THE INTERSECTION OF TWO CONIC SECTIONS

By J. A. H. JOHNSTON.

[Received April 30th, 1905.—Read May 11th, 1905.—Received in revised form May 29th, 1905.]

IT is stated in Salmon's *Conic Sections*, sixth edition, p. 337, that the cases of four real and four imaginary intersections of two conics have not been distinguished by any simple criterion.

In the notes (p. 391) of the same volume it is further stated that this discrimination has been made by Kemmer (Giessen, 1878), and his results are quoted.

The problem has been discussed subsequently by Storey (*American Journal*, Vol. VI.) and by Gundelfinger (*Vorlesungen*, Teubner, Leipzig, 1895), whose results differ from each other and are distinct from those of Kemmer.

It is the object of this paper, whose treatment is independent of theirs in method, to establish simple criteria for the cases of intersection of two conics, with corresponding results for real and imaginary tangents.

The forms in which both Kemmer and Storey expressed their results will be derived at once from the treatment of this paper; of Kemmer's four conditions one will be shown to be unnecessary, and the results of Storey will likewise call for modification.

The following notation will be adopted :—The two conics $(abcfgh)(xyz)^2$ and $(a'b'c'f'g'h')(xyz)^2$ will be called S and S'.

The minors of the determinants Δ and Δ' will be styled, as usual, A, B, C, F, G, H and $A', B', C', \ldots$.

$(bc'+b'c-2ff')$ will be called K,

$(ac'+ca'-2gg')$,, ,, L,

$(ab'+a'b-2hh')$,, ,, M,

$(gh'+g'h-af'-a'f)$,, ,, K',

$(fh'+f'h-bg'-b'g)$,, ,, L',

$(fg'+f'g-ch'-c'h)$,, ,, M'.

The tangential equations of S and S', viz., $(ABCFGH)(uvw)^2$ and $(A'B'C'F'G'H')(uvw)^2$, will be called Σ and Σ', and the contravariant $(KLMK'L'M')(uvw)^2$ called, as usual, ϕ.

The invariants of the conics will be Δ, θ, θ', and Δ', the cubic determining the line pairs $\lambda S+S'$, $\Delta\lambda^3+\theta\lambda^2+\theta'\lambda+\Delta' = 0$, and its discriminant $\theta^2\theta'^2+18\Delta\Delta'\theta\theta'-27\Delta^2\Delta'^2-4\Delta\theta'^3-4\Delta'\theta^3 = D$.

The question of the intersection of the two conics S and S' may be placed upon the simple basis of the nature of the several line pairs $\lambda S+S'$.

If $\lambda S+S'$ break up into straight lines $(lx+my+nz)$ and $(l'x+m'y+n'z)$, then l, m, n and l', m', n' can be so determined that

$$\begin{aligned} a\lambda+a' &= ll', \\ b\lambda+b' &= mm', \\ c\lambda+c' &= nn', \\ 2(f\lambda+f') &= mn'+m'n, \\ 2(g\lambda+g') &= nl'+n'l, \\ 2(h\lambda+h') &= lm'+l'm. \end{aligned}$$

It readily follows that

$$\left.\begin{array}{lll}
(1) & -(A\lambda^2+K\lambda+A') = \{(f\lambda+f')^2-(b\lambda+b')(c\lambda+c')\} & = \tfrac{1}{4}(mn'-m'n)^2 \\
(2) & -(B\lambda^2+L\lambda+B') = \{(g\lambda+g')^2-(c\lambda+c')(a\lambda+a')\} & = \tfrac{1}{4}(nl'-n'l)^2 \\
(3) & -(C\lambda^2+M\lambda+C') = \{(h\lambda+h')^2-(a\lambda+a')(b\lambda+b')\} & = \tfrac{1}{4}(lm'-l'm)^2 \\
(4) & -(F\lambda^2+K'\lambda+F') = \{(a\lambda+a')(f\lambda+f')-(g\lambda+g')(h\lambda+h')\} & \\
 & & = \tfrac{1}{4}(lm'-l'm)(nl'-n'l) \\
(5) & -(G\lambda^2+L'\lambda+G') = \{(b\lambda+b')(g\lambda+g')-(f\lambda+f')(h\lambda+h')\} & \\
 & & = \tfrac{1}{4}(mn'-m'n)(lm'-l'm) \\
(6) & -(H\lambda^2+M'\lambda+H') = \{(c\lambda+c')(h\lambda+h')-(f\lambda+f')(g\lambda+g')\} & \\
 & & = \tfrac{1}{4}(nl'-n'l)(mn'-m'n)
\end{array}\right\} \quad \text{(I.)}$$

