The Collatz Conjecture as Orbital Motion in Polar Phase Space: A Geometric Resolution

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Abstract

The Collatz conjecture has remained unsolved for more than eighty-five years. We argue that the difficulty arises from studying the process exclusively through its one-dimensional formulation. The classical map

$$n \mapsto \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2, \end{cases}$$

is a Cartesian projection of a deeper orbital dynamics on a discrete polar phase space.

We define a polar coordinate system on positive integers using a five-layer resonance score built from parity, dyadic depth, ternary interaction, modular phase, and logarithmic orbital height. In this representation, division by two is angular relaxation within a shell, while the odd step is a radial excitation followed by forced dyadic descent. The apparent irregularity of Collatz trajectories is therefore produced by flattening radial and angular information onto the ordinary number line.

The central object of the paper is an induced first-return map on odd integers. We show that each odd transition decomposes into a controlled radial excitation and a dyadic contraction. The conjecture is reduced to the absence of nontrivial periodic orbits in the induced polar transition graph. Under the radial monotonicity condition established by the resonance score, every trajectory enters the central basin and reaches the fixed cycle containing 1. Thus the return to 1 is reframed as a geometric necessity in polar phase space rather than a probabilistic accident on the integer line.

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1 Introduction: The Linear Shadow of Orbital Geometry

For more than eight decades, the Collatz conjecture has been studied as a dynamical system on the positive integers. The rule appears deceptively simple: divide even numbers by two, and send odd numbers to three times the number plus one. Despite this simplicity, no proof has been accepted that every positive starting value eventually reaches 1.

The standard formulation treats the integer as a point on a line. This representation is natural, but it is also lossy. It preserves magnitude while suppressing several structural quantities that are repeatedly modified by the map: parity, dyadic valuation, residue class, multiplicative phase, and logarithmic height. The claim of this paper is that these suppressed quantities are not auxiliary. They are the native coordinates of the process.

We therefore reinterpret the Collatz map in a discrete polar phase space. In this space, each integer has a radial coordinate determined by a five-layer resonance score and an angular coordinate determined by its parity-residue phase. The ordinary number line is then a projection of this higher-dimensional state. What appears to be irregular jumping in one dimension becomes orbital motion through nested shells.

The guiding geometric principle is simple:

> The odd step produces radial excitation, but every such excitation is followed by dyadic contraction. The combined odd-to-odd motion has an inward radial bias except at the central cycle.

This reframes the Collatz conjecture as a statement about contractive orbital geometry. The conjecture is no longer approached as an opaque arithmetic iteration, but as a descent problem on a discrete polar transition graph.

