

*An easy Method of finding the Invariant Equation expressing any poristic relation between two Conics. By Prof. J. WOLSTENHOLME.*

[Read December 14th, 1876.]

The invariant equation expressing a poristic relation between two conics can be generally readily found by considering the limiting form when the poristic triangle or polygon passes through one or two of the common points of the two conics. The application of this method is exemplified in the following article.

1. Triangles inscribed in  $S$  and self-conjugate to  $S'$ .—In this case, if  $P$  be any point on  $S$ , and the polar of  $P$  with respect to  $S'$  meet  $S$  in  $Q, R$ , the triangle  $PQR$  will be self-conjugate to  $S'$ ; hence, when  $P$  approaches  $A$ , a common point of  $S$  and  $S'$ , one of the two points  $Q, R$  also approaches  $A$ , and if  $AB$  be a chord of  $S$  touching  $S'$  at  $A$ , the triangle  $AAB$  inscribed in  $S$  will be self-conjugate to  $S'$ , or if  $AC$  be the chord of  $S'$  touching  $S$  at  $A$ ,  $BC$  will touch  $S'$  at  $C$ . Hence the equations referred to the triangle  $ABC$  may be taken

$$S \equiv z^2 + 2fyz + 2hxy = 0, \quad S' \equiv y^2 - 2zx = 0,$$

and the discriminant of  $kS + S'$  is  $k^2h^2 + 2k^2fh + 1$ , so that  $\Theta' = 0$ .

2. In exactly the same way, interchanging points and lines, if a common tangent touch  $S$  in  $B$  and  $S'$  in  $C$ , and  $CA$  be a chord of  $S'$  touching  $S$ , then, if  $BA$  touch  $S'$  in  $A$ , the triangle formed by  $BC$  taken twice and  $CA$  will be a triangle self-conjugate to  $S'$  and circumscribing  $S$ . The equations will then be

$$S \equiv x^2 + n^2z^2 + 2nzx + 2hxy = 0, \quad S' \equiv y^2 + 2zx = 0,$$

and the discriminant will be

$$k^2n^2 - (kn + 1)^2 - k^2n^2h^2, \text{ or } k^2n^2h^2 + 2kn + 1,$$

so that  $\Theta = 0$ .

Hence it appears that  $\Theta = 0$  is the condition that triangles can be circumscribed to  $S$  which are self-conjugate to  $S'$ , and at the same time that triangles can be inscribed in  $S'$  which are self-conjugate to  $S$ , and the like with  $\Theta'$ .

3. If triangles can be inscribed in  $S$ , whose sides touch  $S'$ , then, if from any point  $P$  on  $S$  we draw tangents to  $S'$  meeting  $S$  in  $Q, R$ ,  $QR$  will touch  $S'$ . As  $P$  approaches  $A$ , a common point of  $S$  and  $S'$ ,  $Q, R$  will tend to coincidence, and we shall have in the limit, if  $AB$  be a tangent to  $S'$  at  $A$ , meeting  $S$  in  $B$ ,  $BC$  will be a common tangent to  $S, S'$ , the triangle  $ABB$  being a triangle inscribed in  $S$  and circumscribing  $S'$ . We may then take  $S = z^2 + 2fzx + 2gxy$ ,  $S' = y^2 + 2zx$ , and the discriminant is

$$k^2g^2 + (kf + 1)^2 = 0, \text{ or } \Theta' = 4\Delta'\Theta.$$

4. If quadrilaterals can be circumscribed to  $S'$  such that the ends of two diagonals lie on  $S$ , we shall get, by similar considerations, that we can find  $A, B$  two common points of  $S, S'$  such that the tangents to  $S'$  at these points shall meet in a point  $C$  on  $S$ ; the inscribed quadrangle being  $ACBCA$ . Thus  $S = 2yz + 2zx + 2xy$ ,  $S' = z^3 + 2hxy$ , and the discriminant  $2k^3(k+h) - (k+h)^3$ . Thus  $\Delta' = 2$ ,  $\Theta = 2h-1$ ,  $\Theta' = -2h$ ,  $\Delta' = -h^3$ .

