On the Stability of a Liquid Ellipsoid which is rotating about a Principal Axis under the influence of its own attraction. By A. B. Basset, M.A.

[*Read Nov.* 10*th*, 1887.]

1. When a mass of liquid is rotating in a state of steady motion under the influence of its own attraction, the different ellipsoidal forms which its free surface can assume may be classified as follows :—

I. *Maclaurin's Spheroid*, in which the free surface is an oblate spheroid, and the liquid rotates as a rigid body about the axis of the spheroid. If ρ be the density of the liquid, ζ the angular velocity of the spheroid (which in this case is identical with the molecular rotation), it is known that $\zeta^2/4\pi\rho$ must not be greater than $\cdot 1123$, in order that steady motion may be possible, and in this case the free surface may be one or other of two oblate spheroids, which coalesce when $\zeta^2/4\pi\rho = \cdot 1123$.

II. *Jacobi's Ellipsoid*, in which the free surface is an ellipsoid with three unequal axes, and the liquid rotates as a rigid body about the least axis. In this case $\zeta^2/4\pi\rho$ must not be greater than $\cdot 0934$, in order that the ellipsoid may be a possible form of the free surface. Hence, if $\zeta^2/4\pi\rho < \cdot 0934$, there are three ellipsoidal forms, viz., the two Maclaurin's spheroids, and the Jacobian ellipsoid. When $\zeta^2/4\pi\rho = \cdot 0934$, Jacobi's ellipsoid coalesces with the most oblate of the two spheroids, and when $\zeta^2/4\pi\rho$ lies between $\cdot 0934$ and $\cdot 1123$, the ellipsoidal form is impossible.

III. *Dedekind's Ellipsoid*, in which the free surface remains stationary in space, but there is an internal motion of the particles of liquid, due to molecular rotation ζ about lines parallel to the least axis. In this case, if a and b are the greatest and mean axes respectively, $a^2b^2\zeta^2/(a^2+b^2)^2\pi\rho$ must not be greater than $\cdot 0934$; and when the former quantity is equal to $\cdot 0934$, we must have $a = b$, and Dedekind's ellipsoid coalesces with the most oblate of the two Maclaurin's spheroids.

IV. *The Irrotational Ellipsoid*, in which the axis of rotation is the *mean* axis, and the motion is irrotational. In this case the spheroidal form is not possible.

V. An ellipsoid in which there is molecular rotation ζ, and an *independent* angular velocity $\zeta+\Omega$ about the axis to which ζ refers.

In this case the axis will be the *mean* or *least* axis, according as

$$\frac{\zeta}{\Omega} < \text{ or } > \frac{a^2-b^2}{a^2+b^2}\left(1 \pm \frac{2a}{\sqrt{a^2-b^2}}\right).$$

When this inequality becomes an equality, the free surface will be a prolate spheroid rotating about an equatorial axis. This case includes the four preceding cases.

VI. *Riemann's Ellipsoid*, in which the rotation takes place about an instantaneous axis lying in a principal plane. This case includes all the preceding cases; moreover, if the instantaneous axis does not lie in a principal plane, steady motion is impossible.*

In the present paper, I propose to consider the stability of a liquid ellipsoid which in steady motion is rotating about a principal axis, and which is subjected to a disturbance such that the free surface in the beginning of the disturbed motion is an ellipsoid. A disturbance of this character may be communicated by enclosing the liquid ellipsoid in a case which is subjected to an impulsive couple about any diameter together with a deformation of its surface, and is therefore equivalent to a disturbance produced by an impulsive pressure communicated to the free surface of the liquid. The disturbed motion may therefore be investigated by means of Riemann's general equations of motion, a proof of which I have given in Vol. XVII. of the Society's *Proceedings*; and references to the equations of this paper will be denoted as follows: [E].

The method employed is founded upon Riemann's paper, and the present investigation is an amplification of his work upon this portion of the subject.

2. By [E. 21], the potential energy of an ellipsoidal mass of gravitating liquid of mass M and uniform density ρ is

$$U = D - \tfrac{2}{5} M\pi\rho\, abc \int_0^\infty \frac{d\lambda}{P},$$

where $P = \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$, and D is a constant. Let R be the radius of a sphere of equal volume, then

$$U = 0 \quad \text{when} \quad a = b = c = R,$$

therefore $$D = \tfrac{4}{5} M\pi\rho R^2,$$

and $$U = \tfrac{4}{5} M\pi\rho R^2 - \tfrac{2}{5} M\pi\rho\, abc \int_0^\infty \frac{d\lambda}{P} \quad \text{.....................(1).}$$

* For proofs of the foregoing theorems, see—Riemann, *Abhand. König. Wiss., Göttingen*, Vol. IX.; Greenhill, *Proc. Camb. Phil. Soc.*, Vols. III. and IV.