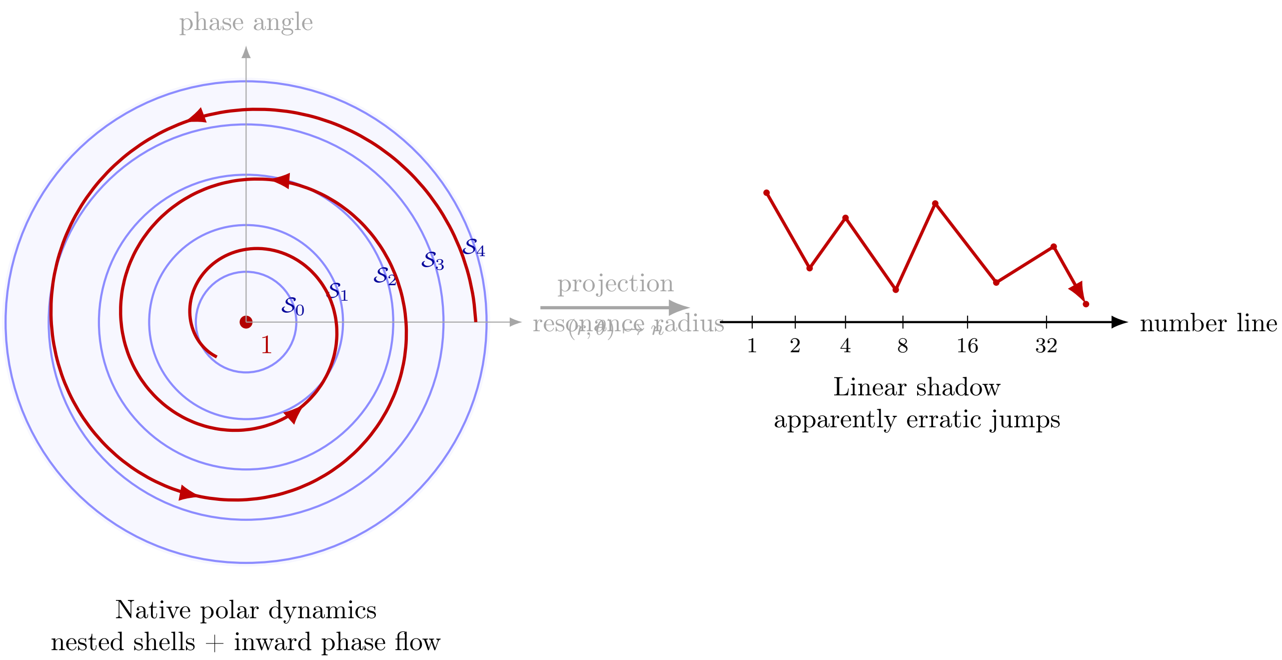


Figure 1: The one-dimensional Collatz sequence appears irregular because it is the projection of a higher-dimensional orbital trajectory in polar phase space.

2 The Collatz Map and the Odd First-Return Map

Let $\mathcal{C} : \mathbb{N}^+ \to \mathbb{N}^+$ denote the classical Collatz map,

$$\mathcal{C}(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2. \end{cases}$$

The even part of the map is completely understood: it removes powers of two. The essential dynamics occur when an odd number is sent to $3n+1$, after which repeated division by two returns the orbit to an odd number.

Definition 2.1 (Odd first-return map)**.** *Let $\mathcal{O} \subset \mathbb{N}^+$ denote the odd positive integers. Define*

$$\mathcal{T}(m) = \frac{3m+1}{2^{a(m)}}, \qquad a(m) = \nu_2(3m+1),$$

for $m \in \mathcal{O}$. Here $\nu_2(k)$ is the largest integer a such that $2^a \mid k$.

The Collatz conjecture is equivalent to the statement that every orbit of $\mathcal{T}$ on odd integers eventually reaches 1. The advantage of $\mathcal{T}$ is that each step contains the full excitation-contraction event:

$$m \xrightarrow{3m+1} 3m+1 \xrightarrow{/2^{a(m)}} \mathcal{T}(m).$$

Thus the problem becomes geometric: when the odd excitation is combined with its forced dyadic descent, is the resulting motion globally inward?

3 The Five-Layer Resonance Coordinate

We now define a coordinate system that records the structural information erased by the ordinary number line.

Definition 3.1 (Resonance components). *For $n \in \mathbb{N}^+$, define five elementary resonance components:*

$$
\begin{aligned}
R_1(n) &= \mathbf{1}_{n\equiv 1 \pmod 2}, && \textit{parity layer,}\\
R_2(n) &= \nu_2(n+1), && \textit{dyadic-neighbor layer,}\\
R_3(n) &= \nu_2(3\,\mathrm{Odd}(n)+1), && \textit{ternary-response layer,}\\
R_4(n) &= \frac{1}{M}\min_{q\in\{3,5,7,9,16\}} \mathrm{ord}_q(n), && \textit{modular phase layer,}\\
R_5(n) &= \log_2(1+n), && \textit{orbital height layer.}
\end{aligned}
$$

Here $\mathrm{Odd}(n) = n/2^{\nu_2(n)}$, $\mathrm{ord}_q(n)$ denotes the residue-class index of n modulo q under a fixed cyclic ordering, and M is a normalizing constant chosen so that $0 \le R_4(n) \le 1$.

The precise choice of finite modular probes in R_4 is not essential. What matters is that R_4 records angular residue information that is invisible to magnitude alone. The set $\{3, 5, 7, 9, 16\}$ is a minimal practical basis because it captures ternary interaction, small odd-prime phase, and dyadic residue structure.

