\{(f_2^2+t_2^2)a_2^2-1\}\\ &= (a_3^2+1)\{(f_3^2+t_3^2)a_3^2-1\}\\ &= 2a_1^2a_2^2a_3^2+a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2-1.\end{aligned}$$

The sphero-Cartesian can now be expressed in terms of the radii of the (F) dirigent circles

$$\begin{aligned}(2a_1^2a_2^2a_3^2+a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2-1)\sec^2 r-(a_1^2+1)(a_2^2+1)(a_3^2+1)\\ +2\sqrt{\{(a_1^2+1)(a_2^2+1)(a_3^2+1)\}}\,(a_1a_2a_3x-1)\sec r.\end{aligned}$$

25. To express the radii of the (J) circles of inversion in terms of those of the (F) dirigent circles. (Casey on "Cyclides," § 244.)

Let $\delta_1, \delta_2, \delta_3$ denote the radii of the (J) circles:

$$\sec^2\delta_1 = \frac{1+f_1^2}{t_1^2} = \left(1+\frac{a_2^2a_3^2}{a_1^2}\right)(a_1^2+1)(a_2^2+1)^{-1}(a_3^2+1)^{-1},$$

$$\tan^2\delta_1 = \frac{1}{a_1^2}\cdot\frac{a_2^2-a_1^2}{1+a_2^2}\cdot\frac{a_3^2-a_1^2}{1+a_3^2}.$$

This may be also expressed in terms of the distances ρ_1, ρ_2, ρ_3 of the centres of the (J) circles

$$\tan^2\delta_1 = \frac{f_1-f_2}{1+f_1f_2}\cdot\frac{f_1-f_3}{1+f_1f_3} = \tan(\rho_1-\rho_2)\tan(\rho_1-\rho_3).$$

IV. *On the Singularities of Sphero-cyclides.*

26. There is in all cases a pair of nodes, which may be real crunodes, or imaginary nodes; not acnodes. Thus sphero-cyclides are discriminated from other spherical binodal quartics.

When certain mutual relations exist between the parameters, which enter into their equations, there may be another node, crunode, or acnode, which unite in a cusp.

Moreover, the two nodes in the line of nodes may unite in a tacnode; and the two nodes (which are imaginary) may coalesce with a third (acnode) to form a triple point, of Salmon's special form 5°. (*Higher Plane Curves*, § 243.) See below, § 30.

Lastly, these two singularities, the tacnode and the triple point, may (in a special case) coalesce, and form a compound singularity, called a tacnode-cusp, of Salmon's special form 4°.

27. To determine the mutual relation which subsists between the parameters in the equation to a family of sphero-cyclides, which have the same line of nodes and the same double cyclic arcs, when the sphero-cyclides are trinodal.

This relation will be drawn as a first discriminating curve (D_1).

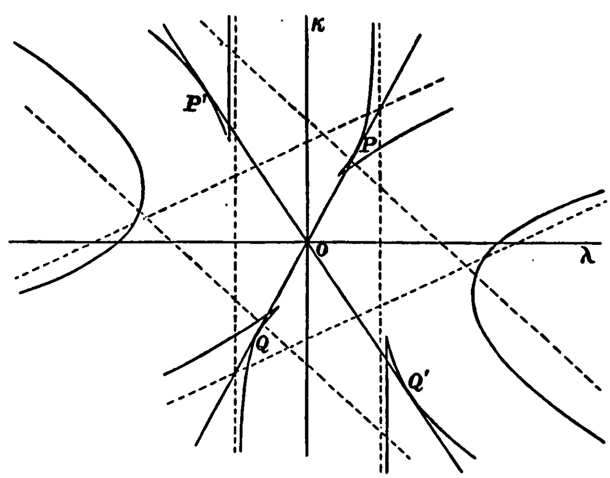

Fig. 3.

Let this equation take the form

$$\phi \equiv \kappa(1+dx)-(1+mx+py)\sec r+\lambda \sec^2 r = 0,$$

where κ, λ are parameters, $\sec^2 r = 1+x^2+y^2$ in Gudermann's system, $(1=0,\ 1+dx=0)$ denote the cyclic arcs, and $(1+mx+py=0)$ the line of nodes.

