that is, λ and μ may be expressed rationally in terms of θ, and consequently the curve is unicursal.

If the discriminant either has two pairs of equal roots, or vanishes, it is clear that μ can be expressed as the ratio of two rational quadratic functions of λ, and that the curve is unicursal; but I am not at present able to make further statement of the peculiarities of these cases.

On the Explicit Integration of certain Differential Resolvents.

By Sir JAMES COCKLE, M.A., F.R.S.

[*Read Nov. 9th*, 1882.]

1. Let y be any root of

$$y^n - xy + x = 0 \qquad (1),$$

and let z be any root of $xz^{n-1} - n^3 z + n^3 = 0 \qquad (2).$

2. Then, if $n = 3$ or 4, we can so determine two numbers λ and μ as that

$$x^{-\lambda} y \text{ and } z^{\mu}$$

shall satisfy the same linear differential equation of the order $n-1$.

3. Put

$$xy^n - y^a + 1 = 0 \qquad (3),$$

then, y being any root of (3), y^m satisfies the n-ordinal*

$$[D]^a \left[\frac{n-a}{a} D + \frac{m}{a}\right]^{n-a} y^m - \left[\frac{n}{a} D + \frac{m}{a} - 1\right]^n x^a y^m = 0 \quad \ldots (4),$$

for all values of m and for all integral values of n and a, provided that $a < n$.

4. In (3) and (4), substitute x^{-1} for x and 1 for a. Then (3) re-

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* I have elsewhere (*Trans. Royal Society, Victoria*, Vol. vii., pp. 192, 193; 1866) shown this. I do not repeat the proof here because means of verifying the result have already been printed in the *Proceedings*. And if, at pp. 216 and 218 of Mr. Harley's Addendum to Mr. Rawson's paper (*Proc. Lond. Math. Soc.*, Vol. ix., 1878), we make the substitution

$$\begin{pmatrix} y^{-1}, & n-a, & -1, & 1, & n, & 1, & -m \\ y, & r, & b, & a, & m, & c, & n \end{pmatrix},$$

wherein the lower line refers to Mr. Harley's formulæ (i.) and (x.), we get (3) and (4) of the text. The results may moreover be transformed into, and confirmed by, a generalization of a theorem of Boole's, simultaneously and independently arrived at by Mr. Harley in England, and by me in Queensland (see *Report of British Association*, Meeting of 1866, pp. 2, 3 of "Notices and Abstracts").

duces to (1), and (4) becomes

$$-D\,[m-(n-1)\,D]^{n-1}\,y^m-[m-1-nD]^n\,x^{-1}y^m=0 \;\;\ldots\ldots(5),$$

which is satisfied by making $y=$ any root of (1).

5. Replacing y, n, a, and x by z, $n-1$, 1, and $\frac{x}{n^3}$ respectively, (3) becomes (2), and, writing μ instead of m, we get, in place of (4),

$$n^3D\,[(n-2)\,D+\mu]^{n-2}\,z^n-[(n-1)\,D+\mu-1]^{n-1}\,xz^n=0 \;\;\ldots\ldots(6),$$

n being an integer >1.

6. Let $m=1$, then (5) reduces to

$$[1-(n-1)\,D]^{n-1}\,y-n\,[-nD-1]^{n-1}\,x^{-1}y=0,$$

no arbitrary constant being introduced after the integration, because $\Sigma y=0$ when $n=3$ or 4.

7. This last $(n-1)$-ordinal is, by the algorithm of factorials, equivalent to $[(n-1)\,D+n-3]^{n-1}\,y-n\,[nD+n-1]^{n-1}\,x^{-1}\,y=0$.

8. Multiply this into $x^{1-\lambda}$, transfer the factor from left to right, and transpose the terms of the result; we get

$$n\,[nD+n\lambda-1]^{n-1}(x^{-\lambda}y)-[(n-1)\,D+(n-1)\,\lambda-2]^{n-1}x\,(x^{-\lambda}y)=0\ldots(7).$$

9. Hence (6) and (7) will, as differential equations, be identical, provided that

$$\mu=(n-1)\,\lambda-1 \;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$n^3D\,[(n-2)\,D+\mu]^{n-2}=n\,[nD+n\lambda-1]^{n-1}\ldots\ldots\ldots\ldots(9).$$

10. But (8) and (9) are satisfied, independently of D,* by

$$n=3,\quad \lambda=\tfrac{1}{3},\quad \mu=-\tfrac{1}{3},$$
$$n=3,\quad \lambda=\tfrac{2}{3},\quad \mu=\;\;\tfrac{1}{3},$$
$$n=4,\quad \lambda=\tfrac{1}{2},\quad \mu=\;\;\tfrac{1}{2}.$$

