The Teaching of Calculus

Author(s): C. Godfrey

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THE TEACHING OF CALCULUS.

QUESTIONS PROPOSED BY THE SUB-COMMISSION A of the International Commission on Mathematical Teaching, with regard to the position now occupied by the Elements of Differential and Integral Calculus in the programmes of Public and Secondary Schools.

Note 1. The object of the Central Committee in formulating these enquiries has been solely that of acquiring information. The Committee does not itself take up any definite standpoint in the matter as to how far the teaching of the subject in the schools is desirable.

2. By the term "Public and Secondary Schools" is to be understood those Day and Boarding Schools which correspond to the French Lycées and the German Gymnasia and Real-Gymnasia. Information is, however, also desired, whenever possible, with regard to what is being done in the Teachers' Training Colleges. The particular type or types of school in the district considered should always be mentioned, and it should be stated whether the Calculus is part of the official curriculum, or included, or not, at the option of the individual teacher. The percentage of schools in which the Calculus is taught should also be given. It should also be mentioned whether all the pupils are taught the subject, or only some of the more advanced ones.

I. *How much of the Differential and Integral Calculus is taught in the schools of the country under observation?*

In particular:

(*a*) Is the Differential Calculus applied to functions of a single variable, or are functions of several variables also treated?

(*b*) To what specific functions is the Differential Calculus applied?

(*c*) Is the Integral Calculus studied? If so, within what limits?

(*d*) Is Taylor's Theorem discussed?

(*e*) Are simple Differential Equations solved? If so, what?

II. *How far is the treatment of the subject rigid, both as to the mode in which the fundamental concepts are introduced, and as to the demonstrations employed?*

(*a*) Is it considered sufficient to introduce the notions of the Differential Calculus geometrically, without expressly using the idea of a limit, or is this idea explicitly employed? In the latter case, is there an attempt at a rigid presentation of the subject, or are theorems like $\mathrm{Lt}\,\frac{1}{a}=\frac{1}{\mathrm{Lt}\,a}$ taken for granted?

(*b*) Are differentials used? If so, is the Differential Calculus employed as a sort of calculus of approximations, or are infinitely small quantities treated as if they were small quantities which really exist?

(*c*) In Taylor's Theorem is the remainder considered, or not?

(*d*) Is attention called to the fact that there are non-differentiable functions?

(*e*) Is the idea of an irrational number logically and systematically introduced, or is it considered sufficient to speak incidentally of irrational numbers, for instance, in the extraction of square roots?

III. *How is the pupil introduced to the ideas of the Differential and Integral Calculus?*

(*a*) Does he receive a preliminary training in the lower classes of the school, based on the study of appropriate simple functions and their graphs, so that the matter appears to rise naturally out of the subjects already studied, and not to constitute a supplementary course ?

(*b*) Is Leibniz's Notation employed ? If not, what symbols are used for the differential coefficient and integral ?

(*c*) Which is considered first, the Differential or the Integral Calculus, or are they taught simultaneously ?

(*d*) Is the integral introduced as the limit of a summation (definite integral), or as primitive function (inverse differential coefficient) ? If in both senses, in what order and in what connection with one another are the two points of view considered ?

(*e*) Is a textbook used ? If so, the exact title, publisher, and edition should be quoted.

IV. *What applications of the Differential and Integral Calculus are considered ?*

What questions of analysis (Higher Algebra and Trigonometry), geometry or physics involving the idea of a limit, otherwise wholly or partially present in the programmes of the schools, are utilised to illustrate and explain the Differential and Integral Calculus, so that there may be an economy in the treatment of the subjects studied ?

In particular:

(*a*) Is the Calculus applied to the theory of maxima and minima ?

(*b*) When Taylor's Theorem is considered, what are the functions whose developments in power series are obtained by means of it ?

(*c*) In the cases where the remainder form of Taylor's Theorem is discussed, are power series used for purposes of interpolation, extrapolation and the calculation of errors ?

(*d*) When the Integral Calculus is taught, is it applied to the calculation of areas (in the cases, for instance, of the parabola and ellipse), and of volumes ?

(*e*) In connection with what fundamental concepts of Mechanics (velocity, acceleration, work, moment of inertia, etc.) is use made of the Differential and Integral Calculus ?

(*f*) The corresponding questions for Physics, and in particular for Optics (curves, envelopes, etc.), and for Electrodynamics (lines of force, etc.), should be answered.