At a singular point,

$$\frac{d\phi}{dx}=0,\quad \frac{d\phi}{dy}=0.$$

For the sphero-cyclide,

$$\kappa d = m \sec r + x\,(1+mx+py)\cos r + 2\lambda x,$$

$$0 = p \sec r + y\,(1+mx+py)\cos r + 2\lambda y\,;$$

whence $\kappa\,(2+dx) = \sec r \;+\; (1+mx+py)\cos r + 2\lambda,$

and $\kappa dy = (my-px)\sec r,$

$$\kappa\,(2+dx) = \left(1-\frac{p}{y}\right)\sec r.$$

All the nodes in the family lie on the sphero-conic

$$\frac{my}{p}\left(x+\frac{2}{d}-\frac{1}{m}\right) = x^2+\frac{2x}{d}-1.$$

The locus of (κ, λ) may be drawn by points for successive values of x from $+\infty$ to $-\infty$. This curve (D_1) has a Newtonian centre, since the expressions for κ, λ both contain $\sec r$ or $(1+x^2+y^2)^{\frac{1}{2}}$.

It has six asymptotes $\lambda \pm p = 0,$

when $x = \dfrac{1}{m}-\dfrac{2}{d}.$

$$\kappa d - 2\lambda x \pm m\,(1+x^2)^{\frac{1}{2}} \pm x\,(1+mx)\,(1+x^2)^{-\frac{1}{2}} = 0,$$

where x has two values from the quadratic

$$x^2-\frac{2x}{d}+1 = 0.$$

It has a pair of points

$$\kappa = \frac{1}{md}\,(m^2+p^2)^{\frac{1}{2}}, \quad \lambda = -\,(m^2+p^2)^{\frac{1}{2}},$$

which correspond to the infinite values of x, y $(my = px)$. It has four pairs of cusps.

The line of nodes $(1+mx+py=0)$ will touch the sphero-conic

$$\kappa\,(1+dx) - \lambda\,(1+x^2+y^2) = 0,$$

if $d^2p^2\kappa^2 + 4\,(m^2+p^2-dm)\,\kappa\lambda = 4\,(m^2+p^2+1)\,\lambda^2.$

The two lines thus defined, as functions of κ and λ, both touch and cut the curve (D_1), as shown in Fig. 3. See remarks on PQ in § 28.

28. To determine the relation between the parameters, when sphero-cyclides with collinear foci are trinodal. (Curve D_1.)

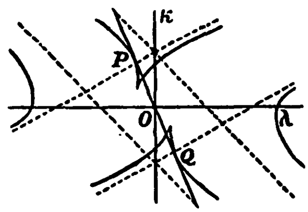

Fig. 4.

In § 27, $p = 0$; and the equation to a family of such quartics is

$$\kappa(1+dx) = (1+mx)\sec r + \lambda \sec^2 r.$$

The sphero-conic (§ 27), on which all the third nodes lie, is resolved into two great circles

$$\text{(I.) } y = 0, \quad \text{(II.) } x = \frac{1}{m} - \frac{2}{d}.$$

The Curve (D_1) in this case consists of two parts, according as

$$\text{(I.) } y = 0, \quad \text{or} \quad \text{(II.) } 1 + mx + 2\lambda \sec r = 0.$$

I. When $y = 0$, κ, λ have the following values:

$$\kappa\,(dx^2+2x-d) = (x-m)\,(1+x^2)^{\frac{1}{2}},$$

$$\lambda\,(dx^2+2x-d)\,(1+x^2)^{\frac{1}{2}} + mdx^3 + 2mx^2 + x + m - d = 0.$$

Two pairs of linear asymptotes are given by the equation

$$\kappa\,(1+dx) = (1+mx)\,(1+x^2)^{\frac{1}{2}} + \lambda\,(1+x^2),$$

where x has two values given by the quadratic

$$dx^2 + 2x - d = 0.$$

A pair of cusps is determined by the value of x, which satisfies both the conditions

$$\frac{d\kappa}{dx} = 0, \quad \frac{d\lambda}{dx} = 0.$$

By taking the logarithmic differential of κ,

$$\frac{2dx+2}{dx^2+2x-d} = \frac{1}{x-m} + \frac{x}{1+x^2}.$$

That is, $$2\,(x+m) + d\,(mx^3 - 3x^2 + 3mx - 1) = 0 \quad \text{............(A)}.$$

