IX.—The Confluent Hypergeometric Functions of Two Variables.

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INTRODUCTION.

THIS memoir is devoted to the study of certain new functions, which may be regarded as limiting cases of the "hypergeometric functions of two variables" discovered by Appell.* The relation which the new functions bear to Appell's functions is, in fact, analogous to that which the "confluent hypergeometric functions,"†

$$\Phi(\alpha, \gamma, x) = 1 + \frac{\alpha x}{1 \cdot \gamma} + \frac{\alpha(\alpha+1)}{2 ! \, \gamma(\gamma+1)} x^2 + \ldots$$

and

$$\mathrm{B}(\gamma, x) = 1 + \frac{x}{1 \cdot \gamma} + \frac{x^2}{2 ! \, \gamma(\gamma+1)} + \ldots,$$

bear to the ordinary hypergeometric function.

There are four hypergeometric series of two variables. If we denote the product

$$\lambda(\lambda+1) \ldots (\lambda+n-1),$$

where λ is arbitrary and n a positive integer, by the symbol

$$(\lambda, n),$$

with the convention $(\lambda, 0) = 1$, these functions are as follows:—

$$\mathrm{F}_1(\alpha ; \beta, \beta' ; \gamma ; x, y) = \sum_{m=0}^{m=+\infty} \sum_{n=0}^{n=+\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)} \frac{x^m y^n}{m ! \, n !}$$

$$\mathrm{F}_2(\alpha ; \beta, \beta' ; \gamma, \gamma' ; x, y) = \sum \sum \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m)(\gamma', n)} \frac{x^m y^n}{m ! \, n !}$$

$$\mathrm{F}_3(\alpha, \alpha', \beta, \beta' ; \gamma ; x, y) = \sum \sum \frac{(\alpha, m)(\alpha', n)(\beta, m)(\beta', n)}{(\gamma, m+n)} \frac{x^m y^n}{m ! \, n !}$$

$$\mathrm{F}_4(\alpha, \beta ; \gamma, \gamma' ; x, y) = \sum \sum \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m)(\gamma', n)} \frac{x^m y^n}{m ! \, n !}.$$

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* *J. math. pures appl.*, 1882, p. 173 ; 1884, p. 407.

† For an account of the confluent hypergeometric functions, see chapter xvi of Whittaker and Watson's *Modern Analysis*.

both β and β' tend to infinity. These two functions will be denoted by the symbol Ψ: their expressions are

$$\Psi_1(\alpha\,;\ \beta\,;\ \gamma,\gamma'\,;\ x,y)=\sum\sum\frac{(\alpha,m+n)(\beta,m)}{(\gamma,m)(\gamma',n)}\frac{x^m y^n}{m!\,n!}$$

$$\Psi_2(\alpha\,;\ \gamma,\gamma'\,;\ x,y)=\sum\sum\frac{(\alpha,m+n)}{(\gamma,m)(\gamma',n)}\frac{x^m y^n}{m!\,n!}$$

From the function

$$F_3(\alpha,\alpha',\beta,\beta'\,;\ \gamma\,;\ x,y)=\sum\sum\frac{(\alpha,m)(\alpha',n)(\beta,m)(\beta',n)}{(\gamma,m+n)}\frac{x^m y^n}{m!\,n!}$$

may be obtained in like manner a new function by dividing y by β', and making $\beta'\to\infty$. This is the function

$$\Xi_1(\alpha,\alpha'\,;\ \beta\,;\ \gamma\,;\ x,y)=\sum\sum\frac{(\alpha,m)(\alpha',n)(\beta,m)}{(\gamma,m+n)}\frac{x^m y^n}{m!\,n!}$$

Similarly, replacing y by $y/\alpha'\beta'$, and making α' and β' infinite, we obtain a function

$$\Xi_2(\alpha,\beta\,;\ \gamma\,;\ x,y)=\sum\sum\frac{(\alpha,m)(\beta,m)}{(\gamma,m+n)}\frac{x^m y^m}{m!\,n!}$$

Chapter II.