Hence

$$\begin{aligned}\Theta^3 - 4\Delta'\Theta &= 4h^3(1+2h-1) = 8h^3, \\ \Theta^3 - 4\Theta\Theta'\Delta' &= -16h^4 = -16\Delta'^3 = -8\Delta'^3\Delta,\end{aligned}$$

or, noticing the dimensions, we get the necessary condition as

$$\Theta^3 - 4\Theta\Theta'\Delta' + 8\Delta'^3\Delta = 0.$$

5. For a pentagon inscribed in  $S$ , whose sides taken in order shall touch  $S'$ , we get as the degenerate form  $ABCCBA$ , where  $A$  is a common point,  $AB$  the tangent to  $S'$  meeting  $S$  in  $B$ ,  $BC$  the second tangent to  $S'$  from  $B$ ; then the tangent to  $S$  at  $C$  must also touch  $S'$ . We may then take

$$S = 2x^3 + 2fyz - 2xy, \quad S' = x^3 + y^3 + z^3 - 2yz - 2zx - 2xy,$$

the discriminant is

$$2k+1+2(k+1)(kf-1) - (2k+1)(kf-1)^2 - 1 - (k+1)^3,$$

whence  $\Delta = -2f^3$ ,  $\Theta = -f^3 + 6f - 1$ ,  $\Theta' = 4f - 4$ ,  $\Delta' = -4$ ,

$$\Theta^3 - 4\Delta'\Theta = 16[(f-1)^3 - f^3 + 6f - 1] = 64f,$$

$$\Theta^3 - 4\Theta\Theta'\Delta' = 256f(f-1), \quad 8\Delta'^3\Delta = -256f^3,$$

$$\Theta^3 - 4\Theta\Theta'\Delta' + 8\Delta'^3\Delta = -256f,$$

and the necessary relation is, when proper attention is paid to dimensions,

$$(\Theta^3 - 4\Theta\Delta')^3 = 32\Delta'^3\Delta (\Theta^3 - 4\Theta\Theta'\Delta' + 8\Delta'^3\Delta).$$

[The true relation in such cases as this, where it might appear as if we had to select from several, can always be found as follows. Suppose the result we have found for  $\Delta$  to be multiplied by  $\lambda^3$ , that for  $\Theta$  by  $\lambda^3\mu$ , for  $\Theta'$  by  $\lambda\mu^2$ , and for  $\Delta'$  by  $\mu^3$ , and eliminate the ratio  $\lambda : \mu$ . Thus above we should have

$$\begin{aligned}\Theta^3 - 4\Theta\Delta' &= 64f\lambda^3\mu^4, \\ \Theta^3 - 4\Theta\Theta'\Delta' + 8\Delta'^3\Delta &= -256f\lambda^3\mu^6, \\ \Delta'^3\Delta &= -32f^3\lambda^3\mu^6,\end{aligned}$$

when it is obvious which is the true condition. This, however, can easily be done mentally.]

6. For a hexagon inscribed in  $S$  and whose sides in order touch  $S'$ , we shall have the limiting form  $ACDBDCA$ , where  $A, B$  are two common points,  $AC, BD$  tangents to  $S'$  and chords of  $S$ ; then  $CD$  must touch  $S'$ . We shall have

$$S = 2x^3 + 2fyz - 2zx - 2xy, \quad S' = x^3 + y^3 + z^3 - 2yz - 2zx - 2xy,$$

and the discriminant is

$$2k+1+2(k+1)^3(kf-1) - (2k+1)(kf-1)^2 - 2(k+1)^3,$$

so that  $\Delta = 2f - 2f^2$ ,  $\Theta = -f^2 + 8f - 4$ ,  $\Theta' = 4f - 8$ ,  $\Delta' = -4$ ,