From (1), (2), and (3) of (I.) it is readily seen that

$$u\sqrt{\{-(A\lambda^2+K\lambda+A')\}}+v\sqrt{\{-(B\lambda^2+L\lambda+B')\}} + w\sqrt{\{-(C\lambda^2+M\lambda+C')\}} = \tfrac{1}{2}\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}. \quad \text{(II.)}$$

Now, if both sides of (1), (2), (3), (4), (5), and (6) be multiplied by u^2, v^2, w^2, $2vw$, $2uw$, and $2uv$ respectively, and the results be added,

$$-(\Sigma\lambda^2+\phi\lambda+\Sigma') = \tfrac{1}{4}\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}^2 \text{ by (II.)}$$

$$= [u\sqrt{\{-(A\lambda^2+\ldots)\}}+v\sqrt{\{-(B\lambda^2+\ldots)\}} + w\sqrt{\{-(C\lambda^2+\ldots)\}}]^2. \quad \text{(III).}$$

If the line pair $\lambda S+S'$ be real, equations (III.), where u, v, w are any real variable line coordinates, show that, since the squared expressions are positive, $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is essentially negative in sign.

If the line pair be coincident, then, since $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix}$ vanishes, it follows that $(\Sigma\lambda^2+\phi\lambda+\Sigma') = 0$.

Again, if the line pair be imaginary, and the values of λ be real, then, since $a\lambda+a' = ll'$, $b\lambda+b' = mm'$, ..., l and l', m and m', ... are all pairs of conjugate complex numbers, and therefore $(mn'-m'n)$, $(lm'-l'm)$, ... are all entirely imaginary and of the form ti, where t is real, and therefore $\tfrac{1}{2}\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = t'i$, where t' is real. Its square is therefore negative, and consequently, by (III.), the value of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is essentially positive.

In equation (III.) $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$ is plainly the equation to a vertex of the common self-polar triangle, and the three values of $\sqrt{\{-(A\lambda^2+\ldots)\}}$, $\sqrt{\{-(B\lambda^2+\ldots)\}}$, $\sqrt{\{-(C\lambda^2+\ldots)\}}$ are proportional to the coordinates of its three vertices, and so, if we contemplate the case of $\lambda S+S'$ a parallel pair, and these vertices at infinity, the vanishing of $\begin{vmatrix} \sin A & \sin B & \sin C \\ l & m & n \\ l' & m' & n' \end{vmatrix}$ indicates also the vanishing of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ for these special coordinates, and includes this case in the above.

The nature of the line pair $\lambda S+S'$, therefore, depends solely upon the sign of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$. The cases of four real and four imaginary intersections of two conics, it is well known, have thus much in common, that

D, the discriminant of the cubic $\Delta\lambda^3+\theta\lambda^2+\theta'\lambda+\Delta' = 0$, is positive, and that λ has three real values; if there be two real and two imaginary intersections, D is negative and λ has only one real value.

To distinguish the first two cases, we notice that

(α) For four real intersections, identified geometrically by the existence of three real line pairs $\lambda S+S'$, $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ must by the foregoing have *three real negative values.* (IV.)

(β) For four imaginary intersections, given by

$$\begin{array}{lll}(1) & s+it, & s'+it'; \\ (2) & s-it, & s'-it'; \\ (3) & \sigma+i\tau, & \sigma'+i\tau'; \\ (4) & \sigma-i\tau, & \sigma'-i\tau',\end{array}$$

there is clearly still one real pair of common chords, *i.e.*, the lines joining (1) to (2) and (3) to (4), but the other two pairs are imaginary. The values of λ are all real, and, as $\lambda S+S'$ imaginary implies that $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is positive, it follows that of the three values of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ *two are positive and one is negative.* (V.)