Now U is evidently positive ; hence the integral must be a maximum when $a = b = c = R$, and will become indefinitely small when any one of the axes of the ellipsoid becomes infinitely small or infinitely large.

Let $2c$ be the axis of rotation, then, employing the notation of the preceding paper, let

$$E = \frac{M}{10}\left\{\frac{\omega_3^2(a^2-b^2)^2}{a^2+b^2} + \frac{4a^2b^2\zeta^2}{a^2+b^2} - 4\pi\rho abc\int_0^\infty \frac{d\lambda}{P}\right\} \quad \text{.........(2).}$$

By [E. 20 and 21], E is the variable part of the energy of a mass of liquid whose free surface is constrained to maintain a fixed ellipsoidal form and which is rotating about the least axis. In steady motion ω_3 and ζ, and therefore E, are certain functions of a, b, c; let E_0 be the value of E in steady motion.

Let a disturbance (which for brevity will be called an ellipsoidal disturbance) be communicated to the liquid by means of an impulsive pressure applied to its free surface, which is such that in the beginning of the disturbed motion the free surface is a slightly different ellipsoid. Then, if $E_0+\delta E$ is the energy of the disturbed motion, we obtain by [E. 20 and 21],

$$\delta E = \frac{M}{10}\left[\dot{a}^2+\dot{b}^2+\dot{c}^2+\frac{\omega_1^2(b^2-c^2)^2}{b^2+c^2}+\frac{\omega_2^2(c^2-a^2)^2}{c^2+a^2}+\frac{4b^2c^2\xi^2}{b^2+c^2} + \frac{4c^2a^2\eta^2}{c^2+a^2}\right]+E-E_0.$$

All the terms in square brackets are positive, and in the beginning of the disturbed motion are small quantities ; hence, if $E > E_0$, these terms must remain small quantities and the free surface can never deviate far from its form in steady motion, and the motion is therefore stable. But, if $E < E_0$, the terms in square brackets may become a finite positive quantity, and the difference $E-E_0$ may become a finite negative quantity, such that the difference between the two sets of terms always remains equal to the infinitesimal quantity δE. When this is the case the free surface may deviate far from its form in steady motion, and the motion may be unstable.

Hence, for the particular kind of disturbance which we are considering, the condition of stability requires that the energy in steady motion should be a minimum. Or, in other words, if the steady motion is stable, it must be impossible by any kind of ellipsoidal disturbance to abstract energy from the system.

3. Let the disturbing pressure be divided into two parts p_1, p_2, the

former of which produces a variation of the axes and no change in the angular momentum, whilst the latter produces no instantaneous variation of the axis but changes the angular momentum. The resultant of p_2 will consist of a single force, which produces a translation of the whole mass of liquid, and which it is unnecessary to consider; and a couple G. If the axis of this couple lie in the principal plane, which is perpendicular to the axis of rotation in steady motion, the energy will be evidently increased by its application; but, if the axis of the couple does not lie in this principal plane, the component of the couple about the axis of rotation may diminish the energy if it acts in the opposite direction to that of rotation, in which case the motion will be unstable.

In Maclaurin's spheroid the component of the couple about the axis of rotation necessarily vanishes, since p_2 always passes through the axis of rotation; the case of Dedekind's ellipsoid, in which the free surface is stationary, will be considered later on.

Hence, so far as the action of p_2 is concerned, Jacobi's ellipsoid, the Irrotational ellipsoid, and the ellipsoids belonging to the general class V., including the prolate spheroid rotating about an equatorial axis, but excluding Dedekind's ellipsoid, are stable whenever the couple component about the axis of rotation of the disturbing pressure either vanishes or is in the same direction as the rotation; but when this is not the case the motion may be unstable.