Its discriminant is $4\,(d^2+1)\,\{(m^2-1)\,d + 2m\}^2 = 0.$

Let $d = \dfrac{2m}{1-m^2}$, or $\tan\delta = \tan 2\mu$, that is, let the distance from the origin of the arc of nodes be half that of the cyclic arc

$$\kappa d\,(x-m)\left(x+\frac{1}{m}\right) = (x-m)\,(1+x^2)^{\frac{1}{2}},$$

$$2\lambda m\,(x-m)\left(x+\frac{1}{m}\right)(1+x^2)^{\frac{1}{2}}+(x-m)\,(2m^2x^2+2mx+m^2+1) = 0.$$

There is but one solution applicable, viz., $x = m$, $y = 0$; and consequently it belongs to the subsequent Case II. The curve drawn with the values of κ, λ, when the factor $(x-m)$ is withdrawn, is inapplicable, and does not constitute a part of the Curve (D_1).

The cubic (A) has only one real solution; it has no equal roots, when $d = \frac{2m}{1-m^2}$, unless $m = 1$, and $d = \infty$. Consequently, there is only one pair of cusps for each sphero-cyclide of this family.

If the line of nodes $(1+mx = 0)$ touch the sphero-conic

$$\kappa\,(1+dx) = \lambda \sec^2 r,$$

then

$$m\kappa\,(m-d) = (m^2+1)\,\lambda.$$

If this ratio $\kappa : \lambda$ be substituted in their preceding values as functions of x,

$$(mx+1)^2\,\{x\,(1+md)+m-d\} = 0 \quad\text{..................(B)}.$$

Hence this line PQ $(\kappa:\lambda)$, in Figs. 3, 4, 5, both touches and cuts Curve (D_1).

To their point of contact there corresponds a sphero-cyclide with a triple point (see § 26); to their point of intersection, a quartic with a tacnode and crunode; moreover, this line PQ discriminates quartics with real crunodes from those with imaginary nodes.

II. For the second portion of Curve (D_1), when

$$x = \frac{1}{m}-\frac{2}{d},\quad 1+mx+2\lambda \sec r = 0,$$

there results the hyperbola

$$\frac{d}{m^2}\,\kappa\lambda = \frac{1}{d}-\frac{1}{m}.$$

For all values of κ, λ thus correlated, the quartic degenerates into two coincident circles, codiametral with the (x) arc,

$$1+mx+2\lambda\,(1+x^2+y^2)^{\frac{1}{2}} = 0.$$

COR.—For the sphero-Cartesian, the conditions for a third node or other singularity are derived, when $d = 0$. (Fig. 5.)

[29. To find the condition for a tacnode-cusp in a sphero-cyclide.

In this case the tangent at the cusp in the Curve (D_1) coincides with the line PQ, which therefore meets (D_1) in three coincident points; the triple point unites with a cusp in the corresponding quartic.

The values of x in (B) are identical, if $2md = m^2-1$, or $\tan\delta+\cot 2\mu = 0$, or $\frac{\pi}{2}+\delta = 2\mu$ (p. 133).

This value of d satisfies (A), the condition of a cusp:

$$(mx+1)\{(m^2+3)x^2-4mx+(1+3m^2)\} = 0.$$

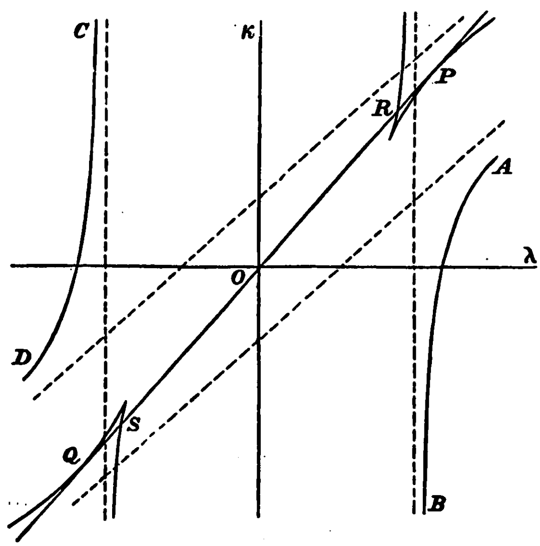

FIG. 5.

The two sphero-Cartesians

$$\sec^2 r \pm (1\pm x)\sec r = 2$$

have each a tacnode-cusp, when $x = \mp 1$ respectively.]

