

AN ANALYSIS OF AN EXPERIMENT IN TEACHING
FIRST YEAR MATHEMATICS.¹

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For several years the writer has been interested in the work of first-year mathematics—the problem of what to teach and how to teach it. With this problem in mind she has taught during the past two years seven classes in college entrance algebra. It is to set forth the salient points in the presentation of the subject, and in the results obtained in what she believes to have been a very satisfactory experiment, that this paper is written.

The text used as a basis for this work was *Fundamentals of High School Mathematics*, by Rugg and Clark, a text of the “unified” type, though conservative, but constructed upon the conviction that ninth grade mathematics needs a complete reconstruction rather than a reorganization. The outstanding features of the course are: (1) Selection of subject matter on the criteria of “Social Worth” and “Thinking” outcomes; (2) The unique arrangement of the material. The first semester algebra develops symbolism and the equation. A very rational presentation of signed numbers and a new method of teaching factoring, constitute the chief features of the second semester algebra. The text was supplemented in the second semester by material chosen from one of the conventional texts.

The primary aim in teaching mathematics is not so much to impart facts as to develop a mode of thought; namely, a problem-solving or scientific attitude of mind, the ability to analyze and see relationships in new situations, to express these relationships, and to determine them. This requires clear ideas and sound associational processes and cannot be developed without much practice in meeting real problem situations. For this reason the central idea throughout the course was that of problem solving.

The underlying meanings of the course in algebra may be classified under two headings; namely, the narrower associations of ideas that have to do with developing the various phases of the subject, and the broader basic concepts of the equation and functionality. The mental processes in these two classes are the same; namely, the association of ideas in past experiences with the new ideas presented by the new technical vocabulary of the subject.

¹An abstract of a discussion given before the Kansas Association of Mathematics Teachers.

Arithmetic deals with quantitative ideas expressed by words or by numbers. Numbers are symbols representing a high degree of abstraction. Algebra, on the other hand, deals with the devices of manipulating these quantitative ideas through the use of letter symbols for numbers. This is a much higher degree of abstraction. We must agree, however, that the mastery of this symbolism plays a very important part in the mastery of the subject.

The first problem involved in learning algebra is, then, the development of its symbolism, how and why to represent numbers by the use of symbols. A student is likely to get on very much better in his use of symbols and in his appreciation of their meaning if the economy of time and of mental energy obtained by their use is stressed. Hence the meaning of symbolism is best fixed by associating with symbols the idea of abbreviation, of "representing" or "standing for."

A clear grasp of the significance of the use of letters to represent numbers can come only by the most gradual transition from the use of words to the use of letters as symbols for numbers. Letters are highly abstract in that they represent no particular content, and the pupil must bridge this gap between his past experiences and the new situation expressed by this abstract symbolism. If he is carried forward too rapidly in the course, or if too many things are attempted, there is introduced an element of guess work and confusion which explains many of the failures in this subject.

Association is the basis of habit formation, and the whole process of learning is a process of establishing a system of habits such that a correct response comes with a given situation. The law of habit formation is repetition, hence the pupil needs not only gradual transition from word symbols to letter symbols, but also much repetition. Therefore the inclusion, as in one traditional text, of " $3x+5x = \text{how many } x?$ " as the fifth problem proposed for solution is unpsychological and unpedagogical. The pupil can make neither the association with previous ideas as $3 \text{ bu.} + 5 \text{ bu.} = \text{how many bu.}?$ nor can he make the transition without much repetition of such material with the intervening step, $3b+5b = \text{how many } b?$ Initial letters help to bridge the gap, since the degree of abstraction seems less than with the use of x or y .

As a means of clarifying algebraic symbolism, perhaps the simplest is the construction and evaluation of simple formulas

from arithmetic and geometry such as those for percentage and its applications, for perimeters of rectangles, for areas of rectangles and triangles, for volumes, etc. Problems from denominate numbers suggest a never ending source of supply. "How many eggs are 6 doz. eggs and 2 doz. eggs?" "Using d for 12, how many eggs in $2d+3d$ eggs?" "Change $7y$ (yards) $+6f$ (feet) to smaller units, that is, to inches." "If $y = 3f$, how many f in $4y+6y-3y$?"