In the case of Dedekind's ellipsoid, by (2),

$$E_0 = \frac{2M}{5}\left\{\frac{a^2b^2\zeta^2}{a^2+b^2} - \pi\rho abc\int_0^\infty \frac{d\lambda}{P}\right\},$$

where

$$\frac{4a^2b^2\zeta^2}{a^2+b^2} = \frac{Aa^2-Cc^2}{a^2} = \frac{Bb^2-Cc^2}{b^2},$$

and the effect of a disturbing couple about the axis of rotation will be to increase the energy by the quantity

$$\frac{M\omega_3^2(a^2-b^2)^2}{10(a^2+b^2)},$$

whence $E > E_0$, and therefore the motion so far as this kind of disturbance is concerned is stable.

4. We must now consider the disturbance p_1 which produces a variation of the axes. From the last two of [E. 16] we obtain

$$(a-b)^2 w = \text{const.} = r, \quad (a+b)^2 w' = \text{const.} = r' \quad \text{.........(3)},$$

whence, from [E. 9],

$$\frac{\zeta}{c} = \frac{r'-r}{2abc}, \quad h_3 = \frac{M}{5}(r'+r) \quad \text{....................(4)}.$$

Also, from [E. 6],

$$h_3 = \frac{M}{5(a^2+b^2)}\left\{(a^2-b^2)^2\omega_3+4a^2b^2\zeta\right\},$$

whence
$$E = \frac{M}{5}\left\{\frac{r^2}{(a-b)^2}+\frac{r'^2}{(a+b)^2}-2H\right\} \quad\text{.................(5)},$$

where
$$H = \pi\rho abc\int_0^\infty \frac{d\lambda}{P}.$$

We must now obtain the value of E_0. Putting $\ddot{a}$, $\ddot{b}$, $\ddot{c}$ each equal to zero in the first three of [E. 16], and taking account of (3), we obtain

$$\left.\begin{aligned} 0 &= \tfrac{1}{2}Cc-\frac{\sigma}{c} \\ \frac{r'^2}{(a+b)^3}+\frac{r^2}{(a-b)^3} &= \tfrac{1}{2}Aa-\frac{\sigma}{a} = \frac{1}{2a}(Aa^2-Cc^2) \\ \frac{r'^2}{(a+b)^3}-\frac{r^2}{(a-b)^3} &= \tfrac{1}{2}Bb-\frac{\sigma}{b} = \frac{1}{2b}(Bb^2-Cc^2) \end{aligned}\right\} \quad\text{......(6)}.$$

Whence (5) becomes

$$\begin{aligned} E_0 &= \tfrac{1}{2}(Aa^2+Bb^2-2Cc^2)-2H \\ &= -H-\tfrac{3}{2}Cc^2 \quad\text{...................................(7)}, \end{aligned}$$

since
$$Aa^2+Bb^2+Cc^2 = 2H.$$

Whence E_0 is a finite *negative* quantity.

The constants r, r' express the fact that the angular momentum and the vorticity* are unchanged during the motion; also since the disturbance p_1 does not change the angular momentum or vorticity, these constants must have the same values as in steady motion.

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* The equation [E. 17], which expresses the fact that the vorticity is constant, may be shortly proved thus:—

Since ξ, η, ζ are independent of x, y, z, the vortex lines must all be parallel to some diameter r of the ellipsoid. Let l, m, n be the direction cosines of r, dS an element of the plane conjugate to r, and ϵ the angle between r and S.

The condition that the vorticity should be constant requires that

$$\text{const.} = \iint \omega\sin\epsilon\, dS = \omega S\sin\epsilon = \omega S pr^{-1},$$

where p is the perpendicular from the centre on to the tangent plane parallel to the plane S. But, since the volume of the ellipsoid is constant, Sp = const., therefore ω/r = const., or

$$\omega^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right) = \text{const.},$$

i.e.,
$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2} = \text{const.}$$

Since the volume of the ellipsoid is constant, the conditions that E may be a minimum require that

$$\left.\begin{aligned} \frac{dE}{da}-\frac{c}{a}\,\frac{dE}{dc}=0 \\ \frac{dE}{db}-\frac{c}{b}\,\frac{dE}{dc}=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

On performing the differentiations it will be found that (8) lead to (6); hence the first conditions are satisfied.

We must now enquire whether, in the general case, E has a minimum value when r and r' are unchanged by the disturbance.