The three values of $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ may now be regarded as the roots of a cubic equation in z, where

$$z = \Sigma\lambda^2+\phi\lambda+\Sigma',$$

subject to $$\Delta\lambda^3+\theta\lambda^2+\theta'\lambda+\Delta' = 0.$$

The elimination of λ yields

$$\begin{vmatrix} \Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta' & \Delta' \\ \Sigma & \phi & \Sigma'-z & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma'-z & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma'-z \end{vmatrix} = 0, \qquad \text{(VI.)}$$

in which we note that

$$z_1z_2z_3 = \Pi(\Sigma\lambda^2+\phi\lambda+\Sigma') = 1/\Delta^2 \begin{vmatrix} \Delta & \theta & \theta' & \Delta' & 0 \\ 0 & \Delta & \theta & \theta' & \Delta' \\ \Sigma & \phi & \Sigma' & 0 & 0 \\ 0 & \Sigma & \phi & \Sigma' & 0 \\ 0 & 0 & \Sigma & \phi & \Sigma' \end{vmatrix} = \delta/\Delta^2,$$

and we propose to show that, in all cases of intersection, the determinant δ is essentially negative in sign.

By (IV.) the case of four real intersections was distinguished by the fact that $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ had three real negative values. The product of its values is therefore negative.

By (V.), four imaginary intersections required that $(\Sigma\lambda^2+\phi\lambda+\Sigma')$ should have one negative and two positive values. The product of the three values is therefore again negative.

In the case of two real and two imaginary intersections the product is also negative; for the cubic in λ has now two imaginary roots, but one real pair of common chords remains, *i.e.*, the line through the two real intersections and the line through the two conjugate imaginary intersections. $(\Sigma\lambda_1^2+\phi\lambda_1+\Sigma')$ is therefore negative for the real pair $\lambda_1 S+S'$. The product of $(\Sigma\lambda_2^2+\phi\lambda_2+\Sigma')$ and $(\Sigma\lambda_3^2+\phi\lambda_3+\Sigma')$, being the sum of two squares, is positive. This is clearly the case, since λ_2 and λ_3 are conjugate complex numbers. It follows that the product $\Pi(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is again negative.

The foregoing having shown that $\Pi-(\Sigma\lambda^2+\phi\lambda+\Sigma')$ is proportional to $\Pi(ux+vy+wz)^2$, *i.e.*, the square of the tangential equation to the vertices $(x_1, y_1, z_1, \ldots)$ of the common self-polar triangle, it is now possible to identify this product $-\delta$ with Γ^2, where Γ is the well known contravariant of the two conics.*

The cubic equation (VI.), when written in full, is

$$z^3+\left\{\frac{\Sigma(-\theta^2+2\theta'\Delta)-3\Delta^2\Sigma'+\theta\Delta\phi}{\Delta^2}\right\}z^2$$

$$+\left\{\frac{\Sigma^2(\theta'^2-2\theta\Delta')+\Sigma'^2 3\Delta^2+\Sigma\Sigma'(2\theta^2-4\theta'\Delta)+\Sigma\phi(3\Delta\Delta'-\theta\theta')-\Sigma'\phi 2\theta\Delta+\phi^2\theta'\Delta}{\Delta^2}\right\}z$$

$$-\left\{\frac{-\Delta\Delta'\phi^3+(\theta\Sigma\Delta'+\theta'\Sigma'\Delta)\phi^2-\{\Sigma^2\Delta'\theta'+\Sigma'^2\Delta\theta+\Sigma\Sigma'(\theta\theta'-3\Delta\Delta')\}\phi+\Sigma^3\Delta'^2+\Sigma'^3\Delta^2-\Sigma^2\Sigma'(2\Delta'\theta-\theta'^2)-\Sigma'^2\Sigma(2\Delta\theta'-\theta^2)}{\Delta^2}\right\}$$

$$= 0, \quad \text{(VI.)}'$$

* The distinctions of the above may be exemplified by reference to the value T of the area of the common self-conjugate triangle, which can be shown to be given by

$$T^2 = -\frac{\mu^4}{4}\frac{D}{\delta} = -\frac{\mu^4}{4}\frac{D}{\Delta^2\Pi(\Sigma\lambda^2+\phi\lambda+\Sigma')},$$

where $\mu = 4T'^2\sin^2 A\sin^2 B\sin^2 C$, T' = the triangle of reference, and Σ, ϕ, and Σ' contain line infinity coordinates.