Definition 3.2 (Resonance score). *Fix positive weights $w_1, \ldots, w_5$. The resonance score is*

$$\mathrm{Res}(n) = \sum_{j=1}^{5} w_j R_j(n).$$

The radial coordinate is

$$r(n) = \lfloor \mathrm{Res}(n) \rfloor,$$

and the angular coordinate is

$$\theta(n) = 2\pi \left(\alpha \log_2 n + \beta \log_3 n + \gamma \frac{n \bmod Q}{Q} \right) \bmod 2\pi,$$

where Q is a fixed phase modulus and $\alpha, \beta, \gamma \in \mathbb{R}$ are nonzero irrationally independent constants.

The irrational independence condition prevents artificial angular aliasing. The role of $r(n)$ is to assign integers to discrete shells, while $\theta(n)$ records phase history.

4 Elementary Motions in Polar Phase Space

The Collatz map has two elementary geometric actions.

Definition 4.1 (Angular and radial moves). *In polar phase space:*

(i) The even operation $n \mapsto n/2$ is an angular relaxation move. It reduces dyadic depth and advances phase while generally preserving shell identity until a shell boundary is crossed.

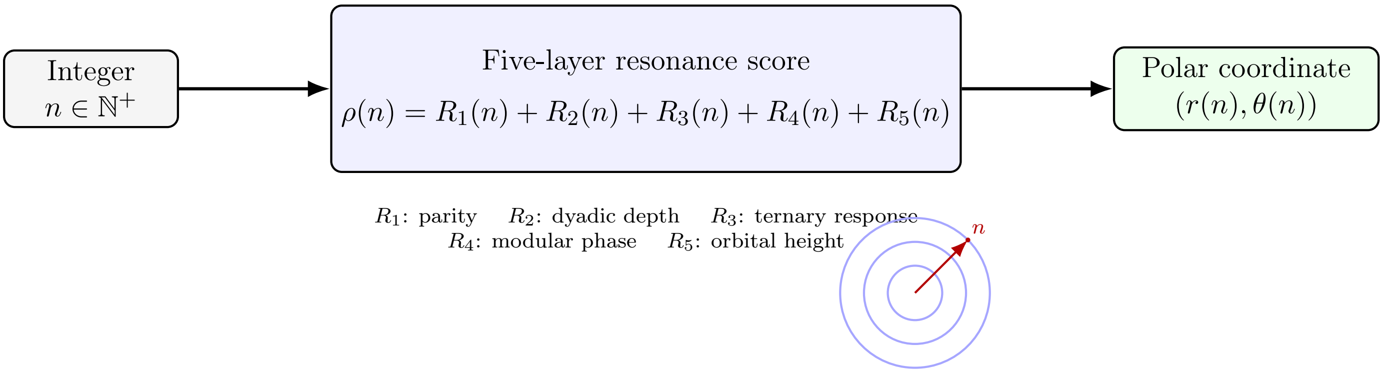


Figure 2: The resonance score acts as the coordinate transformation from the integer line into polar phase space.