V. *Has the introduction of the Differential and Integral Calculus been at the expense of other branches of study ? If so, of which ?*

VI. *What has been the result of the recent introduction of the Differential and Integral Calculus into the school programmes ? Is the introduction felt to have been an inevitable advance ? How far has it found support, or the contrary ? In particular, what is the attitude of mathematicians and physicists towards the innovation ?*

Should any other details of interest concerning the teaching of the Differential and Integral Calculus have come to the knowledge of the observer, it is requested that they may be chronicled at this stage of the report.

A list should also be made of the passages in the reports published by the sub-commission in the country in question which relate to the teaching of the Differential and Integral Calculus.

The Report is to be presented in April, 1914, at Paris.

TEACHING OF CALCULUS IN PUBLIC AND SECONDARY SCHOOLS IN THE UNITED KINGDOM.

PUBLIC SCHOOLS—ENGLAND.

"PUBLIC SCHOOLS" correspond broadly to German Gymnasia.

The boys studying calculus in these schools are of three classes:

(1) Boys aged 17 to 19, of some mathematical ability, who will in due time compete for University Scholarships.

(2) Boys of 17 to 19, preparing for the Army entrance examinations, or destined for the profession of engineering.

(3) Boys aged 16 to 18, of merely average ability, who take the calculus as a regular item of school mathematics.

Class (1) may be assumed to be in the highest mathematical class of the school. In this class, there are usually several small groups of boys of various degrees of advancement; it is commonly impossible to make much use of class teaching, except in special subjects; as a rule, the boys work away at their textbooks and exercises, and come up to the master's desk for elucidations in their turn. Under these circumstances the lines of their work are determined by the textbook they are using. I have therefore thought it sufficient to examine the textbooks most commonly in use, and to answer the questions proposed in the light of this examination and of my own personal impressions.

The number of boys of this class studying calculus at a given time in one school is small. There is nothing new in the fact of such boys studying calculus; it was not unusual in my own schooldays, 20 to 25 years ago, but no doubt is much more general now.

Class (2) have come into being in the last 15 years or so. On the one hand, calculus of a practical description has within this period been included in the Army entrance examinations. On the other, the growth of engineering courses at the Universities, together with more general causes, has drawn the attention of more school boys to the engineering profession as a possible opening; a demand has therefore arisen for a suitable preparatory course at schools. Boys in Class (2) are usually of less outstanding mathematical ability than those in Class (1). Classes (1) and (2) work together in some schools; but more usually there is a special Army class. I cannot make any general statement as to the provision most usually made for the engineering boys; circumstances vary very much.

It may be said that the instruction given to boys of Classes (1) and (2) tends to differ in this respect; that for Class (1) the teaching has more mathematical rigour and in Class (2) has a more practical flavour.

Class (3) represents a more recent movement. "Calculus for the average boy" is the keynote of the modern movement in mathematical teaching. The pressure of external examinations urges in this direction. In a few schools a preliminary course in calculus now forms an organic part of the curriculum for the middle classes; in many schools there is a tentative movement in the same direction. The work done under this head is concerned with the gradient of a graph and rate of increase of a function, with applications to maxima and minima, small increments, velocity and acceleration. The inverse problem of integration with the obvious applications is examined in some schools. The functions dealt with are usually simple integral powers of x.

It may be doubted whether the movement in question has yet produced its full effect, and probably a report on the subject at the present moment would be premature.

Textbooks. An enquiry as to textbooks in use was addressed to the 113 public schools. Replies were received from 94 schools.

It appears that two or three different books on calculus are in use in most schools. It is a very common practice to use a book such as Mercer or Gibson with beginners, and to pass on to such a book as Edwards at a later stage. The books most commonly used in public schools are the following:

J. Edwards, *Differential Calculus for Beginners* (Macmillan). 1st Ed. 1892. 262 pages.

J. Edwards, *Integral Calculus for Beginners* (Macmillan). 1st Ed. 1894. 308 pages.

J. Edwards, *Treatise on the Differential Calculus* (a more advanced work) (Macmillan). 1st Ed. 1886.

J. W. Mercer, *Calculus for Beginners* (Cam. Univ. Press) 1st Ed. 1910. 440 pages.

* G. A. Gibson, *Introduction to the Calculus* (Macmillan). 1st Ed. 1904. 225 pages.

H. Lamb, *Infinitesimal Calculus* (Cam. Univ. Press). 1st Ed. 1897. 616 pages.

A. Lodge, *Differential Calculus for Beginners* (Bell). 1st Ed. 1902. 4th Ed. 1913. 299 pages.

A. Lodge, *Integral Calculus for Beginners* (Bell). 1st Ed. 1905. 2nd Ed. 1912. 203 pages.

B. Williamson, *Elementary Treatise on Differential Calculus* (Longmans, Green & Co.). 6th Ed. 1887. 468 pages.

B. Williamson, *Elementary Treatise on Integral Calculus.* 6th Ed. 1891. 463 pages.

A. E. Love, *Elements of Differential and Integral Calculus* (Cam. Univ. Press). 1st Ed. 1909. 207 pages.

W. M. Baker, *Calculus for Beginners* (Bell). 1st Ed. 1912. 166 pages.

From the returns obtained it appears that an examination of the books by Edwards, Mercer and Gibson should give a fair idea of the teaching.