Let $z = 5E/M$, $R^3 = abc$, $x = a$, $y = b$, then

$$z = \frac{r^2}{(x-y)^2}+\frac{r'^2}{(x+y)^2}-2\pi\rho R^3\int_0^\infty \frac{d\lambda}{\sqrt{(x^2+\lambda)(y^2+\lambda)(R^6/x^2y^2+\lambda)}} \ldots (9)$$

Since a, b, c are positive, and a is never less than b, we have to examine the form of the surface (9) between the planes $y = 0$, $x-y = 0$.

First suppose r is not zero.

When $x=y$, $z=\infty$. If y has any finite value $<$ or $= x$, then, as x increases from y to infinity, z diminishes, and the value of E_0 in steady motion shows that z will vanish and become negative, and when x is very large z is very small. Moreover, z can never become equal to $-\infty$ for any values of x or y, and when x and y are *both* very large z is very small, unless $x-y$ is small.

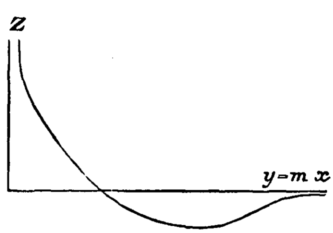

Fig.1.

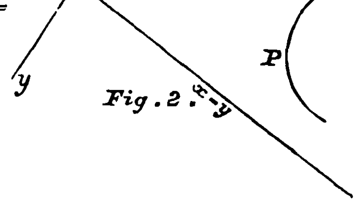

Fig.2.

A general idea of the form of the surface may be obtained from the accompanying figures. Fig. 1 is the curve of section made by the plane $y = mx$, $m < 1$; and Fig. 2 shows the curves of section made by the planes xz and xy. The surface cuts the plane of xy along the curve xP, and the sheet underneath this plane gradually bends upwards towards the plane.

It therefore follows that in this case z must have a minimum value, which is given by (7).

By a similar course of reasoning it may be shown that z has also a minimum value when $\tau = 0$.

The conditions that τ should vanish require that in steady motion

$$w\,(a-b)^2 = 0,$$

which requires either that $a = b$, which is the case of Maclaurin's spheroid; or that $w = 0$, which by [E. 8] is the same as

$$\omega_3 + \frac{2ab\,\Omega_3}{a^2+b^2} = 0,$$

which is a special case of V.

5. The analytical difficulties of examining whether E is a minimum by means of the usual conditions that E_{aa}, E_{bb}, $E_{aa}E_{bb}-E_{ab}^2$ must all be positive, where $E_{aa} = d^2E/da^2$, &c., would be considerable; but in the case of Maclaurin's spheroid, this may be done without much trouble.

Since c is a function of the independent variables a and b, we have (omitting the factor $M/5$, and putting $Q = Aa^2-Cc^2$, $R = Bb^2-Cc^2$)

$$E_a = -\frac{2\tau^2}{(a-b)^3} - \frac{2\tau'^2}{(a+b)^3} + \frac{Q}{a},$$

$$E_b = \frac{2\tau^2}{(a-b)^3} - \frac{2\tau'^2}{(a+b)^3} + \frac{R}{b},$$

$$\left.\begin{aligned} E_{aa} &= \frac{6\tau^2}{(a-b)^4} + \frac{6\tau'^2}{(a+b)^4} + \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc}\right) - \frac{Q}{a^2} \\ &= 6\,(w^2+w'^2) + \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc}\right) - \frac{Q}{a^2} \\ E_{bb} &= 6\,(w^2+w'^2) + \frac{1}{b}\left(\frac{dR}{db} - \frac{c}{b}\frac{dR}{dc}\right) - \frac{R}{b^2} \\ E_{ab} &= 6(w'^2-w^2) + \frac{1}{a}\,\frac{dQ}{db} - \frac{c}{ab}\,\frac{dQ}{dc} \end{aligned}\right\} \quad\text{......(10).}$$

Equations (10) are perfectly general, but in the case of Maclaurin's spheroid
$$w = w' = \tfrac{1}{2}\zeta = Q^{\frac{1}{2}}/2a,$$
and a must be put equal to b after the differentiations have been performed. Whence
$$\left.\begin{aligned} E_{aa} = E_{bb} &= \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc} + \frac{2Q}{a}\right) \\ E_{ab} &= \frac{1}{a}\frac{dQ}{db} - \frac{c}{a^2}\frac{dQ}{dc} \end{aligned}\right\} \quad\text{............(11).}$$