Translation is a vital step in the comprehension of algebraic symbolism and of mathematics as a language. Its outstanding phases are recognition of equality between two quantities and ability to express this equality by means of symbols. As a second step in clarifying the idea of symbolism, much translation from algebraic statements into verbal language as well as the reverse is invaluable. Illustrative examples of each:

I. $n+4 = 13$ is the same as the word statement, "The sum of a certain number and 4 is 13;" $4y = 26$ should be translated "The product of a certain number and 4 is 26," or "Four times a certain number equals 26."

II. (a) The "word" method of stating the example. (a) There is a certain number such that if you add 5 to it the result will be 18. What is the number?

(b) An abbreviated way to write it. (b) What number plus 5 equals 18?

(c) A more abbreviated way to write it. (c) $No.+5 = 18$

(d) The best way to write it. (d) $n+5 = 18$

The functional relation is fundamental in mathematics and is the central organizing principle of algebra. Its method is essentially that of the equation. Although this is conceded by mathematicians, the traditional text does not make it so. In the supplementary text referred to, for example, only 16 per cent of the total problem content of the book is verbal and equational work, while 84 per cent is consumed in formal exercises for the acquiring of manipulatory skill, whereas substantially the reverse is true of the text experimented with. It might be noted here that psychological analyses show, and experience bears out the statement, that skill in the manipulation of an isolated operation does not function effectively when the operation is associated with other operations in new situations. Application is an explicit phase and practice must be given in the

application of the operation to prevent its degenerating into mere rote memory. This neglect to make application of a skill learned, again explains many failures in algebra.

In developing the concept of the equation as representing a balance of values, specific comparison was made with the weighing scale. The meaning of balance is thus fixed and the idea is developed that whatever is done to one member of the equation must be done to the other, to preserve the balance.

The fundamental notion concerning two quantities which "change together" and with it the idea of the "constant" and "variable" are developed in the text used by three methods—the tabular, the graphic, and the equational. The association necessary for connecting the values in the three methods so as to fix the idea that the table, the graph, and the equation set forth the same truths should be clearly brought about. Meanings are fixed only through many responses to situations in which they occur, hence the value of tabulating, graphing and symbolizing by equations the relationships which exist between pairs of variables to give the pupil a grasp of the meaning of functionality. The use of similar triangles and of trigonometric ratios (\tan and \cos) in problems of finding unknown distances to give meaning to ratio and proportion sets up an association with real experiences and takes the topic out of the realm of the abstract into that of the concrete—the concrete including at any stage all the abstractions previously made and assimilated.

The facts in connection with understanding the step-by-step process by which the child-mind learns would, it seems, convince the thoughtful teacher that first semester algebra is not the psychological place to introduce the subject of signed numbers and their use. The pupil has quite enough to do to conquer the high degree of abstraction involved in literal symbolism and the basic idea of the equation. He cannot take ideas so rapidly and assimilate them; and, moreover, he does not need them. He has at his disposal an amount of intuitive knowledge which will enable him to solve equations of the simpler type as he gradually formulates his axioms. Signed numbers were therefore taken up as the opening topic in second semester algebra. To develop the broad concept of "oppositeness" with the use of positive and negative numbers, specific associations must be made: associating the idea with above and below zero on the temperature scale; with assets and liabilities; with before and after on the time scale;

etc.; the central aim being to fix the meaning of sense in numbers, that positive and negative express oppositeness. It should be clear that the same point may lie in the positive or the negative direction from the origin from which the distances are measured; i. e., the zero point.

A rational presentation of the principles governing the use of signed numbers in the four fundamental operations requires specific association of ideas.

1. Associating with readings of the thermometer at different times to introduce addition; for example:

The top of the mercury column of a thermometer stands at zero degrees (0°). During the next hour it rises 3° , and the next 4° . What is the temperature at the end of the second hour?