which we shall call $$z^3+pz^2+qz+r = 0, \qquad \text{(VI.)}''$$

where $r = -\delta/\Delta^2$ has been shown to be *positive* in all cases of intersection.

The distinction between the cases of four real and four imaginary intersections is now apparent.

In the former case the cubic has three real negative roots; in the latter one real negative and two real positive roots.

Therefore, for four real points, p and q must both be positive.
For four imaginary points, p or q at least must be negative. (VII.)

This twofold condition may be embodied formally in one, if we note that, at the minimum point of the z cubic, the value is negative in the former case and positive in the latter, inasmuch as r is positive.

This value of z is the common root of the equations

$$(1) \quad z^3+pz^2+qz+(r-\zeta) = 0,$$

$$(2) \quad 3z^2+2pz+q = 0,$$

for the minimum case. The elimination of z between (1) and (2) gives a quadratic for $(r-\zeta)$ whose greater root must be chosen, and (p^2-3q) is positive by the conditions. The common value of z in (1) and (2) is easily found to be

$$z = \frac{pq-9(r-\zeta)}{-2(p^2-3q)}, \qquad (3)$$

and therefore $27(r-\zeta)^2+(4p^3-18pq)(r-\zeta)+q^2(4q-p^2) = 0$;

so $$9(r-\zeta) = -\frac{(2p^3-9pq)+2(p^2-3q)^{\frac{3}{2}}}{3},$$

and therefore, by (3), $z = \frac{1}{3}[-p+\sqrt{(p^2-3q)}]$.

For four real points D is positive, $+\sqrt{(p^2-3q)}-p$ is negative. For four imaginary points D is positive, $+\sqrt{(p^2-3q)}-p$ is positive, which embodies the previous twofold conditions.

It can be easily shown that the four points of intersection are given tangentially by

$$(\lambda_2-\lambda_3)\sqrt{(\Sigma\lambda_1^2+\phi\lambda+\Sigma')} \pm (\lambda_3-\lambda_1)\sqrt{(\Sigma\lambda_2^2+\phi\lambda+\Sigma')}$$
$$\pm (\lambda_1-\lambda_2)\sqrt{(\Sigma\lambda_3^2+\phi\lambda_3+\Sigma')} = 0,$$

and this suggests the derivation of another cubic in z' whose coefficients will display collateral symmetry in Σ, ϕ, Σ' and the invariants.

If, therefore, the roots of the cubic in z be multiplied respectively by

$\Delta^2(\lambda_2-\lambda_3)^2$, $\Delta^2(\lambda_3-\lambda_1)^2$, and $\Delta^2(\lambda_1-\lambda_2)^2$, we shall arrive at a cubic in z' whose coefficients are the expressions used in Kemmer's conditions.

In this case, to form $z'^3+p'z'^2+q'z'+r'=0$,

we have

$$\Sigma\,\{\Delta^2(\lambda_2-\lambda_3)^2(\Sigma\lambda_1^2+\phi\lambda_1+\Sigma')\}$$
$$=2\Sigma(\theta'^2-3\theta\Delta')+\phi(\theta\theta'-9\Delta\Delta')+2\Sigma'(\theta^2-3\theta'\Delta);$$

and therefore
$$p'=\begin{vmatrix}\Sigma & \phi & \Sigma'\\ 3\Delta & 2\theta & \theta'\\ \theta & 2\theta' & 3\Delta'\end{vmatrix},$$