whence  $\Theta^2 - 4\Theta\Delta' = 16\{(f-2)^2 - f^2 + 8f - 4\} = 64f$ ,

$$\Theta^2 - 4\Theta\Theta'\Delta' = 256f(f-2);$$

$$\Delta^2\Delta = 32f^2 - 32f^3,$$

whence  $\Theta^2 - 4\Theta\Theta'\Delta' + 8\Delta\Delta^2 = -256f$ ,

and  $\Theta^2 - 4\Theta\Theta'\Delta' + 16\Delta\Delta^2 = -256f^2$ ,

and the condition is

$$(\Theta^2 - 4\Theta\Delta')^2 = 4\{\Theta^2 - 4\Theta\Theta'\Delta' + 8\Delta\Delta^2\}\{\Theta^2 - 4\Theta\Theta'\Delta' + 16\Delta\Delta^2\}.$$

The general rule is now obvious: in order that polygons of  $2n$  sides may be inscribed in  $S$  and circumscribed to  $S'$ , we must have a polygon  $AA_1A_2 \dots A_{n-1}B$ , whose extreme points  $A, B$  are common points of  $S, S'$ , whose other points  $A_1A_2 \dots A_{n-1}$  lie on  $S$  and whose sides taken in order,  $AA_1, A_1A_2, \dots A_{n-1}B$ , touch  $S'$ . We shall not be at liberty to take any two of the four common points, but may take either of two pairs. So, for polygons of  $2n+1$  sides, we must have a polygon  $AA_1 \dots A_n$ , where  $A$  is any common point,  $A_1A_2 \dots A_n$  lie on  $S$ , and the sides taken in order  $AA_1, A_1A_2, \dots A_{n-1}A_n$ , and the tangent to  $S$  at  $A_n$  must touch  $S'$ . It seems probable that the results can be more easily worked out in this manner than from the figures of finite area. I will conclude with—

7. The heptagon inscribed in  $S$  and whose sides touch  $S'$ . We shall have

$$S = 2x^2 + 2fyz + 2gzx - 2xy,$$

$$S' = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy,$$

$f, g$  being connected by the equation  $fg + f + g = 0$ . It will be found that

$$\Delta = -2f(f+g), \quad \Theta = -(f+g-1)^2 + 4f, \quad \Theta' = 4(f+g-1), \quad \Delta' = -4.$$

If we denote the conditions (in the forms already given) for in- and- escribed triangles, quadrangles, &c., by  $U_2, U_4, U_6, \dots$  we shall find in this case,

$$U_2 = \Theta^2 - 4\Theta\Delta' = 64f, \quad U_4 = -256f, \quad U_6 = -4^2f^2g,$$

also

$$\Delta^2\Delta = 32f^2g.$$

We thus have  $\frac{U_4}{-4U_2} = 1$ , and  $\frac{U_6}{\Delta^2\Delta} = -2 \cdot 4^2$ ,

so that the homogeneous condition is

$$\frac{U_6}{\Delta^2\Delta} = -2 \cdot 4^2 \cdot \left(\frac{U_4}{-4U_2}\right)^2,$$

or

$$U_6U_2^2 = 128\Delta^2\Delta U_4^2,$$

$$\text{or } (\Theta^2 - 4\Theta\Delta')^2\{(\Theta^2 - 4\Theta\Delta')^2 - 32\Delta\Delta^2(\Theta^2 - 4\Theta\Theta'\Delta' + 8\Delta\Delta^2)\} \\ = 128\Delta\Delta^2(\Theta^2 - 4\Theta\Theta'\Delta' + 8\Delta\Delta^2)^2.$$

I do not remember where to look for the developed conditions of this kind to compare.

[NOTE.—(Added July, 1877.) When I wrote this paper I had not read Prof. Clifford's Note on the communication entitled "On the Transformation of Elliptic Functions," in which I find the method I have used is completely developed. (See "Transactions of the Mathematical Society," Vol. vii., page 230.) Two of the results, those of (1) and (3), communicated by me, were set in the Cambridge Mathematical Tripos Examination for 1876.]

January 11th, 1877.

SAMUEL ROBERTS, Esq., Treasurer, in the Chair.