30. To determine all the varieties of sphero-cyclides with collinear foci by the aid of the first discriminating curve. (Figs. 4, 5.)

Let the line POQ be first considered: for all values of the parameters κ, λ, for points above this line in the first quadrant, and below this line in the third quadrant, all such quartics have two crunodes; for points on the other sides of this line, the two characteristic nodes are imaginary. At the points of intersection R, S, the quartics have a tacnode and crunode; at the points of contact P, Q, a triple point formed by the union of an acnode with two imaginary nodes.*

To the cusp in this Curve (D_1) there corresponds a cusp in the sphero-cyclide, besides the two imaginary nodes; for points on (D_1) on either side of the cusp, crunodes and acnodes correspond.

For points (κ, λ) on Curve (D_1) above R, the Sphero-Cartesians are tricrunodal; and for points beyond P, acnodal with two imaginary nodes.

For points (κ, λ) on either side of Curve (D_1), the companion-curves become bipartite, or unipartite with folia.

The outer lines in the second and fourth quadrants AB, CD are bounding lines: *i.e.*, for points (κ, λ) thereon, the corresponding sphero-cyclides shrink to points; and for exterior points, none correspond.

V. *On the Points of Undulation in Sphero-cyclides.*

31. To determine the mutual relation between the parameters, which enter into their equations, when sphero-cyclides have points of undulation, or the second discriminating curve (D_2).

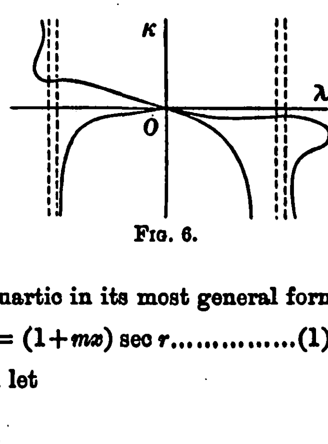

Fig. 6.

At a folium-point, or point of undulation, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$.

Let these tests be applied to this quartic in its most general form

(§ 27) $\qquad \lambda \sec^2 r - \kappa\,(1+dx+ey) = (1+mx) \sec r$ ....(1).

Differentiate thrice successively, and let

$$p = \frac{dy}{dx}.$$

$$2\lambda\,(x+py) - \kappa\,(d+ep) = m \sec r + (x+py)\,(1+mx) \cos r \text{......(2)},$$

$$2\lambda\,(1+p^2)$$
$$= 2m(x+py) \cos r + (1+p^2)(1+mx) \cos r - (x+py)^2\,(1+mx) \cos^3 r \text{...(3)},$$

$$0 = m\,(1+p^2) \cos r - m\,(x+py)^2 \cos^3 r - (1+p^2)\,(x+py)(1+mx) \cos^3 r$$
$$+ (1+mx)\,(x+py)^3 \cos^5 r.$$

---

* [This triple point discriminates sphero-cyclides from other binodal quartics. In the quartic $\qquad \kappa\,(1+dx) = (1+mx)\sqrt{(1+x^2-y^2)} + \lambda\,(1+x^2-y^2)$,
the triple point for the same critical values of κ and λ, as in the text, has one real, and two real coincident branches, caused by the union of a crunode with two real nodes.]

This last condition may be resolved into two factors

$$(1+p^2)\sec^2 r-(x+yp)^2, \text{ and } m\sec^2 r-(x+py)(1+mx) \;\ldots(4).$$

The former factor is rejected, as it should be from its form

$$1+p^2+(px-y)^2,$$

since $$\sec^2 r = 1+x^2+y^2.$$

The second factor (4) gives the real condition: when substituted in (2) and (3),

$$2\lambda(x+py)-\kappa(d+ep) = 2m\sec r \;\ldots\ldots\ldots\ldots(2),$$

$$2\lambda(1+p^2)\sec r = m(x+py)+(1+p^2)(1+mx)\ldots\ldots\ldots(3).$$

The elimination of κ and λ from (1), (2), and (3) gives another condition,

$$(d+ep)\sec^2 r\,[m(x+py)+(1+p^2)(1+mx)]$$
$$-2(x+py)(1+dx+ey)[m(x+py)+(1+p^2)(1+mx)]$$
$$+2(1+p^2)\sec^2 r\,[2m(1+dx+ey)-(d+ep)(1+mx)] = 0.$$