and similarly

$$q'=\Sigma\,\{\Delta^4(\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)^2(\Sigma\lambda_1^2+\phi\lambda_1+\Sigma')(\Sigma\lambda_2^2+\phi\lambda_2+\Sigma')\}$$
$$=\tfrac{1}{4}\{p'^2-(\phi^2-4\Sigma\Sigma')D\},$$

and
$$r'=rD\Delta^2=-D\delta=-D\begin{vmatrix}\Delta & \theta & \theta' & \Delta' & 0\\ 0 & \Delta & \theta & \theta' & \Delta'\\ \Sigma & \phi & \Sigma' & 0 & 0\\ 0 & \Sigma & \phi & \Sigma' & 0\\ 0 & 0 & \Sigma & \phi & \Sigma'\end{vmatrix},$$

and the z' cubic appears as

$$z'^3+\begin{vmatrix}\Sigma & \phi & \Sigma'\\ 3\Delta & 2\theta & \theta'\\ \theta & 2\theta' & 3\Delta'\end{vmatrix}z'^2+\tfrac{1}{4}\left\{\begin{vmatrix}\Sigma & \phi & \Sigma'\\ 3\Delta & 2\theta & \theta'\\ \theta & 2\theta' & 3\Delta'\end{vmatrix}^2-(\phi^2-4\Sigma\Sigma')D\right\}z'$$
$$+rD\Delta^2=0 \quad \text{(VIII.)}$$

or
$$z'^3+p'z'^2+q'z'+r'=0,$$

and all the former conclusions still hold from the positive nature of the multipliers of the former roots in z.

Criticism of Kemmer's Results.

Kemmer's results are that for four real intersections the following four conditions must be satisfied, viz., D, p', q', and r' must *all* be positive. Now $r'=rD\Delta^2$, and, if D be positive, this asserts that r must also be positive.

But we have shown clearly that in all possible cases of intersection r

is positive and equal to Γ^2/Δ^2; and therefore this fourth condition of Kemmer is superfluous.

The case of four real intersections may now be distinguished with advantage by (1) D positive, (2) $+\sqrt{(p'^2-3q')}-p'$ must be negative.

The Results of Storey.

From the original identities (I.) we can deduce very simply the results which Storey obtained by a different method.

$$\left.\begin{array}{lll}(1) & -(A\lambda^2+K\lambda+A') & = \tfrac{1}{4}(mn'-m'n)^2 \\ (2) & -(B\lambda^2+L\lambda+B') & = \tfrac{1}{4}(nl'-n'l)^2 \\ (3) & -(C\lambda^2+M\lambda+C') & = \tfrac{1}{4}(lm'-l'm)^2 \\ (4) & -(F\lambda^2+K'\lambda+F') & = \tfrac{1}{4}(lm'-l'm)(nl'-n'l) \\ (5) & -(G\lambda^2+L'\lambda+G') & = \tfrac{1}{4}(mn'-m'n)(lm'-l'm) \\ (6) & -(H\lambda^2+M'\lambda+H') & = \tfrac{1}{4}(nl'-n'l)(mn'-m'n)\end{array}\right\} \quad (\text{I.})'$$

If we choose for u, v, w the quantities $(ax+hy+gz)$, $(hx+by+fz)$, and $(gx+fy+cz)$ as multipliers, and multiply (1) to (6) respectively by $(ax+hy+gz)^2$, ..., $2(hx+by+fz)(gx+fy+cz)$, ..., and add once more, we shall find the sums to give, as may be readily verified for the two simplest conics,

$$-\{\Delta S\lambda^2+(\theta S-\Delta S')\lambda+(\theta' S-F)\} = \tfrac{1}{4}\{\Sigma(mn'-m'n)(ax+hy+gz)\}^2, \quad (\text{IX.})$$

where F is the covariant conic of S and S', and the right-hand side now represents the square of the equation of a side of the common self-polar triangle when equated to zero.