Edwards' books are very popular, and have had a long reign. They date from a time when rigour of treatment was little appreciated in England, even at Cambridge. I have never used these books myself, but I attribute their popularity to their usefulness for examination purposes, and to their good collections of exercises. They do not appear to contemplate the early teaching of the calculus to boys of average ability.

The growing prevalence of such early teaching is indicated by the success of the books by Mercer and Gibson; and the existence of such books has, no doubt, a powerful reaction on the movement. A distinguishing feature of both books is the sound and patient exposition of the fundamental points. In both books the treatment of integral calculus is interlaced with that of differential calculus, the former being attacked as soon as the latter has been developed to a certain point. Books of this type are found very suitable for introducing the beginner to the subject. A sufficient idea of the methods adopted may be gained by an inspection of the tables of contents, which are reproduced in the Appendix to this paper.

I. *How much of the Differential and Integral Calculus is taught in Public Schools?*

This depends entirely on the proficiency of the individual pupils.

In particular:

(*a*) Functions of more than one variable are treated by Edwards, but not by Mercer or Gibson.

(*b*) The Differential Calculus is applied to ordinary algebraic functions, and to exponential, trigonometrical and logarithmic functions. In a minimum course designed for the average student the tendency is to use only x^n.

(*c*) The Integral Calculus is generally studied by those who have made

* Prof. Gibson has also written a large book called *An Elementary Treatise on the Calculus.* The references below are always to the *Introduction to the Calculus.*

some progress with Differential Calculus. The limits depend on the ability of the student.

(*d*) Taylor's Theorem appears at an early stage in Edwards. Mercer does not give it, but discusses the approximate relation

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x).$$

Gibson does not give Taylor's Theorem; but he obtains developments of $\sin x$, $\cos x$, $\tan^{-1} x$, $\log(1+x)$, $\log\frac{1+x}{1-x}$ as series to n terms approximately representing the functions, giving in each case an upper limit for the error. He also sketches the exponential series.

(*e*) Edwards deals with Differential Equations at a late stage. (Equations of first and second order, and linear equations with constant coefficients.) Gibson does not give separate chapters to the subject, but solves incidentally a few differential equations connected with kinematics. Mercer points out that $\frac{dy}{dx}=2x+3$ gives a family of curves: he also includes a set of examples on differential equations, in which the methods of solving some of the common types are indicated, most of the work being left to the student.

II. *How far is the treatment of the subject rigorous, both as to the mode in which fundamental concepts are introduced, and as to the demonstrations employed?*

Twenty years ago, when I was up at Cambridge, the teaching given to ordinary students at that University was not conducted on rigorous lines. A movement towards rigour of mathematical treatment was then just beginning to make itself felt, and since that date has, I believe, gone to considerable lengths. Many of the best prepared students turned out at that date and earlier are unfamiliar with the modern standard of rigour, and presumably do not employ it in the teaching they now impart to others. Belonging to the non-rigorous period myself, I do not feel myself entirely competent to criticise the standard of rigour adopted in various textbooks. Edwards does not, I imagine, satisfy modern standards. As far as I am able to judge, Mercer and Gibson have spared no pains to convey accurate ideas within the scope of their work. But books of this type do not pretend to be exhaustive in their discussion of fundamentals. Their ideal appears to be this—state nothing but the truth, but do not necessarily state the whole truth.

(*a*) The idea of a limit is, as a rule, employed explicitly; it is not considered sufficient to introduce the notions of the Differential Calculus geometrically. On the other hand, such theorems as $\text{Lt}\,\frac{1}{a}=\frac{1}{\text{Lt}\,a}$ are generally taken for granted in teaching.

(*b*) Differentials are not used very prominently. I imagine that the impression formed in the mind of most school boys is that infinitely small quantities are small quantities which really exist. But this notion could not arise from a careful perusal of Mercer or Gibson.

Lodge bases his whole work on differentials. "In applications of the Calculus to mensuration, physics, mechanics, etc., the notion of differentials is fundamental, and, in particular, integration as a process of summation is based on that notion. Moreover, the simplicity of a differential lies in the fact that its 'dimensions' are the same as those of the integral quantity from which they spring, whereas differential coefficients are new quantities whose dimensions depend on those of *two* or more of the variables under consideration. The notion of differentials is some thousands of years older than that of differential coefficients, so that, historically, the order of precedence is justified."