By the aid of (4) this eliminant may take the form

$$2m(1+dx+ey)[(1+p^2)\sec^2 r-(x+py)^2]$$
$$= (d+ep)\sec^2 r\;[(1+p^2)(1+mx)-m(x+py)],$$

or a still simpler form by the aid of (4),

$$2m(1+dx+ey)-(d+ep)(1+mx) = 0 \;\ldots\ldots\ldots\ldots(5).$$

By eliminating p from (4) and (5), the locus of the points of undulation in a family of sphero-cyclides is seen to be a sphero-conic.

$$2m-d+mdx+2mey = \frac{e}{y}[m(1+y^2)-x] \;\ldots\ldots\ldots\ldots(6).$$

By giving y successive values from $+\infty$ to $-\infty$, single values of x, and therefrom of κ, λ, so that the required discriminating curve (D_2) is drawn. The curve (D_2) has the origin for a Newtonian centre, since κ, λ are determined as factors of $\sec r$ or $(1+x^2+y^2)^{\frac{1}{2}}$.

It has an asymptote, when

$$d+ep = 0;$$

and $$1+dx+ey = 0, \qquad ex+(m-d)y = me.$$

For these values $\kappa = \infty$, and λ is known from (3).

It has a second asymptote, also parallel to the (κ) axis, when

$$mdy+e=0,\quad x=\infty,\quad \text{and}\quad ep=d;$$

for these values

$$\kappa=\infty,\quad 2\lambda(d^2+e^2)=m(d^2+2e^2).$$

There is a node at the origin, corresponding to the value $(1+mx=0)$, since it yields two values of y.

Only half of the curve is drawn in Fig. 6.

32. To determine the mutual relation between the parameters, when sphero-cyclides with collinear foci have points of undulation. Curve (D_3).

In this case $e=0$ in § 31; and the locus of the points of undulation in such a family is a great circle, co-diametral with the (y) arc,

$$2m-d+mdx=0 \quad \ldots\ldots\ldots\ldots\ldots(6).$$

The limiting values of y are ∞ and 0; negative values of y give the same negative value of p from (4), and therefore the same points, since κ and λ are functions of

$$(x+yp)\ \text{and}\ (1+p^2).$$

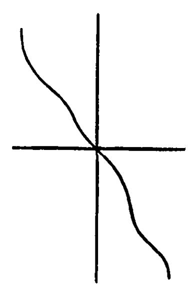

Fig. 7.

When $y=\infty$, $p=\infty$; and $\kappa=0=\lambda$ determines the origin.

When $y=0$, $p=\infty$; and κ, λ have these finite values,

$$(3)\quad 2\lambda(1+x^2)^{\frac{1}{2}}=1+mx;\qquad (2)\quad 2\lambda m\frac{1+x^2}{1+mx}-d\kappa=2m(1+x^2)^{\frac{1}{2}};$$

and the curve (D_3) terminates abruptly. The origin is its centre.

If $d=0$, the form fails, since (2) and (3) are incompatible, and by (6) $m=0$. Sphero-Cartesians have therefore no point of undulation.

The points of undulation are limiting forms of folia or depressions, characterised by two points of inflexion. Accordingly, for values of (κ, λ) on one side or the other of the Curves (D_3), in Figs. 6, 7, the companion sphero-cyclides have two points of inflexion or none.

VI. *On the Sphero-du-cyclide.*

33. This is the polar or complementary curve of the sphero-cyclide; its modes of generation and the forms of its equations are derived at

once by dualising (§ 36). It is a class-quartic with two bitangents, real or imaginary, and therefore of the eighth order.

The sphero-du-cyclide has two double foci, which are the single foci of the focal conic of the sphero-cyclide.

It is generated in four ways, as the envelope of a variable small circle, whose concentric great circle rolls on a dirigent sphero-conic (F), and which small circle is so related to a fixed small circle (J), that their common tangents are quadrants in length.

The four dirigent conics (F) are doubly concyclic; their two common cyclic arcs are double in the sphero-du-cyclide, thence generated.

34. Def.—The *quadranto-circle* of a spherical triangle ABC has the common orthocentre of ABC and its polar triangle $A'B'C'$ for its own pole, and is such that two corresponding sides BC, $B'C'$ of the primitive and polar triangles meet each other and the quadranto-circle in points A'', A'', which are 90° distant from A.