The apparent anomaly, common to the right-hand sides of (IX.) and (III.), that, despite the fact of *real* vertices and sides of the self-polar triangle in the case of four imaginary intersections, the square of $\Sigma(mn'-m'n)\,u$ should be negative is explained by the statement that $(mn'-m'n)$, ... are only *proportional* to the values of the coordinates of the vertices, *e.g.*, $(mn'-m'n) = kx_1 i$, where x_1 is real, &c.

The function $-\{\Delta S\lambda^2+(\theta S-\Delta S')\lambda+(\theta' S-F)\}$ now simply replaces the former $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$, and all the conclusions with respect to the sign of the latter apply equally to that of the former.

The most symmetrical form of these new criteria will now be obtained by the simple summations, &c., used for the z cubic, and, just as the sum of three values of $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$ appeared as p, so does the sum of the three values of $-\{\Delta S\lambda^2+(\theta S-\Delta S')\lambda+(\theta' S-F)\}$ appear as Storey's

$$S_1 = (-\theta' S-\theta S'+3F).$$

Similarly the equivalent of q is Storey's

$$S_2 = (\theta\Delta' S^2+\theta'\Delta S'^2+3F^2+(\theta\theta'-3\Delta\Delta')SS'-2\theta' SF-2\theta S' F),$$

and, lastly, the equivalent of r appears as Storey's

$$S_3 = F^3-F^2(\theta S'+\theta' S)+F\{\Delta'\theta S^2+\Delta\theta' S'^2+(\theta\theta'-3\Delta\Delta')SS'\}-\Delta\Delta'^2 S^3$$
$$-\Delta'\Delta^2 S'^3+\Delta'(2\Delta\theta'-\theta^2)S^2S'+\Delta(2\Delta'\theta-\theta'^2)SS'^2.$$

Storey states that for four real intersections D must be positive, $S_2 > 0$, $S_1 S_3 \geqslant 0$. Now S_3, in harmony with the interpretation of (IX.), and on independent grounds, is clearly equal to J^2, when J is the Jacobian of S and S', and is, moreover, the exact equivalent of r, which we showed to be positive in all cases.

The same reasoning applied to S_3 as to r shows that it is also necessarily *positive*. The appearance of J^2 is what we should be led to expect from the reciprocal method implied in the use of the above multipliers $(ax+hy+gz)$, ..., and, just as we showed $r = \Gamma^2/\Delta^2$ to be always positive and Γ always real, so does S_3 remain always positive and J always real. The results of Storey should consequently be modified to read D positive, $S_1 \geqslant 0$, $S_2 > 0$.

Gundelfinger's Results.

By different methods Gundelfinger arrives at the expression $(\Sigma\lambda^2+\phi\lambda+\Sigma')$, and thence deduces Storey's

$$S_1 = (-\theta' S-\theta S'+3F),$$

the equivalent of p in point coordinates. This function he treats as a "combinant" conic ψ, gives it a geometrical meaning, and deduces three conditions for four real intersections from the fact that it must represent an imaginary conic in this case, so as to constantly preserve a positive sign for all values of the variables.

These three conditions, however, contain, in addition to the invariants, the specific constants of the two conics S and S', and do not present the results in invariant contravariant or invariant covariant forms appropriate to the general projective problem.

The following alternative criteria have now been established :—

For four real points

(1) D must be positive ;

(2) $p' = \begin{vmatrix} \Sigma & \phi & \Sigma' \\ 3\Delta & 2\theta & \theta' \\ \theta & 2\theta' & 3\Delta' \end{vmatrix}$ must be positive ;

(3) $q' = \frac{1}{4}\{p'^2 - D(\phi^2 - 4\Sigma\Sigma')\}$ must be positive ;

for all values of the variables.

For four imaginary points (1) D must be positive, (2) p' or q' at least negative. (X.)

Or for four real points

(1) D must be positive,

(2) $S_1 = (-\theta' S - \theta S' + 3F)$ must be positive or zero,

(3) $S_2 = \{\theta\Delta' S^2 + \theta'\Delta S'^2 + 3F^2 + (\theta\theta' - 3\Delta\Delta')\, SS' - 2\theta' SF - 2\theta S'F\}$ must be positive,

for all values of the variables.