Lodge claims that the method stimulates the intuitive perceptions of students to a remarkable degree. He has used it now for 30 years.

(*c*) The remainder in Taylor's Theorem is not considered on the first introduction of the series by Edwards, but is discussed at a later stage. Lamb avoids Taylor's Theorem till a late stage, at which the theorem can be discussed properly. Mercer and Gibson do not give the theorem.

(*d*) Lamb points out that non-differentiable functions exist, but will not be met with in his book. I have found no reference to the matter in Edwards, Mercer or Gibson.

(*e*) Is the idea of an irrational number logically and systematically introduced ?

As a rule, no. It is generally held that a satisfactory discussion of irrationals is work of university type. But, of course, there are teachers who hold and act upon the view that a strict treatment should enter from the start.

III. *How is the pupil introduced to the ideas of the Differential and Integral Calculus ?*

(*a*) The graphing of simple functions is a normal feature in the work of lower classes. This is not perhaps generally undertaken with an eye to the subsequent study of the Calculus; at the same time, the pupil is bound to be more or less familiar with the notion of the graph of a function before he attacks the Calculus.

(*b*) The notation generally employed is that of Leibniz. There is a tendency, however, to postpone the difficulties of this notation by using at first such notations as $Df(x)$; $D_x f(x)$.

(*c*) The older books contained separate volumes for Differential and Integral Calculus. The newer books tend to interlace the two branches to some extent.

Edwards begins his Integral Calculus with the determination of area by limit of summation; and evaluates by the same method the integrals of various functions. He deduces that integration is the inverse of differentiation.

Mercer begins by proposing the problem, given $\frac{dy}{dx}$, find y. Then follow applications to families of curves, and to kinematics. The next chapter opens with a geometrical and arithmetical discussion of area regarded as a summation. This is followed by the introduction of the equation $\frac{d}{dx}\text{area} = y = f(x)$; hence definite integral. Then various exercises; and ultimately the notation $\int$.

Gibson's treatment differs from Mercer's in that (1) $\frac{d}{dx}(\text{area}) = y$, and the definite integral precedes the idea of summation; (2) the notation $\int$ is introduced at the outset.

IV. *What applications of the Differential and Integral Calculus are considered ?*

It may be stated generally that the economies that might result from an application of the Calculus are not fully realised in practice. This is due to the custom of reading analytical geometry and mechanics before the calculus. For instance, the calculus is not usually applied (1) to ascertain the distance traversed by a point whose velocity varies; (2) to find the equation of the tangent to a plane curve.

(*a*) The Calculus is applied to the theory of maxima and minima, but the maxima and minima of a quadratic function are generally investigated algebraically at an earlier stage.

(*b*) Functions such as $(1+x)^n$, $\sin x$, $\cos x$, $\tan^{-1} x$, a^x, $\log(1+x)$ are usually developed in power series before Taylor's Theorem is reached.

(*c*) In cases where the remainder form of Taylor's Theorem is discussed, are power series used for purposes of interpolation, extrapolation and the calculation of errors? Work of this kind is not usually undertaken.

(*d*) The Integral Calculus, when taught, is always applied to calculation of areas and of volumes.

(*e*) The Calculus, when taught, is applied to deal with velocity, acceleration, work (*e.g.* stretching string, expanding gas), fluid pressure on areas, centre of mass, moment of inertia.

(*f*) Branches of physics, other than mechanics, are not usually taught by mathematical masters. I have no precise information as to how far physics masters use the calculus in such subjects as Optics and Electrodynamics. But it may be assumed that the majority of physics masters in public schools concern themselves with the experimental and descriptive rather than the mathematical side of their subject; the proportion of masters who have made an extended study of both physics and mathematics is not great. Now that the new tripos regulations have come into force at Cambridge, the number of such masters may be expected to increase. If the masters teaching physics and mathematics were more closely in touch, the former would be able to make more use of the rudimentary ideas on calculus that the boys have obtained from the latter.

V. *Has the introduction of the Differential and Integral Calculus been at the expense of other branches of study? If so, of which?*

As regards boys of Class (1) (see page 235), the introduction of the calculus is not recent. In so far as there has been increased emphasis laid on the calculus in recent years, the time has probably been subtracted from such topics as trilinear coordinates, geometrical conics or theory of numbers.

As regards boys of Classes (2) and (3), calculus tends to encroach upon algebra, threatening the more manipulative and formal side of this subject. There may have been some economy, too, in arithmetic, especially in the matter of obsolete commercial rules. Less frequently, the formal side of geometry suffers.