The great circles, which are concentric with the four (J) small circles, taken by threes, constitute four spherical triangles, which are self-conjugate with respect both to the (F) conics and the (J) circles. Each triad has the fourth great circle for its quadranto-circle.

The other properties of the du-cyclide are in like manner readily derived from those of the sphero-cyclide.

35. The transformations of § 11 have their counter-part, in transforming from a sphero-conic to a sphero-du-cyclide.

I. The dual formulæ (1), (2) become

$$\sin r = \cos(r-p)\sin\delta, \quad \tan r = \frac{\cot p}{\operatorname{cosec} p \operatorname{cosec}\delta - 1} \quad \text{.........(1)},$$

and of anallagmatic inversion

$$\frac{1-\tan\frac{p_1}{2}}{1+\tan\frac{p_1}{2}} \cdot \frac{1-\tan\frac{p_2}{2}}{1+\tan\frac{p_2}{2}} = \left\{\frac{1-\tan\frac{\delta}{2}}{1+\tan\frac{\delta}{2}}\right\}^2 \quad \text{...............(2)}.$$

II. If U, V, W denote the fixed circles J_1, J_2, J_3, which have the property that their common tangent arcs are quadrants in length, and if the dirigent conic (F) be denoted by the point equation

$$(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2 = 0,$$

then the sphero-du-cyclide, thence generated (§ 33), has an equation of the same form $\quad (a, b, c, f, g, h \between U, V, W)^2 = 0.$

The symbol U has the dual form of that given in § 11,

$$U = \cos AP - \cos \delta_1.$$

AP now denotes the inclination of an arbitrary circle (ξ, η) to the great circle (l, m) concentric with (J_1), whose radius is $\frac{\pi}{2} - \delta_1$, and

equation $$U \equiv \frac{l\xi + m\eta + 1}{\sqrt{(l^2+m^2+1)}\sqrt{(\xi^2+\eta^2+1)}} - \cos \delta_1 = 0.$$

36. Universally in Analytical Spherics, the formulæ of any curve are applicable to its dual or polar curve, by writing p', q', r', the line-coordinates of a great circle referred to $A'B'C'$, the polar triangle, for α, β, γ, the point-coordinates of the dual point or spherical centre referred to ABC, the primitive triangle.

In Gudermann's system, where the triangles ABC, $A'B'C'$ coincide, being tri-quadrantal, $-\xi, -\eta$ must be substituted for x, y, in dualising.

All the equations in this memoir are equally applicable to this dual curve, when the point and line-coordinates are thus transformed.

VII. *On Doubly Bi-confocal Sphero-cyclides.*

37. Dr. Casey has applied the methods of sphero-conics to sphero-cyclides by the transformation given in § 11. ("On Cyclides," §§ 270—273, 308—311.)

I will apply it to transform the general equation of confocal sphero-

conics $$lp^2 + mq^2 + nr^2 + \lambda\,(apP + bqQ + crR) = 0,$$

and to prove that all sphero-cyclides, which have the same pair of double foci, but different single foci, pass through four concircular points, which lie on a small circle, whose pole or spherical centre is the orthocentre of the triangle of reference, or centre of the J circle from which it is generated.*

We have to interpret

$$a^2U^2 + b^2V^2 + c^2W^2 - 2bcVW\cos a - 2caWU\cos b - 2abUV\cos c,$$

* [Doubly bi-confocal bicircular quartics and cyclides do not intersect. The function $(apP + bqQ + crR)$, when quadrically transformed, gives the centre of J or the orthocentre of the centres of J_1, J_2, J_3 as a point-circle

$$\Sigma\,(a^2\alpha^2\cos^2 A - 2bc\beta\gamma\cos A\cos B\cos C),$$

not situate on the curve. A system of doubly bi-confocal Cartesians and Cartesian cyclides is expressed by the relation $S \propto D$, where S is a circle or sphere, and D the distance of a point on the locus from the centre of the (F) focal circle or sphere.]

where, by § 11, II., in Dr. Casey's notation,

$$p \propto U \propto \cos at - \cos aP \propto \mu \sec a - a',$$

$$q \propto V \propto \mu \sec b - \beta', \quad r \propto W \propto \mu \sec c - \gamma'.$$