Now
$$Q = 2\pi\rho\, abc \int_0^\infty \left(\frac{1}{c^2+\lambda} - \frac{1}{a^2+\lambda}\right)\frac{\lambda\, d\lambda}{P},$$
therefore (omitting $2\pi\rho\, abc$)
$$\frac{dQ}{db} = \int_0^\infty \left(\frac{1}{a^2+\lambda} - \frac{1}{c^2+\lambda}\right)\frac{\lambda b\, d\lambda}{(b^2+\lambda)\, P},$$
$$\frac{dQ}{dc} = \int_0^\infty \left(\frac{1}{a^2+\lambda} - \frac{3}{c^2+\lambda}\right)\frac{\lambda c\, d\lambda}{(c^2+\lambda)\, P}.$$

Therefore, when $a = b$,
$$a\frac{dQ}{db} - c\frac{dQ}{dc} = \int_0^\infty \frac{\left[2\lambda^2 c^2 + \lambda\left\{8a^2c^2 - (a^2+c^2)^2\right\} + 2a^4c^2\right]\lambda\, d\lambda}{(a^2+\lambda)^2\,(c^2+\lambda)^2\, P}.$$

The numerator of this expression can never become negative, since it is positive when $\lambda = 0$, and the roots of the equation for λ, obtained by equating it to zero, are imaginary. Hence
$$a\frac{dQ}{db} > c\frac{dQ}{dc}.$$

Now dQ/db, and therefore dQ/dc, are negative; it therefore follows from (11) that E_{aa} will be positive if $E_{aa} - E_{ab}$ is positive. Now
$$a(E_{aa} - E_{ab}) = \frac{dQ}{da} - \frac{dQ}{db} + \frac{2Q}{a},$$
$$\frac{dQ}{da} = 2Aa + a^2\frac{dA}{da} - c^2\frac{dC}{da},$$
$$\frac{dQ}{db} = a^2\frac{dA}{db} - c^2\frac{dC}{da};$$
and
$$\frac{dA}{da} = -3a\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^3\,(c^2+\lambda)^{\frac{1}{2}}} = 3\frac{dQ}{d}$$

Therefore the condition becomes

$$2Aa^2 - Cc^2 - a^4 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^3 (c^2+\lambda)^{\frac{1}{2}}} > 0.$$

Now, if e be the excentricity of the meridian section of the spheroid (the factor $2\pi\rho a^2 c$ being omitted),

$$A = \frac{1}{a^3 e^3} \left\{ \sin^{-1} e - e\sqrt{1-e^2} \right\},$$

$$C = \frac{2}{a^3 e^3} \left\{ \frac{e}{\sqrt{1-e^2}} - \sin^{-1} e \right\},$$

$$\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^3 (c^2+\lambda)^{\frac{1}{2}}} = \frac{1}{a^5 e^5} \left[\tfrac{3}{4} \left\{ \sin^{-1} e - e\sqrt{1-e^2} \right\} - \tfrac{1}{2} e^3 \sqrt{1-e^2} \right],$$

whence the condition becomes

$$\sin^{-1} e \left\{ 2 - e^2 - \frac{3}{8e^2} \right\} + \sqrt{1-e^2} \left(\frac{3}{8e} - \frac{5e}{8} \right) > 0.$$

The above expression is positive for all values of e lying between 0 and 1, both inclusive, whence Maclaurin's spheroid is stable.

6. In the last edition of Thomson and Tait's "Natural Philosophy," Vol. I. Part II., pages 329 and 333, it is stated that Maclaurin's spheroid is unstable if the excentricity exceeds that of the spheroid which coalesces with the limiting Jacobian ellipsoid; that is, when

$$e > \sin 54°21'27'' \text{ or } \cdot 8127.$$

Unfortunately, no proof of this statement is given, and if the analysis of the present paper be correct, it follows that the disturbance which produces instability, cannot be what I have called an ellipsoidal disturbance, but must be such that the boundary in the beginning of the disturbed motion must be a surface which is not an ellipsoid but one slightly differing therefrom.

Apparently, however, Poincaré does not agree with Sir W. Thomson's result, for on p. 379 of an elaborate memoir* he says :—"Les ellipsoïdes de révolution, qui sont plus aplatis que celui qui est en même temps un ellipsoïde de Jacobi, mais dont l'aplatissement reste inférieur à une certaine limite, sont stables si le fluide est parfaitement dépourvu de viscosité ; ils ne sont plus si le fluide est visqueux et si

* "Sur l'Équilibre d'une Masse Fluide animée d'un mouvement de Rotation. *Acta Mathematica*, Vol. vii., p. 259.

peu qu'il soit." It should be noticed that Poincaré considers a disturbance of a much more general character than I have done.