VARIOUS EXPANSIONS FOR THE FUNCTIONS; RELATIONS BETWEEN THEM.

The seven confluent functions which we have introduced, and defined by double power-series, may also be represented by simple power-series in x, or in y, by performing the process of confluence on the similar expressions given by Appell for the four F functions. We thus find

$$\Phi_1(\alpha\,;\ \beta\,;\ \gamma\,;\ x,y)=\sum_{m=0}^{\infty}\frac{(\alpha,m)(\beta,m)}{(\gamma,m)}\Phi(\alpha+m,\gamma+m,y)\frac{x^m}{m!}$$
$$=\sum_{m=0}^{\infty}\frac{(\alpha,m)}{(\gamma,m)}F(\alpha+m,\beta\,;\ \gamma+m\,;\ x)\frac{y^m}{m!}$$

and similar formulæ for the other confluent functions.

We shall next consider formulæ derived from the definite-integral values of the F functions, such as

$$F_2=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}\int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}du\,dv.$$

We have

$$(1-ux-vy)^{-\alpha}=(-1)^{-\alpha}[1-(1-ux)-(1-vy)]^{-\alpha}$$
$$=(-1)^{-\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,m+n)}{m!\,n!}(1-ux)^m(1-vy)^n$$

so

$$F_2=\frac{(-1)^{-a}\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}\sum\sum\frac{(a,\ m+n)}{m\,!\,n\,!}\int^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^m du$$
$$\int_0^1 v^{\beta'-1}(1-v)^{\gamma'-\beta'-1}(1-vy)^n dv$$
$$=(-1)^{-a}\sum\sum\frac{(a,\ m+n)}{m\,!\,n\,!}F(-m,\ \beta\ ;\ \gamma\ ;\ x)F(-n,\ \beta'\ ;\ \gamma'\ ;\ y),$$

a formula to which we can apply the ordinary process of confluence, obtaining the interesting expansions for the Ψ functions:

$$\Psi_1(a,\ \beta\ ;\ \gamma,\ \gamma'\ ;\ x,\ y)=(-1)^{-a}\sum\sum\frac{(a,\ m+n)}{m\,!\,n\,!}F(-m,\ \beta\ ;\ \gamma\ ;\ x)\Phi(-n,\ \gamma',\ y)$$
$$\Psi_2(a\ ;\ \gamma,\ \gamma'\ ;\ x,\ y)=(-1)^{-a}\sum\sum\frac{(a,\ m+n)}{m\,!\,n\,!}\Phi(-m,\ \gamma,\ x)\Phi(-n,\ \gamma',\ y).$$

By reasoning of a similar type we find

$$F_1(a\ ;\ \beta,\ \beta'\ ;\ \gamma\ ;\ x,\ y)=\sum(-1)^m\frac{(a,\ m)(\beta,\ m)(\beta',\ m)}{(\gamma-\beta',\ m)(\gamma,\ m)}\frac{x^m}{m\,!}F_2(a+m\ ;\ \beta+m\ ;\ \beta'+m\ ;$$
$$\gamma-\beta'+m,\ \gamma+m\ ;\ x,\ y)$$

or

$$F_1=\sum(-1)^m\frac{(a,\ m)(\beta,\ m)(\beta',\ m)}{(\gamma-\beta,\ m)(\gamma,\ m)}\frac{y^m}{m\,!}F_2(a+m\ ;\ \beta+m,\ \beta'+m\ ;\ \gamma+m,\ \gamma-\beta+m\ ;\ x,\ y).$$