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* When $n=3$ or 4, then $(n-2)^{n-2}=n^{n-3}$, and (9) is of the degree $n-2$ in D. Hence (9) will be satisfied identically, if, treating D as a symbol of quantity, we can find more than $n-2$ values of D which shall satisfy (9). Let $D=0,\ \frac{\mu}{2-n}$; then (9) takes the forms $0=n\,[n\lambda-1]^{n-1}$ and $0=\left[\frac{2-n\lambda}{n-2}\right]^{n-1}$, both of which equations are satisfied by $\lambda=\frac{1}{n}$ or $\frac{2}{n}$, when $n=3$; and by $\lambda=\frac{2}{n}$, when $n=4$. If $n=4$, we require a third equation in order to establish the identity. Put $D=\frac{1}{n}-\lambda$, and (9) becomes $n^3\left(\frac{1}{n}-\lambda\right)\left[\lambda-\frac{2}{n}\right]^{n-2}=0$. Therefore $\lambda=\frac{2}{n}=\frac{1}{2}$ is an appropriate value; and there is no other. Other convenient assumptions, for instance, $D=\frac{1}{n}$ or $D=-\lambda$, which severally reduce (9) to

$$n\left[(n-1)\,\lambda-\frac{2}{n}\right]^{n-2}=[n\lambda]^{n-1}\quad\text{and}\quad[-1]^{n-1}=-n^2\,[\lambda]^{n-1},$$

confirm the foregoing results. And an actual development of (9), in the particular cases $n=3, 4$, is thus verified.

11. Hence, for these values of n, λ, μ, the expressions $x^{-\lambda}y$ and z^r satisfy the same $(n-1)$-ordinal.

12. It follows that the values of the arbitrary constants denoted by suffixed K's can always be so adjusted as to yield

$$x^{-\lambda}y_r = K_1 z_1^r + \ldots + K_{n-1} z_{n-1}^r,$$

or

$$y_r = x^{\lambda}(K_1 z_1^r + \ldots + K_{n-1} z_{n-1}^r) \quad \ldots\ldots\ldots\ldots\ldots (10),$$

where y_r is any root of (1).

13. When $n=2$ the process must be slightly modified. Put $a=1$, $m=1$; then (3) and (4) become

$$xy^2 - y + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$D(D+1)y - [2D]^2 xy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and from (12) we get successively

$$(D+1)y - 2(2D-1)xy = 2c_1$$

and the complete integral

$$y = \frac{c_1}{x} + \frac{c_2}{x}\sqrt{1-4x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

14. Take $y=\frac{1}{2}$, then $x=-2$, and by division we find that the other value of y is -1. Hence $\frac{1}{2} = -\frac{c_1}{2} + \frac{3c_2}{2}$, $-1 = -\frac{c_1}{2} - \frac{3c_2}{2}$, therefore $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$, and, with these values of c_1 and c_2, (13) is the solution of (11), the radical being taken as two-valued.

15. So long as we do not, for the purpose of defining the arbitrary constants, assume the general solution of the given equation, it is for the most part immaterial what values we assign to y. But $y=1$ must be excluded, because it would give $x=0$. And, when $n=2$, $y=2$ gives $x=\frac{1}{4}$, and, the other value of y being 2, we can only determine c_1 $(=\frac{1}{2})$. But any form of y which gives a unique value to x will suffice for the purpose. Let $y = 1 \pm \sqrt{-1}$; then $x=\frac{1}{4}$, and, as before, $c_1 = \frac{1}{2} = c_2$. Corresponding remarks apply to the cases $n=3$ or 4.

16. Take the cubic $\quad y^3 - xy + x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$

and, first, suppose that $\lambda = \frac{1}{3}$, $\mu = -\frac{1}{3}$. Then, by Art. 12, each of its roots is the product of $x^{\frac{1}{3}}$ into a linear and homogeneous function of the cube roots of the reciprocals of the roots of the quadratic

$$xz^2 - 27z + 27 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

17. But these roots may be taken as known, for the substitution of $27x$ for x in (15) changes (15) into (11), the general solution of which is effected in Arts. 13, 14. I add that, if in (15) we replace

z by $-\frac{27}{x^2}z$, we get the ordinary reducing quadratic, viz.,

$$z^2+xz+\frac{x^3}{27}=0.$$

18. In (14) put $y=-1$; then $x=\frac{1}{2}$, and the other two values of y are $\frac{1}{2}(1\pm\sqrt{-1})$. From (15), we get $z=27\pm15\sqrt{3}$. Hence, by (10), the expression

$$K_1(54+30\sqrt{3})^{-\frac{1}{3}}+K_2(54-30\sqrt{3})^{-\frac{1}{3}},$$

which is equivalent to

$$K_1(3+\sqrt{3})^{-1}+K_2(3-\sqrt{3})^{-1} \text{ or } \tfrac{1}{6}\{3(K_1+K_2)+(K_1-K_2)\sqrt{3}\},$$

can be made to yield the three values -1, $\frac{1}{2}(1\pm\sqrt{-1})$. And the three systems of relations

$$K_1,\ K_2=-\rho,\ -\rho^2,$$

where ρ is a cube root of 1, yield those values.