If it starts at 0° , rises 3° , then falls 4° , what is the reading?

2. Associating with saving or losing a given amount of money for a given time, introducing multiplication:

If you save \$5.00 a month ($+\5), how much better off will you be six months from now ($+6$)? Evidently you will be \$30 better off ($+\30). Thus $+5$ times $+6 = +30$.

If you are wasting \$5.00 a month ($-\5.00), how much better off will you be in 6 months from now ($+6$)? Evidently you will be \$30 worse off ($-\30). Thus -5 times $+6 = -30$.

3. Associating with the making of change or with readings on a thermometer at different places, introducing subtraction:

"If a customer gives the clerk 50 cents in payment for a 27 cent purchase, the clerk begins at 27 cents and counts out enough money to make 50 cents." The clerk begins at the subtrahend, 27 cents, and counts to the minuend, 50 cents.

On a certain day the mercury stands at -4° in Chicago and at $+13^{\circ}$ in St. Louis. How much warmer is it in St. Louis, or what is the difference between $+13^{\circ}$ and -4° ? Naturally, we do the same thing the clerk does, begin at the subtrahend and count to the minuend; i. e., we count from -4° to $+13^{\circ}$, giving us $+17^{\circ}$. The difference is called positive because we counted upward. If we counted downward, the difference would be called negative.

The same reasoning may then be applied to the abstract number scale finally dropping even this. Division is taught as the opposite of multiplication: $+8 \div -2 = -4$, because $(-2)(-4) = +8$, etc.

Perhaps one of the greatest stumbling blocks in beginning algebra is the subject of factoring. It has been estimated that

39% of all recurring errors made by pupils indicate positive inability in particular types of factoring. The traditional presentation of the "57 varieties" or cases of "Special Products and Factoring" results in much loss of valuable time, largely because they are so arranged that the learning of one actually inhibits the learning of the others. The reason for this is that the pupils have acquired skill in the several "cases" which they cannot generalize. An analysis of the learning process shows that if such a generalization could be made in the initial presentation of the topic, much of the difficulty would be removed. As a matter of fact the general quadratic trinomial, ax^2+bx+c , will handle all of the "cases" except that of "common factor," which can be taught in connection with multiplication and division, and $a^3\pm b^3$ which may be omitted until third semester algebra.

Visual imagery is the predominating feature in the handling of this subject, hence to present many times to the pupil the

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product of two general binomials $(2x+3)(3x+4) = ?$ thus stressing the importance of the middle term greatly facilitates his grasp of the meaning of the process of factoring ax^2+bx+c . Expansion of such products may be motivated by using it as a tool to find areas of rectangles. In all work the pupil should be required to translate his operations into words. Words are the instruments of abstraction and they aid the pupil to analyze the situation. Experience will soon lead him to make classifications for himself; for example, when the binomials are alike he has a perfect square, the bx term being twice the product of the given terms, a fact of importance to him when teaching him to solve the quadratic equation by "completing the square"—or, again, if the binomials are alike except for sign, the bx term is zero. He will observe the converse statements for himself. As soon as the pupil's response to the situation $(\quad)(\quad) = ?$ becomes automatic he should reverse the operation and factor, and again appeal to visual imagery as suggested by such a form

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as $15x^2-14x-8 = (\quad)(\quad)$. Such a method is well adapted to bring about "logical thinking," and moreover has the added advantage of saving about three-fourths of the time usually allotted to this subject.

It might be remarked here that analysis of the learning processes shows that the most economical place to teach an opera-

tion is in connection with its application. So here, after the five or six recitations necessary for the presentation, factoring is best taught in the solution of equations and in fractions.

Such a procedure as the foregoing does not build any habits which inhibit any other learning and gives the pupil a power of generalization which develops a grasp of the larger aspects of his subject, a thing to be cultivated in every situation with which he is confronted.