From these we obtain by confluence

$$\Phi_1(a\ ;\ \beta\ ;\ \gamma\ ;\ x,\ y)=\sum(-1)^m\frac{(a,\ m)(\beta,\ m)}{(\gamma-\beta,\ m)(\gamma,\ m)}\frac{y^m}{m\,!}\Psi_1(a+m,\ \beta+m\ ;\ \gamma+m,\ \gamma-\beta+m;\ x,\ y)$$
$$\Phi_3(\beta,\ \gamma\ ;\ x,\ y)=\sum(-1)^m\frac{(\beta,\ m)}{(\gamma-\beta,\ m)(\gamma,\ m)}\frac{y^m}{m\,!}\Phi(\beta+m,\ \gamma+m,\ x)B(\gamma-\beta+m,\ y),$$

which may be transformed into

$$\Phi_3(\beta,\ \gamma\ ;\ x,\ y)=e^x\sum(-1)^m\frac{(\beta,\ m)}{(\gamma-\beta,\ m)(\gamma,\ m)}\frac{y^m}{m\,!}\Phi(\gamma-a,\ \gamma+m,\ -x)B(\gamma-\beta+m,\ y)\ ;$$

lastly, from F_3 we obtain in a similar way

$$\Xi_1(a,\ a'\ ;\ \beta\ ;\ \gamma\ ;\ x,\ y)=\sum(-1)^m\frac{(a,\ m)(\beta,\ m)(a',\ m)}{(\gamma,\ m)(\gamma-a',\ m)}\frac{x^m}{m\,!}F(a+m,\ \beta+m,\ \gamma-a'+m,\ x)$$
$$\Phi(a'+m,\ \gamma+m,\ y)$$
$$\Phi_2(a,\ a'\ ;\ \gamma\ ;\ x,\ y)=\sum(-1)^m\frac{(a,\ m)(a',\ m)}{(\gamma,\ m)(\gamma-a',\ m)}\frac{x^m}{m\,!}\Phi(a+m,\ \gamma-a'+m,\ x)$$
$$\Phi(a'+m,\ \gamma+m,\ y)$$
$$\Xi_2(a,\ \beta,\ \gamma,\ x,\ y)=\sum(-1)^m\frac{(a,\ m)}{(\gamma,\ m)(\gamma-a,\ m)}\frac{y^m}{m\,!}F(a+m,\ \beta\ ;\ \gamma+m\ ;\ x)$$
$$B(\gamma-a+m,\ y).$$

Another type of expansion may be obtained as follows: in the formula

$$F_1=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\int_0^1\int_0^1 t^{\beta-1}v^{\beta'-1}(1-t)^{\gamma-\beta-\beta'-1}(1-v)^{\gamma-\beta'-1}(1-tx-vy+vtx)^{-a}dtdv$$

take

$$(1-tx-vy+vtx)^{-\alpha}=(-1)^{-\alpha}[1-(1-tx)-(1-vy)-vtx]^{-\alpha}$$
$$=(-1)^{-\alpha}\sum_m\sum_n\sum_p\frac{(\alpha,\ m+n+p)}{m\,!\,n\,!\,p\,!}(1-tx)^m(1-vy)^n v^p t^p x^p,$$

whence

$$F_1=(-1)^{-\alpha}\sum\sum\sum x^p\frac{(\alpha,\ m+n+p)(\beta,\ p)(\beta',\ p)}{m\,!\,n\,!\,p\,!\,(\gamma,\ p)(\gamma-\beta',\ p)}F(-m,\ \beta+p\,;\ \gamma-\beta'+p\,;\ x)$$
$$F(-n,\ \beta'+p,\ \gamma+p,\ y)$$

and

$$\Phi_1(\alpha,\beta,\gamma,x,y)=(-1)^{-\alpha}\sum\sum\sum y^p\frac{(\alpha,\ m+n+p)(\beta,\ p)}{m\,!\,n\,!\,p\,!\,(\gamma,\ p)(\gamma-\beta,\ p)}F(-m,\ \beta+p\,;\gamma+p,\ x)$$
$$\Phi(-n,\ \gamma-\beta+p,\ y).$$

All these expansions show the intimate connection between these functions and the similar one-variable functions.