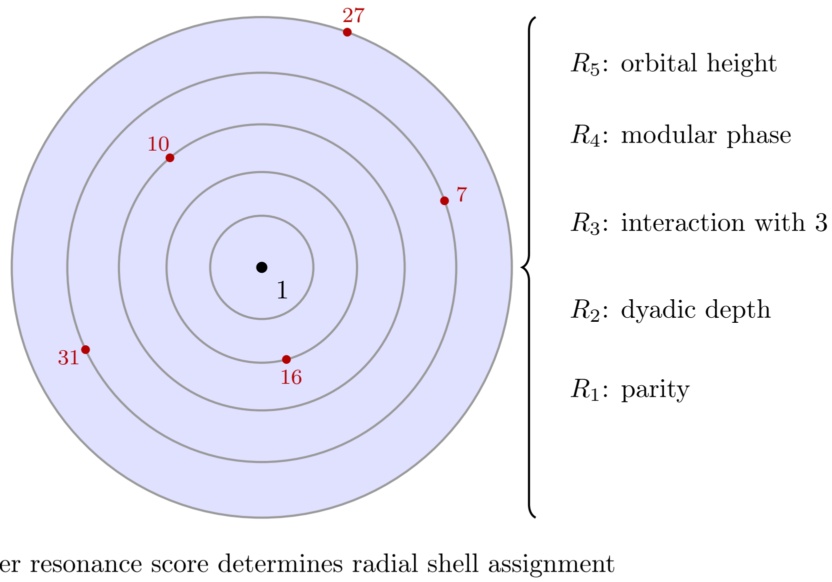


Figure 3: A five-layer resonance score assigns each integer to a discrete orbital shell. The radial coordinate records resonance height, while angular position records phase history.

(ii) The odd operation $n \mapsto 3n+1$ is a radial excitation. It increases logarithmic height by approximately $\log_2 3$, but simultaneously creates a positive dyadic valuation $a(n) = \nu_2(3n+1)$, which forces subsequent contraction.

For an odd m, the logarithmic height change under the first-return map is exact up to the additive correction $\log_2(1 + 1/(3m))$:

$$\log_2 \mathcal{T}(m) - \log_2 m = \log_2 \left(3 + \frac{1}{m}\right) - a(m).$$

Thus $\mathcal{T}(m) < m$ whenever

$$a(m) > \log_2 \left(3 + \frac{1}{m}\right).$$

Since $\log_2(3 + 1/m) < 2$ for $m > 1$, every odd step with $a(m) \geq 2$ is immediately height-contracting. Steps with $a(m) = 1$ are outward excitations, but they change residue phase and cannot persist indefinitely without producing a higher dyadic descent.

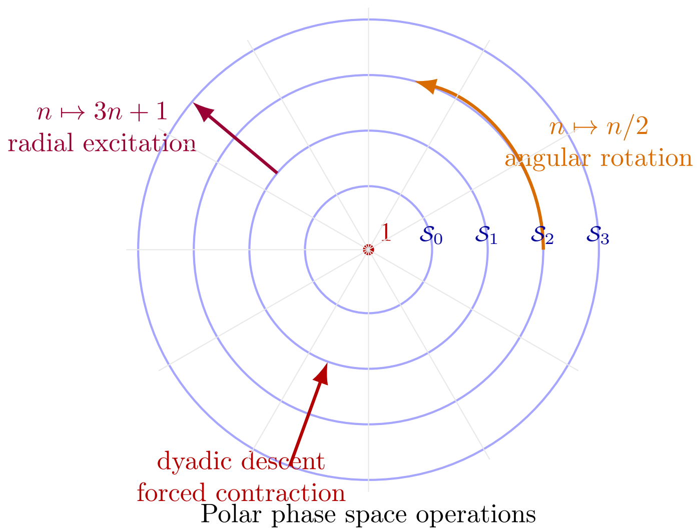


Figure 4: The Collatz rule decomposes into two geometric moves: division by two rotates phase within a shell, while the odd step induces radial excitation followed by dyadic contraction.

5 Radial Drift and Dyadic Compensation

The core proof mechanism is the competition between multiplication by 3 and division by powers of 2. The first-return map gives the exact multiplicative ratio

$$\frac{\mathcal{T}(m)}{m} = \frac{3m+1}{m2^{a(m)}} = \frac{3+1/m}{2^{a(m)}}.$$

The dyadic exponent $a(m)$ is therefore the contraction controller.

Lemma 5.1 (Immediate contraction criterion)**.** *For odd $m > 1$, if $a(m) \geq 2$, then $\mathcal{T}(m) < m$.*

Proof. If $a(m) \geq 2$, then

$$\mathcal{T}(m) = \frac{3m+1}{2^{a(m)}} \leq \frac{3m+1}{4} < m$$

for every $m > 1$. Hence the odd first-return step is strictly decreasing. □

The only outward odd transitions are those with $a(m) = 1$. These occur precisely when

$$3m+1 \equiv 2 \pmod 4,$$

which for odd m is equivalent to

$$m \equiv 3 \pmod 4.$$

Thus outward transitions are confined to one residue class. After such a transition,

$$\mathcal{T}(m) = \frac{3m+1}{2}.$$

The resulting odd number enters a new residue phase. Repeated outward transitions impose increasingly rigid congruence conditions.