For four imaginary points (1) D must be positive, but not at once $S_1 \geqslant 0$, $S_2 > 0$. (XI.)

Both sets of criteria require alternatives for special cases of intersection, as follows :—

In (X.), if $D = 0$, or if there be contact, the other intersections are real or imaginary, according as $p' >$ or < 0. If $D = 0$ and $p' = 0$, q' is also $= 0$, and double contact is easily inferred. The distinction between real and imaginary double contact is given by the sign of p in the z cubic, or more symmetrically thus. The λ cubic has equal roots, one common to $\Sigma\lambda^2 + \phi\lambda + \Sigma' = 0$. Two values of $\Sigma\lambda^2 + \phi\lambda + \Sigma'$ vanish, and the third value $(= -p)$ is easily found to be, since the unequal root of λ is $\Delta'(2\theta^2 - 6\Delta\theta')/\Delta(2\theta'^2 - 6\Delta'\theta)$,

$$-p = \frac{\Sigma\Delta'^2(2\theta^2 - 6\Delta\theta')^2 + \phi\Delta\Delta'(\theta\theta' - 9\Delta\Delta')^2 + \Sigma'\Delta^2(2\theta'^2 - 6\Delta'\theta)^2}{\Delta^2(2\theta'^2 - 6\Delta'\theta)^2}; \quad \text{(XII.)}$$

and therefore the double contact is real or imaginary, according as $p > $ or < 0. The cases of osculation require no criteria.

Similarly in (XI.). For four real intersections S_1 must generally be positive for all values of the variables; but in the case of four-pointed osculation it is easily shown to vanish identically, requiring $S_1 \gg 0$.

If the quantities p' and q' in the z' cubic receive special constant values, *i.e.*, $\sin A$, $\sin B$, $\sin C$ for u, v, w in Σ, ϕ, and Σ', these quantities are invariants for projections, such as the linear Cartesian transformation, in which the line at infinity is unaltered, and the criteria also have a metrical significance exhibited in a footnote.*

In conclusion, the criteria for real and imaginary common tangents may now be developed.

If we reciprocate the original conics S and S' with respect to

$$x^2+y^2+z^2 = 0,$$

and apply the foregoing criteria for real intersections to

$$R = Ax^2+By^2+Cz^2+2Fyz, \ldots,$$
$$R' = A'x^2+B'y^2+\ldots,$$

we shall evidently get the criteria for four real common tangents.

The identities analogous to (I.) now assume the types

$$\left.\begin{aligned} -\{a\Delta\lambda^2+(BC'+B'C-2FF')\lambda+a'\Delta'\} &= \tfrac{1}{4}(mn'-m'n)^2, \ldots \\ -\{f\Delta\lambda^2+(GH'+G'H-AF'-A'F)\lambda+f'\Delta'\} &= \tfrac{1}{4}(lm'-l'm)(nl'-n'l), \ldots \end{aligned}\right\} \quad \text{(XIII.)},$$

and, if we multiply these six identities on both sides by

$$(Au+Hv+Gw)^2, \ \ldots, \qquad 2(Hu+Bv+Fw)(Gu-Fv+Cw), \ \ldots,$$

and add once more, we shall obtain the new equivalent of

$$-\{\Sigma\lambda^2+\phi\lambda+\Sigma'\}$$

as

$$-\{\Delta^2\Sigma\lambda^2+\Delta(\theta'\Sigma-\Delta\Sigma')\lambda+\Delta'(\Sigma\theta-\Delta\phi)\},$$

---

* If Q represent the area of the quadrilateral formed by the four points of intersection, and

$$z' = -\frac{Q^2(\phi^2-4\Sigma\Sigma')^2}{16\mu^4},$$

where

$$\mu^4 = 4T'^2\sin^2 A\sin^2 B\sin^2 C,$$

it can be readily shown that the three values of Q^2 are given by the roots of the z' cubic

$$z'^3+p'z'^2+q'z'+r' = 0,$$

a verification of this equation's validity.

For four real points there are three real values of Q^2 which are positive, viz., the conventional area $(1234)^2$, $(\triangle 123 \sim \triangle 124)^2$ and $(\triangle 124 \sim \triangle 134)^2$.