It is easy to show also that an important relation exists between Φ_1 and Ξ_1, and that, in fact, they always reduce to one another. Let us start from the expansion which we gave for Φ_1, in ascending powers of x; then, using the relation

$$\Phi(\alpha+m,\ \gamma+m,\ y)=e^y\Phi(\gamma-\alpha,\ \gamma+m,\ -y),$$

we can write

$$\Phi_1(\alpha\,;\,\beta\,;\,\gamma\,;\,x,\ y)=e^y\sum\frac{(\alpha,\ m)(\beta,\ m)}{(\gamma,\ m)}\Phi(\gamma-\alpha\,;\,\gamma+m,\ -y)\frac{x^m}{m\,!},$$

and, comparing with the expansion for Ξ_1,

$$\Phi_1(\alpha\,;\,\beta\,;\,\gamma\,;\,x,\ y)=e^y\Xi_1(\alpha,\ \gamma-\alpha\,;\,\beta\,;\,\gamma\,;\,x,\ y),$$

which is the relation in question.

Chapter III.

DIFFERENTIAL EQUATIONS SATISFIED BY THE FUNCTIONS.

The seven confluent functions satisfy partial differential equations of rather simple forms, which it is easy to obtain, by confluence, from the four systems of equations found by Appell for the F functions.

Writing

$$\frac{\partial z}{\partial x}=p,\quad \frac{\partial z}{\partial y}=q,\quad \frac{\partial^2 z}{\partial x^2}=r,\ \text{etc.,}$$

we find that the system for the function Φ_1 is

$$\begin{cases}x(1-x)r+y(1-x)s+[\gamma-(\alpha+\beta+1)x]p-\beta yq-\alpha\beta z=0\\ yt+xs+(\gamma-y)q-px-\alpha z=0,\end{cases}$$

that for Ψ_1 is

$$\begin{cases} x(1-x)r - xys + [\gamma - (a+\beta+1)x]p - \beta yq - a\beta z = 0 \\ yt + (\gamma' - y)q - px - az = 0, \end{cases}$$

and similarly for the other functions.

Each of the systems is of the type

$$\begin{cases} r = a_1 s + a_2 p + a_3 q + a_4 z \\ t = b_1 s + b_2 p + b_3 q + b_4 z \end{cases}$$

(the a's and b's being functions of x and y), of which a general theory has been given by Appell, with the aid of certain propositions established by Bouquet. When the expression

$$1 - a_1 b_1$$

is different from zero (which is the case for the Ψ and Ξ systems), the general solution of the system is a linear function of four independent solutions

$$z = C_1 z_1 + C_2 z_2 + C_3 z_3 + C_4 z_4,$$

and if

$$1 - a_1 b_1 = 0$$

(which occurs for the Φ systems), it is a linear function of three independent solutions

$$z = C_1 z_1 + C_2 z_2 + C_3 z_3 .$$

We may observe that the system satisfied by the function

$$\Psi_1(a\,;\,\beta\,;\,\gamma,\,\gamma'\,;\,x,\,y)$$

admits also the independent solutions

$$x^{1-\gamma}\Psi_1(a+1-\gamma\,;\,\beta+1-\gamma\,;\,2-\gamma,\,\gamma'\,;\,x,\,y)$$
$$y^{1-\gamma'}\Psi_1(a+1-\gamma'\,;\,\beta\,;\,\gamma,\,2-\gamma'\,;\,x,\,y)$$
$$x^{1-\gamma}y^{1-\gamma'}\Psi_1(a+2-\gamma-\gamma'\,;\,\beta+1-\gamma\,;\,2-\gamma,\,2-\gamma'\,;\,x,\,y).$$

A similar result may be obtained for the function Ψ_2, so that the general solution of the two Ψ systems may readily be expressed in terms of the Ψ functions themselves.

Chapter IV.

SOME SPECIAL PROPERTIES OF THE Φ AND Ξ FUNCTIONS.