For brevity, μ stands for $\sqrt{(\cos a \cos b \cos c)}$,

$$6va' = a\alpha + b\beta \cos c + c\gamma \cos b.$$

Hence $\quad P = ap - bq \cos C - cr \cos B \propto \mu\nu \cos B \cos C - 6v\alpha,$

$$Q \propto \mu\nu \cos C \cos A - 6v\beta, \quad R \propto \mu\nu \cos A \cos B - 6v\gamma.$$

Here also, for brevity, ν denotes $\tan a \tan b \tan c$, $6v = ab \sin C$.

$$\begin{aligned} apP + bqQ + crR \propto\; & a(\mu \sec a - a')(\mu\nu \cos B \cos C - 6v\alpha) \\ & + b(\mu \sec b - \beta')(\mu\nu \cos C \cos A - 6v\beta) \\ & + c(\mu \sec c - \gamma')(\mu\nu \cos A \cos B - 6v\gamma) \end{aligned}$$

$$\begin{aligned} \propto (6v)^2 + abc\,(\tan a \cos B \cos C + \tan b \cos C \cos A + \tan c \cos A \cos B) \\ - 12v\sqrt{(\cos a \cos b \cos c)}\,(\alpha \tan a + \beta \tan b + \gamma \tan c). \end{aligned}$$

This is the equation to a small circle, concentric with the J circle of § 18.

$$abc\,\Sigma\,(\tan a \cos B \cos C) = (6v)^2 - abc \tan a \tan b \tan c \cos A \cos B \cos C.$$

The reductions used here depend on the identity

$$\tan b \cos C + \tan c \cos B = \tan a\,(1 - \tan b \tan c \cos B \cos C).$$

The transformed equation to this family of sphero-cyclides is

$$\begin{aligned} l(\mu \sec a - a')^2 + m(\mu \sec b - \beta')^2 + n(\mu \sec c - \gamma')^2 \\ + \lambda\,\{2\,(6v)^2 + abc \tan a \tan b \tan c \cos A \cos B \cos C \\ + 12v\mu\,(\alpha \tan a + \beta \tan b + \gamma \tan c)\}^2. \end{aligned}$$

The small circle intersects the sphero-cyclide in four points, and not in eight, unless the antipodal points be added, since, by expressing the constants in terms of $\Sigma\alpha \tan a$, the first line becomes a sphero-conic.

[38. Sphero-cyclides are cut by planes in four concircular points. As in § 37, any sphero-cyclide (§ 6)

$$a \sec^2 R + (bu_1 + v_1) \sec R + v_2 = 0$$

is intersected by any circle,

$$\sec R = lx + my + n = w,$$

in the same four points as the sphero-conic

$$aw_1^2 + (bu_1 + v_1)\, w_1 + v_2 = 0.]$$

ERRATA.—In § 4, *read* "The line of nodes in the sphero-cyclide is the polar of the centre of any (J) circle with respect to the polar conic of the corresponding (F) focal conic."

In § 6, Cor., *read* "If the cyclide has the imaginary circle at infinity as a cuspidal edge, or if $v_2 \equiv c(x^2+z^2)+dy^2$, the generated sphero-cyclide is called by Dr. Casey a Sphero-Cartesian."

Prof. Hart has already published his Memoir on the "Five Focal Quadrics of a Cyclide," in the *Messenger of Mathematics*, Vol. XIV., pp. 1—8. See § 5.

On the Limits of Multiple Integrals. *By* HUGH MACCOLL, B.A.

[*Read November* 13*th*, 1884.]

In my first paper on the "Calculus of Equivalent Statements" (*Proceedings*, Vol. IX., Nos. 124, 125), I showed how the limits of integration in a multiple integral might always be ascertained whenever we had enough data for the purpose. I now propose to show how the expression of the limits thus ascertained may often be simplified and reduced to a form more convenient for integration.

DEFINITION.—When we have an elementary statement of the form $x_{m'.n}$ or $y_{m'.n}\,x_{r'.s}$ or $z_{m'.n}\,y_{r'.s}\,x_{u'.v}$, &c., presenting the *nearest* limits (or true limits of integration), we may, for brevity's sake, take any of these symbols to denote, *not* the statement itself, but the *integral* which has the limits of integration indicated by the statement.

Thus $y_{m'.n}\,x_{r'.s}$ will be a mere abbreviation for

$$\int_{y_n}^{y_m} dy \int_{x_s}^{x_r} dx.$$