The Conic through Five Given Points

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for the "shine" and "cosh" with which we are so familiar, a substitution on which certain examining bodies appear to lay special stress.

In conclusion, a vast amount of work lies before our Association waiting to be done. Progress must of necessity be slow, but I hope that in the years to come we shall be successful in our attempts to raise mathematics to a higher level in Great Britain than it has ever occupied in the past. And I again repeat that this much desired end will conduce in no uncertain degree to the welfare and prosperity of our nation.

THE CONIC THROUGH FIVE GIVEN POINTS.

The following discussion may be of interest on account of the way in which it leads to the axial and focal properties.

I. Let the locus of a point P such that $A(CDEP)=B(CDEP)$, where A, B, C, D, E are fixed points, be called the conic (AB, CDE). Then it is clear that permuting A, B *or* C, D, E does not affect the conic, and that the conic passes through C, D, E, and also through A, B. (Give BP the four positions BC, BD, BE, BA and then give AP the position AB.)

Since AP and BP correspond homographically the same conic may be defined as (AB, FGH) where F, G, H are any three positions of P except A, B.

II. If F is any point on (AB, CDE) then Pascal's theorem can be proved for any hexagon whose vertices are A, B, C, D, E, F if B is next but one to A, and thus A or B may be interchanged with any one of C, D, E. For instance if A is to be interchanged with C we put A, B, C 1st, 3rd and 5th as in $ADBECF$. This is a Pascal hexagon, and it follows that F lies on (BC, ADE). Thus the conic is not affected by *any* permutation of the five points after which it is named, and we may call it $(ABCDE)$.

III. If any circle has been drawn through AB, the points P, Q where this circle meets the conic can be constructed with the ruler only as follows.

Let AC, AD, AE, BC, BD, BE meet the circle in C_1, D_1, E_1, C_2, D_2, E_2.

Then
$$\begin{aligned} A(C_1D_1E_1P) &= A(CDEP) \\ &= B(CDEP) \\ &= B(C_2D_2E_2P). \end{aligned}$$

Hence P is a double point in the homography in which C_1, D_1, E_1 correspond to C_2, D_2, E_2; the other double point is Q, and thus P, Q are the intersections with the circle of the Pascal line of the hexagon $C_1D_2E_1C_2D_1E_2$.

IV. By III. we may suppose the conic to be named after five points $ABCDE$, of which four are on a circle, say A, B, C, D. Take another circle through AB and let it meet the conic again in P, Q. In the figure of III. we then have
$$\angle D_2C_1A = D_2BA = DBA = DCA.$$

So that C_1D_2 is parallel to CD, as also is C_2D_1. Hence PQ also must be parallel to CD; PQ is therefore in a fixed direction whatever circle through A, B is chosen.

V. As a particular case of IV. suppose C, D both at infinity: then it still follows that
$$D_2C_1A = DBA,\quad D_1C_2B = DAB,$$
and hence C_1D_2, D_1C_2 are parallel and PQ is parallel to them, making the same angle with AC that AD makes with AB.

VI. Still taking $ABCD$ cyclic, suppose that AB meets CD in T, and let the circle, whose centre is T and radius TA, meet CD in K, L. Let G, H be the points where AK, AL meet the conic. (G, H can be found by the known construction depending on Pascal's theorem.) Construct as in III. the point I, where the circle ABG meets the conic again. Then GI is parallel to CD, and therefore AB, GI are equally inclined to AG and AG, BI are parallel. If any other circle through A, G meets the conic in P, Q, then PQ must be parallel to BI, that is, to AG, and all chords of the conic parallel to AG are therefore bisected by a line perpendicular to them. The same holds for chords parallel to AH, and the conic has therefore two axes of symmetry in general. It follows from V. that G, H cannot both be at infinity, and hence the conic has at least one axis.

VII. If PK, PL, PM, PN are drawn perpendicular to AB, CD, AD, CB, then, as P varies, $P(ABCD)$ is proportional to $PM.PN/PK.PL$, and this is true whether $ABCD$ is cyclic or not.

VIII. If in VII. we suppose that $ABCD$ is cyclic we have that

$$PM.PN+PK.PL$$

$\propto$ the square of the tangent from P to the circle $ABCD$.

Proof. Let PB, PD meet the circle again in E, F, and let EE', FF' be diameters.

Then $\dfrac{PL}{FC}=\dfrac{PM}{FA}=\dfrac{PD}{FF'}$ by similar figures $PLMD$, $FCAF'$,

and $\dfrac{PK}{EA}=\dfrac{PN}{EC}=\dfrac{PB}{EE'}$ by similar figures $PKNB$, $EACE'$.

Also $EA.FC+EC.FA=EF.AC$, and thus

$$\frac{PK.PL+PM.PN}{EF.AC}=\frac{PB.PD}{EE'.FF'}.$$

But $EF/BD=PF/PB$, whence $PK.PL+PM.PN\propto PD.PF$.

Hence if P is on a conic through A, B, C, D, the square of the tangent from P to the circle $ABCD$ is proportional to $PM.PN$ or $PK.PL$.