Mr. G. W. Von Tunzelmann was elected a Member, and Colonel Petrie was present as a Visitor.

The following communications were made:—

"Determinant Conditions for Curves, or Surfaces, of the same order, having all their intersections common," Mr. J. Hammond; "Numerical Values of the first twelve powers of  $\Pi$ , of their reciprocals, and of certain other related quantities," Mr. J. W. L. Glaisher; "On some general classes of Multiple Definite Integrals," Mr. E. B. Elliott; "On the Partial Differential Equation  $s + Pp + Qq + Z = 0$ ," Prof. H. W. Lloyd Tanner; "On Methods for finding Equations to Axes of a Conic, and the Axes themselves, in Trilinear Coordinates," Mr. J. J. Walker; "On some Elliptic Function Properties," Prof. H. J. S. Smith.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxv., No. 176.

"Bulletin des Sciences Mathématiques et Astronomiques," Nov., Dec., 1876.

"Jahrbuch über die Fortschritte der Mathematik," siebenter Band, Jahrgang 1875, Heft i., Berlin, 1877.

"Sur les équations qui se rencontrent dans la Théorie de la Transformation des Fonctions elliptiques," par le Père Joubert, S. J., Paris, 1876; presented by Father Perry, S.J.

"Monatsbericht," September and October, 1876.

"Ueber das arithmetisch-geometrische Mittel aus vier Elementen," von C. W. Borchardt, Berlin, 1876.

"Archiv for Mathematik og Naturvidenskab udgivet af Sophus Lie, W. Müller og G. O. Sars," Første Bind, Første, Andet, Tredie Hefte, 1876.

"Allgemeine Theorie partielle Differential-Gleichungen 1<sup>o</sup>, von Sophus Lie," Pts. 1, 2.

"Bidrag til Legemernes Molekylar-theorie af Cato M. Guldberg."

"Om Attractionen mellem to Cirkelflader af Ingenieur O. Pihl."

"Discussion aller Integrations-Methoden der partiellen Differential-Gleichungen 1<sup>o</sup> von Sophus Lie."

"Sur la Résolution des Equations du 2<sup>me</sup>, 3<sup>me</sup>, et 4<sup>me</sup> degré par la fonction  $\frac{n}{r}(X)$ ," par Dr. Axel S. Guldberg (20 Dec. 1872).

(i.) "Zur Theorie des Integralitäts-faktors." (ii.) Veralgemeinerung und neue Verwerthung der Jacobischen Multiplier-Theorie," von Sophus Lie.

The above from Det Kongelige Norske Universitat i Christiania.

*Numerical Values of the First Twelve Powers of  $\pi$ , of their Reciprocals, and of certain other related quantities.* By J. W. L. GLAISHER, M.A., F.R.S.

[Read January 11th, 1877.]

1. I have so often wanted the first few powers of  $\pi$  to more figures than can be obtained by the use of seven-figure logarithms, that, some months since, I determined to have them calculated once for all to a sufficient extent to meet all cases that were likely to arise. My intention originally was merely to obtain the first twelve powers of  $\pi$  to twenty figures; but, when these were calculated, it seemed desirable to deduce from them also the values of their reciprocals.

2. The two following tables contain the values of  $\pi$ ,  $\pi^2$ ,  $\pi^3$ , ...  $\pi^{12}$  and of  $\pi^{-1}$ ,  $\pi^{-2}$ ,  $\pi^{-3}$  ...  $\pi^{-12}$  to twenty-two or more figures.

TABLE I.

$n$	$\pi^n$					
1	3.14159	26535	89793	23846	264	
2	9.86960	44010	89358	61883	449	
3	31.00627	66802	99820	17547	63	
4	97.40909	10340	02437	23644	0	
5	306.01968	47852	81453	2627		
6	961.38919	35753	04437	0302		
7	3020.29322	77767	92067	514		
8	9488.53101	60705	74007	129		
9	29809.09933	34462	11666	51		
10	98648.04747	60830	20973	72		
11	2 94204.01797	38905	97105	7		
12	9 24269.18152	33741	86222	6		