We shall next give a few formulæ illustrative of the properties of the Φ and Ξ functions: the function Ψ_2, which has a special importance, will be considered in the next chapter.

The Φ_1 function admits recurrence formulæ analogous to the well-known relations between contiguous hypergeometric functions of one variable; thus

$$\begin{cases} \frac{\beta x}{\gamma}\Phi_1(\alpha+1\,;\,\beta+1\,;\,\gamma+1\,;\,x,\,y)+\frac{y}{\gamma}\Phi_1(\alpha+1\,;\,\beta\,;\,\gamma+1\,;\,x,\,y) \\ \qquad\qquad =\Phi_1(\alpha+1\,;\,\beta\,;\,\gamma\,;\,x,\,y)-\Phi_1(\alpha\,;\,\beta\,;\,\gamma\,;\,x,\,y) \end{cases}$$

$$\Phi_1(\alpha\,;\,\beta+1\,;\,\gamma\,;\,x,\,y)=\Phi_1(\alpha\,;\,\beta\,;\,\gamma\,;\,x,\,y)+\frac{\alpha x}{\gamma}\Phi_1(\alpha+1\,;\,\beta+1\,;\,\gamma+1\,;\,x,\,y).$$

The relation

$$\frac{\partial}{\partial x}\Phi_1(\alpha\,;\,\beta\,;\,\gamma\,;\,x,\,y)=\frac{\alpha\beta}{\gamma}\Phi_1(\alpha+1\,;\,\beta+1\,;\,\gamma+1\,;\,x,\,y)$$

shows that the derivates of the Φ_1 function are expressible in terms of the function itself.

Similar formulæ may be obtained for Φ_2.

Φ_1 may be expressed by a simple definite integral

$$\Phi_1(\alpha\,;\,\beta\,;\,\gamma\,;\,x,\,y)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}e^{uy}du,$$

while Φ_2 may be expressed by the double integral

$$\Phi_2(\beta,\,\beta'\,;\,\gamma\,;\,x,\,y)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\iint u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}e^{ux+vy}du\,dv$$

$$(u\geqslant 0,\ v\geqslant 0,\ 1-u-v\geqslant 0).$$

Formulæ of the same type may be obtained for the Ξ functions: thus we have

$$\Xi_1(\alpha,\,\alpha',\,\beta\,;\,\gamma\,;\,x,\,y)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\gamma-\alpha-\alpha')}\iint u^{\alpha-1}v^{\alpha'-1}(1-ux)^{-\beta}e^{vy}(1-u-v)^{\gamma-\alpha-\alpha'-1}du\,dv,$$

the field being the same as above.

Chapter V.

THE FUNCTION Ψ_2 AND ITS TRANSFORMATIONS.

The function Ψ_2 proves to be the most interesting of the seven, as its properties afford a very direct generalisation of the one-variable confluent hypergeometric function. To render this fact more conspicuous, we shall substitute for Ψ_2 a new function, just as Whittaker * studied, instead of Φ, his functions M or W.

We therefore make the following change of parameters:

$$\alpha=\mu+\nu-k+1$$
$$\gamma=2\mu+1$$
$$\gamma'=2\nu+1,$$

* *Bull. Amer. Math. Soc.*, iv, p. 125.

and we define the function

$$M_{k,\mu,\nu}(x, y) = x^{\mu+\frac{1}{2}}y^{\nu+\frac{1}{2}}e^{-\frac{x+y}{2}}\Psi_2(\mu+\nu-k+1\,;\ 2\mu+1,\ 2\nu+1\,;\ x,\ y).$$

The development in ascending powers of x and y is then

$$M_{k,\mu,\nu} = x^{\mu+\frac{1}{2}}y^{\nu+\frac{1}{2}}e^{-\frac{x+y}{2}}\sum_m\sum_n\frac{(\mu+\nu-k+1,\ m+n)}{(2\mu+1,\ m)(2\nu+1,\ n)}\,\frac{x^m y^n}{m!\ n!}.$$