IX. In VIII. let AB, and therefore also CD, be perpendicular to an axis, which meets AB, CD in U, V, and let G be the foot of the perpendicular from P to this axis. Then from VIII. we have $t^2=e^2GU.GV$ where e is a constant and t the tangent from P to the circle $ABCD$. Let O be the centre of this circle and take another circle through A, B with centre O_1; let t_1 be the tangent from P to the new circle. Then

$$t_1^2-t^2=2OO_1.GU$$
$$\text{and } t_1^2=GU(e^2GV+2OO_1)$$
$$=e^2GU.GV_1$$

if V_1 is a point on UV such that $e^2VV_1=2OO_1$.

The conic may therefore be defined equally well by reference to the new circle and the lines AB, C_1D_1, where C_1D_1 is the perpendicular to UV through V_1. The points where C_1D_1 meets the new circle lie on the conic. Similarly we may move U forward to U_1 if at the same time we again move the centre of the circle forward to O_2 so that $e^2.UU_1=2O_1O_2$ and now make C_1D_1 the radical axis of the new and old circles. It is not necessary that the new circle should meet the conic in real points.

X. Let r, r_1, r_2 be the radii of the three circles, with centres O, O_1, O_2. We have $r_1^2-r^2=UO_1^2-UO^2$, since the radical axis of the first two passes through U, and similarly $r_2^2-r_1^2=V_1O_2^2-V_1O_1^2$.

Thus $$r_2^2-r^2=V_1O_2^2-V_1O_1^2+UO_1^2-UO^2,$$

while $$OO_2=\tfrac{1}{2}e^2(VV_1+UU_1)=e^2.GG_1$$

if G, G_1 are the middle points of UV and U_1V_1.

After some reduction the value of r_2^2, unless $e=1$, becomes

$$e^2(1-e^2)XG_1.G_1X'+\tfrac{1}{4}e^2.U_1V_1^2$$

where X, X' are two points on the axis such that

$$(1-e^2)(GX+GX')=2GO,$$

$$e^2(1-e^2)GX\,.\,GX'=\tfrac{1}{4}e^2\,.\,UV^2-r^2.$$

Making U_1, V_1, and therefore G_1, to coincide with X or X' we have $r_2=0$; the circle reduces to a point and we have the focus and directrix property.

When $e=1$, the value of $r_2{}^2$ becomes $2GO\,.\,XG_1+\frac{1}{4}U_1V_1{}^2$ where X is a point in the axis such that

$$2GO\,.\,XG=r^2-\tfrac{1}{4}UV^2.$$

Here $r_2=0$ when U_1, V_1, G_1 coincide with X. A. C. DIXON.

MATHEMATICAL NOTES.

253. [I. 2. b.] Cf. Note 249, p. 167.

If n be a prime, a any number prime to n, p the smallest value of the integer m for which a^m-1 is divisible by n and n^s the highest power of n which divides a^p-1, we may write

$$a^p=1+bn^s+c_1n^{s+1}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where $$0<b<n.$$

Now if P be the smallest value of m for which a^m-1 is divisible by n^{s+1}, P must be a multiple of $p=\lambda p$, say. But from (i),

$$a^{\lambda p}=1+\lambda b\,.\,n^s+\text{a multiple of } n^{s+1}.$$

Therefore λb is divisible by n and (since $b<n$) $\lambda=n$.

Further, since in the expansion of $(A+B+C)^n$ the coefficient of every term except A^n, B^n, or C^n is divisible by n,

$$a^{np}=1+bn^{s+1}+n\text{ (a multiple of } n^{s+1})$$

[unless $n=2$, $s=1$]

$$=1+bn^{s+1}+c_2n^{s+2}.$$

Similarly $$a^{n^2p}=1+bn^{s+2}+c_3n^{s+3},$$

and generally $$a^{n^qp}=1+bn^{s+q}+c_qn^{s+q}$$

for all positive integral values of q.

Thus n^qp is the smallest value of m for which a^m-1 is divisible by n^{s+q}.

The above furnishes an answer to Mr. Wiles's question in Note 249 (*M. G.* Dec. 1907, p. 167). From this we can shew that for a composite number N, if a be a number prime to N, then N^qP is the smallest value of m for which a^m-1 is divisible by N^{s+q}, s, q being integers readily found by an examination of the powers to which the various prime factors of N are raised in $a^{p'}-1$, where p' is the smallest value of m for which a^m-1 is divisible by N.

Since we know that $a^{n-1}-1$ is divisible by n it follows that p is a factor of $n-1$, and the exceptions to the rule as enunciated by Mr. Wiles are determined by finding values of n and a for which $a^{n-1}-1$ is divisible by n^2, n^3, ... n^s.

If a be given, this amounts to finding values of m for which a^m-1 is divisible by x^p, where $x>m$ and $p>1$. This can probably be done only by trial: thus

3^5-1 is divisible by 11^2.
5^1-1 .... 2^2.
7^4-1 .... 5^2,