19. Secondly, let $\lambda=\frac{2}{3}$, $\mu=\frac{1}{3}$. We have substantially the same result, but with a different definition of the arbitrary constants. In fact the suppositions

$$\frac{K_1x^{\frac{1}{3}}}{z_1^{\frac{1}{3}}}+\frac{K_2x^{\frac{1}{3}}}{z_2^{\frac{1}{3}}}=C_1x^{\frac{1}{3}}z_1^{\frac{1}{3}}+C_2x^{\frac{1}{3}}z_2^{\frac{1}{3}},\quad z_1z_2=\frac{27}{x},$$

give
$$K_1z_2^{\frac{1}{3}}+K_2z_1^{\frac{1}{3}}=3\rho\,(C_1z_1^{\frac{1}{3}}+C_2z_2^{\frac{1}{3}}).$$

20. In (14), put $y=2$; then $x=8$ and the other two values of y are $-1\pm\sqrt{5}$. From (15), we get $z=\frac{27\pm3\sqrt{-15}}{16}$. Hence, by (10), the expression

$$C_1(108+12\sqrt{-15})^{\frac{1}{3}}+C_2(108-12\sqrt{-15})^{\frac{1}{3}},$$

which is equivalent to

$$C_1(-3+\sqrt{-15})+C_2(-3-\sqrt{-15}),$$

or
$$-3(C_1+C_2)+(C_1-C_2)\sqrt{-15},$$

can be made to yield the three values 2, $-1\pm\sqrt{5}$. And the three systems of relations
$$C_1,\ C_2=-\frac{\rho}{3},\ -\frac{\rho^2}{3},$$

yield those values. This conclusion accords with that of Art. 19, wherein we may put $\rho=1$.

21. Take the quartic $y^4-xy+x=0$..............................(16).
Here $\lambda=\frac{1}{3}=\mu$, and by Art. 12 each of the roots is the product of

$x^{\frac{1}{4}}$ into a linear and homogeneous function of the square roots of the roots of the cubic $xz^3-64z+64=0$ ....(17).

22. In (16) put $y=-2$; then $x=-\frac{16}{3}$, and (16) becomes

$$\tfrac{1}{3}(y+2)\,[2y^3+(y-2)^3]=0 \quad\text{..........................(18)},$$

so that the other three values of y are

$$2\,(1+\rho\sqrt[3]{2})^{-1} \text{ or } \tfrac{2}{3}\,(1-\rho\sqrt[3]{2}+\rho^2\sqrt[3]{4}).$$

23. From (17), we get

$$z=2\rho\sqrt[3]{2}-\rho^2\sqrt[3]{4}, \text{ and } \sqrt{z}=\frac{1}{\sqrt{3}}\,(2+\rho\sqrt[3]{2}-\rho^2\sqrt[3]{4}),$$

as will be seen on squaring.

24. Hence, by (10), the expression

$$\frac{4\sqrt{-1}}{3}\,[K_1(2+\omega\sqrt[3]{2}-\omega^2\sqrt[3]{4})+K_2(2+\omega^2\sqrt[3]{2}-\omega\sqrt[3]{4}) +K_3(2+\sqrt[3]{2}-\sqrt[3]{4})],$$

wherein ω is an unreal cube root of 1, can be made to yield the four values of y. Write i for $\sqrt{-1}$; then the four systems of relations

$$4iK_1,\ 4iK_2,\ 4iK_3=-1,\ -1,\ -1;\ -1,\ 1,\ 1;\ 1,\ -1,\ 1;\ 1,\ 1,\ -1,$$

yield those values.

25. The foregoing method postulates the extraction of roots, but it does not (even in the case of a quadratic) postulate any algebraical process by which an expression can be made a perfect algebraical power. So that, in applying Tschirnhausen's process to the transformation of a general quartic into (16), we do not need to travel outside this paper.

26. In (17) change z into $-\dfrac{4z}{x}$; we get $z^3-4xz-x^2=0$, Descartes' cubic, which the change of z into $2z$ converts into Ferrari's cubic, and that of z into $4z$ into Euler's cubic. In (11) insert the y of (13); we get $c_1=\frac{1}{2}$, $c_2=\pm\frac{1}{2}$, which is right, the radical being taken as having either value.

On Unicursal Twisted Quartics. *By* R. A. Roberts, M.A.

[*Read Nov. 9th,* 1882.]

I propose to consider, in this paper, some properties of the unicursal twisted quartic curve, namely, the intersection of a quadric and a cubic which contains two non-intersecting generators of the quadric. In the course of my investigation, I shall almost entirely make use of