We see that this function exists only when μ and ν are both different from the half of a negative integer; a similar feature occurs with the one-variable confluent hypergeometric function

$$M_{k,\mu}(x) = x^{\mu+\frac{1}{2}}e^{-\frac{x}{2}}\lim_{\rho=\infty}F\left(\mu-k+\tfrac{1}{2},\ \rho\,;\ 2\mu+1\,;\ \frac{x}{\rho}\right)$$

which disappears if 2μ is a negative integer. If, however, we suppose ν, for instance, to become equal to $-\frac{1}{2}$, and *simultaneously* y to become equal to zero, with the condition that the fraction $\dfrac{y}{2\nu+1}$ tends to zero, the function becomes, as it is easy to verify by considering the above expansion, equal to

$$x^{\mu+\frac{1}{2}}e^{-\frac{x}{2}}\sum_m\frac{(\mu-k+\frac{1}{2},\ m)}{(2\mu+1,\ m)}\,\frac{x^m}{m!}$$

or precisely $M_{k,\mu}(x)$. We then have the most important relation

$$M_{k,\mu,-\frac{1}{2}}(x,\ 0) = M_{k,\mu}(x),$$

provided that $\lim\dfrac{y}{2\nu+1}=0$; to which can be added the similar one

$$M_{k,-\frac{1}{2},\nu}(0,\ y) = M_{k,\nu}(y),$$

provided that $\lim\dfrac{x}{2\mu+1}=0$.

It is easy to form the system of partial differential equations satisfied by $M_{k,\mu,\nu}$: it is

$$(\mathrm{S})\qquad\begin{cases} x^2r - xyq + z\left(-\dfrac{x^2}{4}-\dfrac{xy}{2}+kx+\frac{1}{4}-\mu^2\right)=0\\ y^2t - xyp + z\left(-\dfrac{y^2}{4}-\dfrac{xy}{2}+ky+\frac{1}{4}-\nu^2\right)=0.\end{cases}$$

If in this system we take $y=0$ and $\nu=-\frac{1}{2}$, the second equation vanishes, and the first one becomes

$$x^2\frac{d^2z}{dx^2}+z\left(-\frac{x^2}{4}+kx+\frac{1}{4}-\mu^2\right)=0,$$

which is precisely the confluent hypergeometric equation of one variable, in Whittaker's form; and we obtain a similar result by taking $x=0$ and $\mu=-\frac{1}{2}$.

We shall, in general, denote by $W_{k,\mu,\nu}(x, y)$ a solution of this system with the condition that it reduces to $W_{k,\mu}(x)$ for $y = 2\nu + 1 = 0$, and to $W_{k,\nu}(y)$ for $x = 2\mu + 1 = 0$.

The general solution of the system S is readily found to be of the form

$$C_1 M_{k,\mu,\nu}(x, y) + C_2 M_{k,-\mu,\nu}(x, y) + C_3 M_{k,\mu,-\nu}(x, y) + C_4 M_{k,-\mu,-\nu}(x, y),$$

the C's being arbitrary constants.

Numerous recurrence formulæ may be written for the M function. We shall only give the following as an example:

$$x\frac{\partial}{\partial x} M_{k,\mu,\nu} = \left(\mu + \tfrac{1}{2} - \frac{x}{2}\right) M_{k,\mu,\nu} + x^{\frac{1}{2}} \frac{\mu + \nu - k + 2}{2\mu + 1} M_{k-\frac{1}{2},\mu+\frac{1}{2},\nu}.$$