Independently of any mathematical analysis, it seems almost certain that Maclaurin's spheroid must become unstable when the excentricity exceeds a certain limit. For, suppose the spheroid is shaped like an orange, and let a small jet of air be directed for a short time to some point on its surface. The effect of this will be to cause waves to diverge from the point of application of the jet, which will travel over the surface, but the motion will not be otherwise affected. But, if the shape of the spheroid resembles that of a thin disc, the probable effect of the jet will be to cause the liquid to curl itself up, or possibly to break up into two or more detached portions, and the motion will be thoroughly unstable. It appears to me that the disturbed motion might be investigated by a more simple process than has been employed by Poincaré, by assuming that in the beginning of the disturbed motion the equation of the free surface is of the form

$$\nu = \gamma + \Sigma A_{mn} P_n(\mu) \cos m\phi,$$

where ν, μ are elliptic coordinates of a point, and γ is the value of ν in steady motion, and proceeding upon the lines of my former paper; but any investigation of this kind must form the subject of a future communication.

7. The motion of a liquid spheroid which rotates about its axis of figure has been fully discussed by Dirichlet, whose equations have been deduced on p. 261 of my former paper, the density of the liquid being there taken as unity. From [E. 22] it follows that, if the rotating liquid is inclosed in a case (which may be either a prolate or an oblate spheroid), and the case is removed, it will be impossible for the free surface to retain the spheroidal form unless *initially* $\zeta^2/2\pi\rho < 1$, where ρ is the density; and that, if this condition is not satisfied, the free surface during the subsequent motion will assume some other revolutional form. Also, if $2c$ be the length of the axis of figure, and the free surface is initially spheroidal, it will cease to be so, if at any period of the subsequent motion

$$\frac{\zeta^2}{2\pi} > 1 + \frac{3\dot{c}^2}{8\pi\rho c^2}.$$

The following errata in my former paper should be noticed :—

In equation (1), the second member should be $= 0$.

Page 259, line 9, the following expression should be added to the right-hand side, viz.:

$$\dot{a}^2/a^2-(w-w')\left\{(a-b)\,w-(a+b)\,w'\right\}/a+(v-v')\left\{(c-a)\,v+(c+a)\,v'\right\}/a.$$

Page 261, line 6, *read* $w=w'=\tfrac{1}{2}\zeta$.

,, ,, equation (23), *read* $3\dot{a}^2/a^4$ *instead of* $3\dot{a}^2/4\dot{a}^4$.

Geometry of the Quartic. *By* R. Russell, M.A.

[*Read Nov.* 10*th*, 1887.]

The system of points, the properties of which I intend to discuss, arose from an attempt to interpret geometrically the sextic covariant of a quartic.

I consider a quartic whose coefficients may be any whatever, real or imaginary. Its roots are of the form a_1+ib_1, a_2+ib_2, a_3+ib_3, a_4+ib_4. These are represented as follows:—Assume any two rectangular axes and take the point whose coordinates are a_1, b_1; that point may be considered to represent the complex quantity a_1+ib_1. We see, therefore, that the four roots of a quartic may be represented by four points in a plane.

I. If α, β, γ, δ be the four roots of the quartic, the factors of the sextic covariant are the numerators of

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta};\quad \frac{1}{z-\gamma}+\frac{1}{z-\alpha}-\frac{1}{z-\beta}-\frac{1}{z-\delta};$$
$$\frac{1}{z-\alpha}+\frac{1}{z-\beta}-\frac{1}{z-\gamma}-\frac{1}{z-\delta};$$

z of course denoting a quantity $x+iy$.

Let us consider the roots of the quadratic

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta}=0;$$

and let z represent a root of it. Now $z-\alpha$ defines the length and direction of the line joining z and α, and therefore $\dfrac{1}{z-\alpha}$ defines a line whose length is the reciprocal of that line, and whose direction is the re-

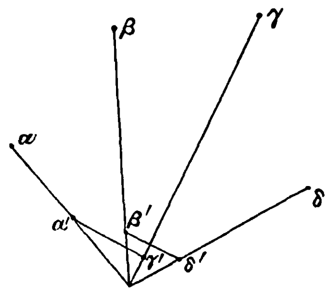

Fig. 1.