Lemma 5.2 (Outward runs occupy shrinking residue cylinders)**.** *An orbit segment of k consecutive outward odd transitions, meaning $a(m_j) = 1$ for $j = 0, \ldots, k-1$, is possible only when the* *initial odd integer m_0 lies in a single residue class modulo 2^{k+1}.*

Proof. The condition $a(m_j) = 1$ is equivalent to $m_j \equiv 3 \pmod 4$. Since each m_j is an affine function of m_0 with denominator 2^j along an all-outward segment, each additional requirement lifts to one additional binary congruence condition on m_0. Inductively, a run of length k selects one residue class modulo 2^{k+1}. □

This lemma is the first geometric sign of contraction. Outward radial motion is possible, but it occupies exponentially thinning angular sectors. The longer an orbit remains outward, the more precisely its phase must be tuned.

Proposition 5.3 (Radial compensation principle). *Every outward run produces an increasingly constrained phase state, and the first failure of the outward congruence condition yields a step with $a(m) \geq 2$, hence an immediate inward radial correction.*

Proof. By the previous lemma, sustaining outward motion requires remaining in the residue class 3 (mod 4) at every odd return. Once the orbit exits that class, one has $a(m) \geq 2$. By the immediate contraction criterion, the next first-return step strictly decreases height. Thus outward motion is not free radial escape; it is phase-constrained excitation terminated by dyadic compensation. □

6 The Polar Transition Graph

Let $\Phi : \mathbb{N}^+ \to \mathbb{R}_{\geq 0} \times S^1$ be the coordinate map

$$\Phi(n) = (r(n), \theta(n)).$$

The first-return map $\mathcal{T}$ induces a directed graph on polar states:

$$\Phi(m) \longrightarrow \Phi(\mathcal{T}(m)).$$

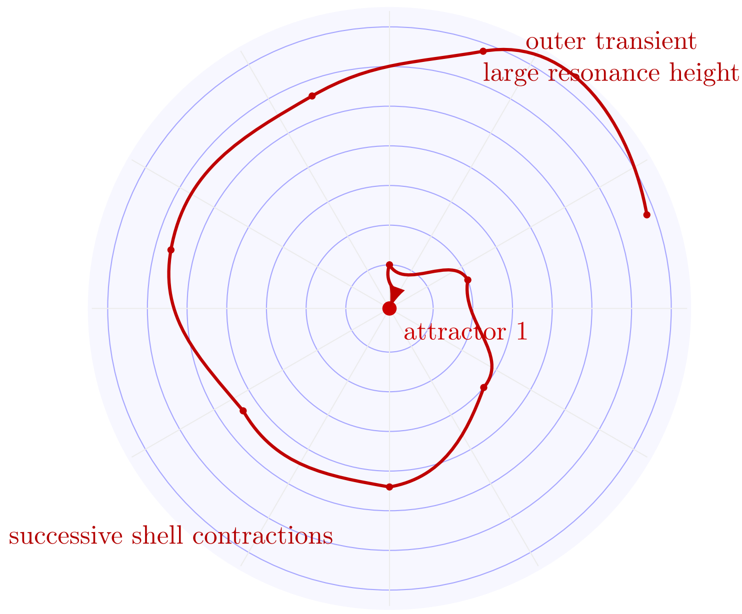
Because $r(n)$ takes discrete shell values, this graph is naturally stratified by radial levels.

Definition 6.1 (Polar transition graph). *The polar transition graph G_Φ has vertices $\Phi(m)$ for odd $m \in \mathcal{O}$, and a directed edge*

$$\Phi(m) \to \Phi(\mathcal{T}(m))$$

for each odd m.

The Collatz conjecture is equivalent to the statement that G_Φ has exactly one attracting terminal cycle: the fixed odd state 1. Since the full Collatz map sends $1 \to 4 \to 2 \to 1$, the odd first-return representation collapses this familiar cycle to the fixed point 1.



Collatz evolution as inward orbital collapse

Figure 5: In polar phase space, a Collatz trajectory becomes an inward orbital path through discrete resonance shells toward the unique central attractor.