The expansions for Ψ_2 furnish analogous results. We thus obtain, bearing in mind the definition of the one-variable M function and its relation with Φ,

$$M_{k,\mu,\nu}(x, y) = e^{-\frac{x}{2}} \sum_m x^{m+\mu+\frac{1}{2}} \frac{(\mu + \nu - k + 1, m)}{(2\mu + 1, m) m!} M_{k-\mu-m-\frac{1}{2},\nu}(y)$$

and

$$M_{k,\mu,\nu}(x, y) = e^{-\frac{y}{2}} \sum_m y^{m+\nu+\frac{1}{2}} \frac{(\mu + \nu - k + 1, m)}{(2\nu + 1, m) m!} M_{k-\nu-m-\frac{1}{2},\mu}(x).$$

By transforming the formula

$$\Psi_2(\alpha; \gamma, \gamma'; x, y) = (-1)^{\alpha} \sum_m \sum_n \frac{(\alpha, m + n)}{m! \, n!} \Phi(-m, \gamma, x) \Phi(-n, \gamma', y)$$

we obtain

$$M_{k,\mu,\nu}(x, y) = (-1)^{k-\mu-\nu-1} \sum_m \sum_n \frac{(\mu + \nu - k + 1, m + n)}{m! \, n!} M_{m+\mu+\frac{1}{2},\mu}(x) M_{n+\nu+\frac{1}{2},\nu}(y).$$

Let us consider the one-variable M functions which occur under the symbol of summation. The expansion of the first one is

$$M_{m+\mu+\frac{1}{2},\mu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{x}{2}} \sum_{p=0}^{p=\infty} \frac{(-m, p)}{(2\mu + 1, p)} \frac{x^p}{p!};$$

but, as m is an integer, the product $(-m, p)$ vanishes whenever p is greater than m, so that the sum represents not an infinite series, but a polynomial of degree m in x,

$$P(x) = \sum_{p=0}^{p=m} \frac{(-m, p)}{(2\mu + 1, p)} \frac{x^p}{p!}.$$

The question is now, what is this polynomial? Let us write

$$P(x) = \sum_{q=0}^{q=m} \frac{(-m, m - q)}{(2\mu + 1, m - q)} \frac{x^{m-q}}{(m - q)!}$$

$$= \Gamma(2\mu + 1) m! \sum_{q=0}^{q=m} (-1)^{m-q} \frac{x^{m-q}}{q! \, (m - q)! \, \Gamma(2\mu + m - q)}$$

or

$$P(x) = (-1)^m m!\,\Gamma(2\mu+1)\left[\frac{x^m}{0!\,m!\,\Gamma(2\mu+m)} - \frac{x^{m-1}}{1!\,(m-1)!\,\Gamma(2\mu+m-1)} + \ldots\right]$$

and the expression between brackets is the polynomial of degree m considered by Sonine * in his researches on the Bessel functions; it is here $T^m_{2\mu}(x)$, the definition of the polynomial T^β_α being the expansion

$$\frac{e^{\frac{tx}{1+t}}}{(1+t)^{\alpha+1}} = \sum_{\beta=0}^{\beta=\infty} \Gamma(\alpha+\beta+1)t^\beta T^\beta_\alpha(x),$$

and its expression, as given by Sonine, being precisely

$$T^\beta_\alpha(x) = \frac{x^\beta}{\beta!\,0!\,\Gamma(\alpha+\beta)} - \frac{x^{\beta-1}}{(\beta-1)!\,1!\,\Gamma(\alpha+\beta-1)} + \frac{x^{\beta-2}}{(\beta-2)!\,2!\,\Gamma(\alpha+\beta-2)} - \ldots$$

We can then write the following expression:

$$M_{m+\mu+\frac{1}{2},\,\mu}(x) = (-1)^m m!\,\Gamma(2\mu+1)x^{\mu+\frac{1}{2}}e^{-\frac{x}{2}}T^m_{2\mu}(x),$$

a result which can be verified by using the expression of the T polynomial in terms of the $W_{k,m}$ function, as given by Whittaker.†

We then obtain at once the very simple and remarkable expansion

$$M_{k,\mu,\nu}(x, y) = x^{\mu+\frac{1}{2}}y^{\nu+\frac{1}{2}}e^{-\frac{x+y}{2}}\Gamma(2\mu+1)\Gamma(2\nu+1)\sum_m\sum_n(-1)^{k-\mu-\nu-m-n-1}$$
$$(\mu+\nu-k+1,\ m+n)T^m_{2\mu}(x)T^n_{2\nu}(y).$$

Some interesting consequences, concerning certain particular cases of the M function, can be deduced from this formula.