Of the seven classes upon which this text was tried out, four were first semester algebra, with a total enrollment of seventy-two students and three were second semester algebra with an enrollment of forty-six students. The students were older than ordinary ninth grade students, the average age being about twenty years, and most of them had not had preparation beyond the fifth grade. About one-third were ex-soldiers receiving Federal Aid for Vocational Training. Of the entire one hundred eighteen students, only five have thus far entered a curriculum requiring mathematics beyond the two units required for entrance, and it is not likely that more than a half dozen more will do so. Of the five mentioned, three took third semester algebra and entered the three hour college algebra course given for students offering one and one-half years of algebra for entrance, while two passed from the second semester algebra into the five hour college algebra course given for students offering only one year of algebra for entrance. All passed their college algebra, three with grades above average. It is worthy of note that of these three two were in the five hour course.

It is the belief of the writer that the very gradual exposition of the text, i. e., keeping "the content of the course just a step in advance of the developing content of the student's mind," in the first semester algebra, gave the classes the momentum which enabled them to begin second semester algebra with the operations upon signed numbers and yet complete the usual work of first year algebra in the allotted time. This she attributes largely to the accumulation of reasoning ability. It was her experience also that the procedure herein set forth not only saved time but that the better methods of presentation gave the students a more complete grasp and insight into the fundamental notions and devices of the subject than had been the result to those receiving the traditional presentation. The proportion of failures in these classes was from 5 per cent to 10 per cent lower than in classes given instruction in conventional texts.

The special aim in first year algebra should be "to give the pupil the ability to use the tools of quantitative thinking—namely, the equation, the formula, and the graph." The experience with these students in college algebra would certainly seem to show that emphasis on problem solving nevertheless tends to develop greater skill in the manipulation of these tools than does emphasis on the side of manipulation alone.

Furthermore, the "social worth" of the material presented was far greater for the one hundred or more students who did not take further mathematics than that of the usual traditional text, and this belief was confirmed by the more mature members of the classes. They felt they "had something practical,—something we can use."

AVERAGE ALTITUDE OF PENNSYLVANIA IS 1,100 FEET.

The approximate mean elevation of Pennsylvania is 1,100 feet, according to the United States Geological Survey, Department of the Interior. The highest point now known is Negro Mountain, in Somerset County, which is 3,220 feet above the sea level. Until 1919, when the topographic survey of the area including Negro Mountain was made, Blue Mountain in Bedford County (elevation 3,136 feet) was thought to be the highest point in the State. It is barely possible, as the topographic surveys are extended, that still higher points may be found.

The surface of Delaware River where it leaves Pennsylvania is at sea level and is the lowest point in the State.

PETROLEUM IN ALASKA.

Petroleum was one of the first useful minerals found in Alaska, but the earliest attempts at its systematic development in the Territory were confined to a very brief oil boom that began in 1901 but that soon collapsed, owing to the rapid development of oil in California. All the oil lands in Alaska were withdrawn from entry in 1910, and patent has been granted to only one claim, which is in the Katalla field, where a few productive oil wells have been drilled. In spite of the small developments, however, Alaska has produced about 56,000 barrels of petroleum, all of it taken from the Katalla field. This oil has been consumed locally, most of it by a small refinery near Katalla.

The passage of the oil and gas leasing act of February 25, 1920, started small stampedes to all accessible places where oil seepages were known and led to the staking of many claims, some of them in areas where no indications of oil have been found. Up to September, 1920, the Juneau land office had received 178 applications for oil-leasing permits, covering in all 388,673 acres of land, which by no means includes all the land staked, most of which will no doubt be found worthless as oil land. Systematic drilling for oil will probably be begun in Alaska this year, and oil fields will no doubt be developed in Alaska, but the geology of the Territory, so far as known, does not indicate that any startling discoveries will be made.

Investigations of oil in Alaska were made in 1903 by the United States Geological Survey, Department of the Interior, which has since then from time to time devoted considerable attention to this subject. The Survey has just published, as Bulletin 719, a report entitled "Petroleum in Alaska," by George C. Martin.