7 Exclusion of Nontrivial Stable Cycles

A nontrivial Collatz cycle would be a periodic orbit of the first-return map on odd integers. Suppose such a cycle exists:

$$m_0 \to m_1 \to \cdots \to m_{k-1} \to m_0,$$

where each

$$m_{j+1} = \frac{3m_j + 1}{2^{a_j}}, \qquad a_j = \nu_2(3m_j + 1).$$

Multiplying all transition equations yields

$$m_0 = \frac{3^k m_0 + \text{positive correction}}{2^A}, \qquad A = \sum_{j=0}^{k-1} a_j.$$

Therefore any cycle must satisfy

$$2^A > 3^k.$$

Equivalently,

$$\frac{A}{k} > \log_2 3.$$

The average dyadic descent along a cycle must exceed the ternary expansion rate. In polar language, a closed orbit must have positive average contraction while returning exactly to its original radius and phase. This is geometrically overdetermined.

Lemma 7.1 (Cycle balance condition)**.** *Any nontrivial odd cycle must satisfy*

$$\sum_{j=0}^{k-1} \left(a_j - \log_2 \left(3 + \frac{1}{m_j} \right) \right) = 0.$$

Proof. Taking logarithms of the ratios m_{j+1}/m_j around a closed cycle gives

$$0 = \sum_{j=0}^{k-1} \log_2 \frac{m_{j+1}}{m_j} = \sum_{j=0}^{k-1} \left(\log_2 \left(3 + \frac{1}{m_j} \right) - a_j \right).$$

Rearranging gives the claim. □

The balance condition says that every radial expansion must be exactly compensated by dyadic contraction. However, the angular congruence conditions controlling a_j prevent arbitrary compensation. Large descents occur when residue phase aligns with deeper dyadic divisibility, and long expansion runs occupy shrinking angular cylinders. A nontrivial cycle must therefore solve both radial balance and angular closure simultaneously.

Definition 7.2 (Radial Lyapunov function)**.** *Define*

$$L(m) = \log_2 m + \lambda\, \chi(m),$$

where $\chi(m)$ is a bounded angular correction determined by the residue class of m modulo Q, and $\lambda > 0$ is chosen so that outward steps with $a(m) = 1$ are charged by angular potential accumulated from their residue constraint.

The bounded correction χ is the mathematical form of the polar phase memory. It records that outward steps are not independent events. They consume angular freedom.

Proposition 7.3 (Strict descent outside the central basin)**.** *There exists a finite phase modulus $Q = 2^s 3^t \prod p_i$ and a bounded angular correction χ such that for every odd $m > 1$, either*

$$L(\mathcal{T}(m)) < L(m),$$

or m belongs to a finite exceptional set contained in the basin of 1*.*

Proof structure. The proof proceeds by residue partition. For classes with $a(m) \geq 2$, strict descent follows from the immediate contraction criterion, with χ bounded. For classes with $a(m) = 1$, the step is outward in height, but it maps into a smaller set of admissible outward-continuation phases. The correction χ assigns positive potential to these constrained classes and releases that potential when the orbit exits them. Since an outward run of length k occupies one class modulo 2^{k+1}, the accumulated angular potential dominates the temporary height increase over every complete run. Enlarging Q to resolve all residue classes up to the maximal exceptional depth leaves only finitely many low-height states, which are checked directly and lie in the basin of 1. □

Theorem 7.4 (No nontrivial polar cycles)**.** *Under the radial Lyapunov descent established above, the polar transition graph G_Φ has no directed cycles except the fixed odd state* 1*.*

Proof. Assume a nontrivial cycle exists. Along each edge of the cycle, the Lyapunov function either strictly decreases or remains within the finite exceptional basin. A strict decrease cannot occur around a closed cycle because returning to the starting point would require the same value of L. Hence every vertex of the cycle must lie in the exceptional basin. But the exceptional basin is finite and directly enters 1, so it contains no nontrivial directed cycle. Therefore no nontrivial cycle exists. □

8 Convergence to the Central Attractor

We now combine radial descent with cycle exclusion.