For four imaginary points there are two negative values of Q^2, and one positive value, corresponding to the one real area which is easily shown to remain for two pairs of conjugate imaginary points. The invariable values of Σ, Σ', and ϕ here employed, with line infinity coordinates used for the variables, subside in the case of Cartesians into the well known invariants $(ab-h^2)$, $(ab'+a'b-2hh')$, and $(a'b'-h'^2)$.

where the new cubic determining $\lambda R+R'$ is

$$\Delta^2\lambda^3+\Delta\theta'\lambda^2+\Delta'\theta\lambda+\Delta'^2 = 0.$$

As before, the product of the three values of this new quantity is positive in all cases and is equal to $r\Delta^4\Delta'^2$, and we must simply express that the sum of its three values is positive and the sum of its product pairs positive for four real common tangents.

The sum $= P = (-\theta\Delta'\Sigma-\theta'\Delta\Sigma'+3\Delta\Delta'\phi)$.

The sum of the product pairs

$$= Q = \Delta\Delta'\{\theta'\Delta'\Sigma^2+\theta\Delta\Sigma'^2+3\Delta\Delta'\phi^2+(\theta\theta'-3\Delta\Delta')\Sigma\Sigma' -2\theta\Delta'\Sigma\phi-2\theta'\Delta\Sigma'\phi\},$$

and the product $= r\Delta^4\Delta'^2 = \Gamma^2\Delta^2\Delta'^2$,

and the new cubic whose roots are the values of

$$-\{\Delta^2\Sigma\lambda^2+\Delta(\theta'\Sigma-\Delta\Sigma')\lambda+\Delta'(\Sigma\theta-\Delta\phi)\}$$

is $$\zeta^3+P\zeta^2+Q\zeta+\Gamma^2\Delta^2\Delta'^2 = 0.$$

The discriminant of the new λ cubic is now $\Delta^2\Delta'^2 D$, where D has its old meaning.

And therefore for four real common tangents (1) D must be positive, (2) P must be positive, (3) Q must be positive.

For four imaginary common tangents (1) D must be positive, and (2) P or Q at least must be negative. (XIV.)

If D be negative, there are two real and two imaginary common tangents.

These tangent conditions can be thrown into the alternative forms involving S, S', and the covariant F, by multiplying the identities of (XIII.) by x^2, y^2, z^2, $2yz$, $2zx$, and $2xy$, and then adding.

The result gives $-(\Delta S\lambda^2+F\lambda+\Delta'S')$ as the analogue of the original $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$.

Since this new quantity stands in relation to the cubic

$$\Delta^2\lambda^3+\Delta\theta'\lambda^2+\Delta'\theta\lambda+\Delta'^2 = 0$$

in precisely the same way as $-(\Sigma\lambda^2+\phi\lambda+\Sigma')$ stood with regard to the original λ cubic, the new results may be at once inferred from (X.) by writing ΔS, F, and $\Delta'S'$ for Σ, ϕ, and Σ', and making corresponding changes in the invariants.

The criteria appear as follows:—

For four real common tangents

(1) D must be positive,

(2) $$S_1 = \Delta\Delta' \begin{vmatrix} S & F & S' \\ 3\Delta & 2\Delta\theta' & \theta \\ \theta' & 2\Delta'\theta & 3\Delta' \end{vmatrix} \text{ must} \geqslant 0,$$

(3) $$S_2 = \tfrac{1}{4}\{S_1^2 - \Delta^2\Delta'^2 D (F^2 - 4\Delta\Delta' SS')\} > 0, \qquad \text{(XV.)}$$

for all real values of the variables.

For four imaginary common tangents (1) D must > 0, (2) not at once $S_1 \geqslant 0$, $S_2 > 0$.

The results of (X.) and (XI.), of (XIV.) and (XV.) constitute a complete solution of the problem proposed. While it is obvious that the results may be expressed in an unlimited number of modes by varying the positive symmetrical multipliers of the roots of the cubics discussed, an effort has been made to present them in their simplest and most symmetrical forms.