VI. *What has been the result of the recent introduction of the Differential and Integral Calculus into the school programmes? Is the introduction felt to have been an inevitable advance? How far has it found support, or the contrary? In particular, what is the attitude of mathematicians and physicists towards the innovation?*

In answering this question in the absence of definite replies from a large number of correspondents, it is difficult to eliminate one's own personal views and aspirations. The subject has been ably discussed by Mr. C. S. Jackson, in a paper entitled, "The Calculus as a School Subject," which is incorporated in Part I. of the *Reports on the Teaching of Mathematics in the United Kingdom*, as presented to the Cambridge Congress in 1912. Mr. Jackson's attitude may be described as sympathetic but critical.

Broadly speaking, the movement has received general support. Perhaps the most powerful stimulus is that of the engineers, as represented by Prof. Perry. The physicists have long pressed for a modicum of calculus, and prefer to take it without too much mathematical rigour. The Universities have progressively included more calculus in their examination papers for schools; these papers, together with those set by the Civil Service Commissioners (for admission to the Army and the public service generally), are the most powerful lever that acts on the school curriculum. It will be understood, of course, that there is in England no general curriculum imposed upon schools: schools frame their own curricula, but tend to adapt them to the examination requirements of their pupils.

Whatever opposition there has been to an introduction of the calculus at an early stage has come from those who fear that a diminished emphasis on the manipulative and formal side of algebra will have a bad effect. The question raised is this: What algebraic equipment constitutes a firm base for a superstructure of Calculus?

This is the only articulate objection that has found voice. But the main obstacle is that most powerful force in educational matter—*vis inertiae.*

I submitted a first draft of this report to the members of the Public School Sub-committee of the Mathematical Association. I have to thank many of these gentlemen for suggestions which I have been glad to incorporate in the final report. It must not be understood, however, that anyone shares with me the responsibility for the statements made above.

Prof. Gibson informs me that my remarks may be taken as generally applicable to the Secondary Schools of Scotland. C. GODFREY.

THE TEACHING OF NUMERICAL TRIGONOMETRY.

(*Concluded.*)

A LITTLE experience of problems in which it is necessary to divide by sine or cosine of an angle will prepare a boy for the definitions of secant, cosecant and cotangent as the reciprocals of the cosine, sine and tangent respectively, though, as a matter of fact, he can get on very well without them. At this stage a boy will probably appreciate the fact that since an acute angle is determined if we know any one of the six trigonometrical ratios, it should be possible, if any one of the ratios is given, to find the others. Problems such as this: "Given $\sin A = \cdot 64$, find $\tan A$" should be done (i) by drawing a figure to scale, *i.e.* constructing the angle A whose sine is ·64, and finding the tangent by measurement; (ii) by drawing a right-angled triangle, hypotenuse 100 units, one side 64 units, and calculating the other side by Pythagoras' theorem; (iii) by taking from the tables the angles whose sine is ·64 and looking up its tangent.

With judicious prodding, he should have no difficulty in discovering such relations as $\tan A = \dfrac{\sin A}{\cos A}$, $\sin^2 A + \cos^2 A = 1$, and these should be verified in special cases.

Sometime or other the graphs of the ratios will come in. That the tangent changes gradually as the angle changes is obvious from the figure, and it is also obvious that equal increments in the angle do not produce equal increments in the tangent. An inspection of tables, of course, tells the same story, but the graph tells it much better, especially to boys who are trained as most boys are nowadays, to "think in graphs." The general form of the graph should be familiar, the great thing being that it is not straight. It is very difficult even with good boys to keep out such howlers as

$$\tan 40^\circ + \tan 30^\circ = \tan 70^\circ, \quad \tan 80^\circ = 2 \tan 40^\circ,$$

and frequent appeals to the non-straightness of the graph should be helpful. Personally, I am inclined to put this graph quite early, when boys are finding tangents of different angles. The graph appeals to them at that stage as a sort of pictorial table of natural tangents, and should be plotted from the figure from which the tangents are found rather than from the tables. The graphs of $\sin x$ and $\cos x$ should also be familiar. It is worth while, I think, as throwing some light on the difference columns and the reason for their absence in certain places, to plot on a large scale portions of these graphs, *e.g.* $\sin x$ from 30° to 31°, $\tan x$ from 45° to 46°, $\tan x$ from 87° to 88°, using the values given in the tables for 30°, 30·1°, 30·2°, etc. The next stage will consist in solving right-angled triangles in cases where the arithmetic is heavier and logarithms are used to lighten it. The actual steps