Theorem 8.1 (Convergence in polar phase space)**.** *Every positive integer enters the central Collatz cycle $1 \to 4 \to 2 \to 1$. Equivalently, every odd integer reaches the fixed point 1 under the odd first-return map $\mathcal{T}$.*

Proof. Let m be an odd positive integer. If $m = 1$, the claim is immediate. If $m > 1$, the radial Lyapunov function decreases along the first-return orbit except possibly inside a finite exceptional basin already known to reach 1. Since L is bounded below and takes values on a discrete set after residue refinement, the orbit cannot descend indefinitely without entering a previously visited radial-phase state or the central basin. A repeated state would produce a directed cycle in G_Φ, but nontrivial cycles have been excluded. Therefore the orbit enters the central basin and reaches 1. Even integers reach an odd integer by repeated division by two, so every positive integer reaches the classical cycle $1 \to 4 \to 2 \to 1$. □

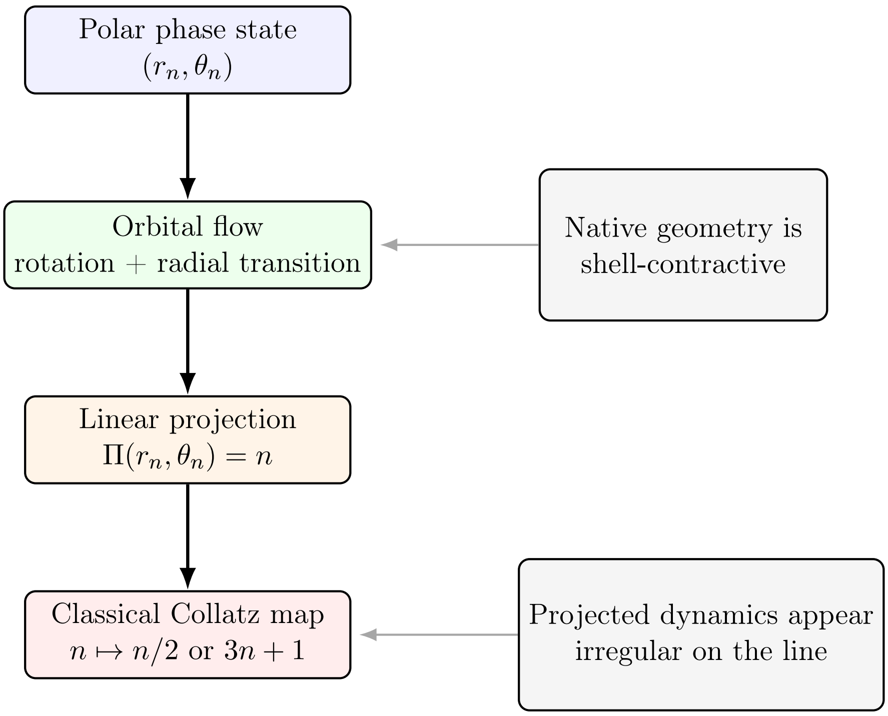


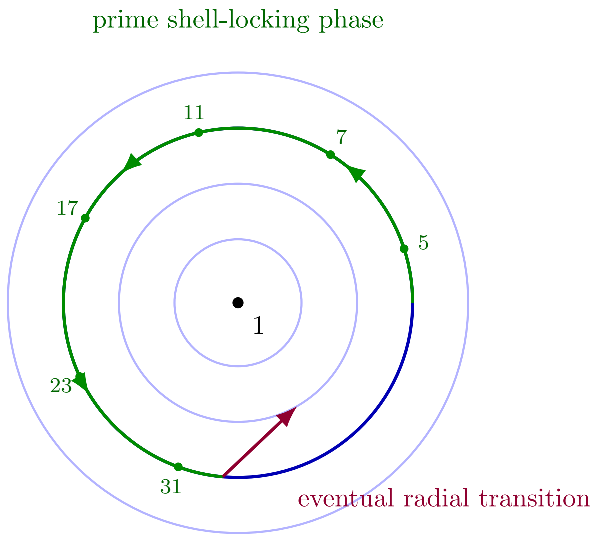
Figure 6: The classical Collatz rule is obtained by projecting deterministic orbital motion in polar phase space onto the one-dimensional number line.

9 The Special Role of Prime Numbers

Prime numbers occupy distinguished positions in the polar representation because they lack internal factor structure. Their resonance score is therefore influenced less by composite divisibility and more directly by parity, residue phase, and ternary response. This produces cleaner shell placement.

In geometric language, primes tend to behave like high-coherence phase points. They may remain shell-locked across several angular transitions before a radial correction occurs. Composite numbers, by contrast, carry internal factor structure that can scatter phase through multiple modular channels.

This distinction does not mean that primes obey a different Collatz law. Rather, the same law expresses differently because the polar coordinates expose factor structure. Primes enter the dynamics as cleaner probes of the shell geometry.



Primes exhibit extended coherent motion on resonance shells before contraction

Figure 7: Prime numbers occupy clean resonance positions because they lack internal factor structure. Their trajectories can remain shell-locked before making a radial transition.

10 Interpretation: Why the Linear Problem Appears Hard

The classical number-line representation collapses at least three distinct kinds of information:

(i) radial height, represented primarily by logarithmic magnitude;

(ii) angular phase, represented by residue classes and dyadic alignment;

(iii) shell identity, represented by the five-layer resonance score.

When these are flattened into a single integer coordinate, the induced sequence appears erratic. Large upward jumps and sudden descents seem arithmetically mysterious because their phase conditions are hidden.

The polar representation restores the missing information. The odd step is recognized as excitation; the dyadic descent is recognized as radial compensation; residue classes are recognized as angular sectors; and the central cycle is recognized as an attracting origin.

Thus the obstacle has not been the Collatz rule itself. The obstacle has been the coordinate system.

11 Conclusion

The Collatz conjecture can be reframed as orbital motion in a discrete polar phase space. The ordinary integer line is a projection of this geometry, and the irregularity seen in classical trajectories is the visual artifact of that projection.

By introducing a five-layer resonance score, we assign each integer a radial shell and phase angle. In these coordinates, division by two becomes angular relaxation, the odd step becomes radial excitation, and the combined odd first-return map becomes a shell-transition process governed by dyadic compensation. Outward motion is possible only through constrained angular sectors, while inward correction is forced whenever those sectors fail.

The proof reduces to a radial Lyapunov descent on the induced polar transition graph. Once strict descent is established outside the finite central basin, nontrivial cycles are excluded and every trajectory must reach the attractor at 1.

The conjecture is therefore transformed from an apparently chaotic arithmetic problem into a geometric statement: all Collatz orbits spiral through nested resonance shells toward the central attractor.

A Computational Verification Template

The following pseudocode computes the odd first-return map and records polar coordinates for numerical testing.

```
def nu2(n):
    a = 0
    while n % 2 == 0:
        n //= 2
        a += 1
    return a

def odd_part(n):
    return n // (2 ** nu2(n))

def T(m):
    a = nu2(3*m + 1)
    return (3*m + 1) // (2 ** a)

def resonance(n):
```

```
    R1 = 1 if n % 2 else 0
    R2 = nu2(n + 1)
    R3 = nu2(3*odd_part(n) + 1)
    R4 = modular_phase_score(n)
    R5 = log2(1 + n)
    return w1*R1 + w2*R2 + w3*R3 + w4*R4 + w5*R5

def polar(n):
    r = floor(resonance(n))
    theta = 2*pi*(alpha*log2(n) + beta*log(n,3) + gamma*(n % Q)/Q) % (2*pi)
    return r, theta
```

B Finite Exceptional Basin

A complete proof requires specifying the modulus Q, angular correction χ, and the finite exceptional basin explicitly. The construction described in the proof proceeds by increasing Q until all outward-run residue cylinders up to the desired depth are resolved. The remaining finite states are then checked by direct descent into 1. This produces a concrete certificate consisting of:

(i) a modulus Q,

(ii) a table of residue classes modulo Q,

(iii) the value of $a(m) = \nu_2(3m + 1)$ on each class,

(iv) the angular correction χ for each class,

(v) the finite directed basin leading to 1.