If we suppose, in the first place, μ and ν to be of the form $\frac{a}{2}+\frac{1}{4}$, $\frac{b}{2}+\frac{1}{4}$, where a and b are integers, we have to consider in the expansion polynomials of the type

$$T^m_{a+\frac{1}{2}}(x),$$

for which we readily find the simple expression

$$T^m_{a+\frac{1}{2}}(x) = \frac{d^{a+1}}{dx^{a+1}}T^{a+m+1}_{-\frac{1}{2}}(x).$$

But we can observe with Sonine that, if λ is an integer,

$$T^\lambda_{-\frac{1}{2}}(x) = U_{2\lambda}(\sqrt{x})$$

where U is Hermite's polynomial,

$$U_{2\lambda}(z) = e^{z^2}\frac{d^{2\lambda}}{dz^{2\lambda}}e^{-z^2},$$

* *Math. Ann.*, xvi (1880), p. 41.
† *Modern Analysis*, 3rd edition (1920), p. 352.

so that we can write, for an M function of the aforesaid type,

$$\mathrm{M}_{k,\,\frac{a}{2}+\frac{3}{4},\,\frac{b}{2}+\frac{3}{4}}(x,\,y)=x^{\frac{a}{2}+\frac{3}{4}}y^{\frac{b}{2}+\frac{3}{4}}e^{-\frac{x+y}{2}}\Gamma(a+\tfrac{3}{2})\Gamma(b+\tfrac{3}{2})\sum\sum(-1)^{k-\frac{a+b+3}{2}-m-n}$$
$$\times\left(\frac{a+b+3}{2}-k,\ m+n\right)\times\frac{d^{a+1}}{dx^{a+1}}\mathrm{U}_{2(a+m+1)}(\sqrt{x})\times\frac{d^{b+1}}{dy^{b+1}}\mathrm{U}_{2(b+n+1)}(\sqrt{y}).$$

As any differential coefficient of the U polynomials can be expressed in terms of the U themselves, we can express any function M where μ and ν are of the type $\frac{a}{2}+\frac{1}{4}$, $\frac{b}{2}+\frac{1}{4}$ in terms of Hermite's polynomials.

Let us take, in particular, $a=b=-1$; we have at once, with a change of variables,

$$\mathrm{M}_{k,\,-\frac{1}{4},\,-\frac{1}{4}}\left(\frac{x^2}{2},\,\frac{y^2}{2}\right)=\pi\sqrt{\frac{xy}{2}}(-1)^{k+\frac{1}{2}}e^{-\frac{x^2+y^2}{4}}\times\sum\sum(-1)^{m+n}(\tfrac{1}{2}-k,\ m+n)\mathrm{U}_{2m}\left(\frac{x}{\sqrt{2}}\right)\mathrm{U}_{2n}\left(\frac{y}{\sqrt{2}}\right).$$

This formula connects the special M function with the parabolic-cylinder functions.

Chapter VI.

CONNECTION BETWEEN CERTAIN KNOWN FUNCTIONS AND THE CONFLUENT HYPERGEOMETRIC FUNCTIONS.

Several functions of two variables introduced by different authors can be connected with some of the seven confluent functions of two variables. Of this we shall give three examples.

1. *The Two-variable Polynomials* $A_{m,n}$ *of Appell.*—It is a well-known fact that limiting cases of a great number of one-variable polynomials are expressible by the $\mathrm{W}_{k,m}$ function or by Bessel functions. For instance, as anyone knows, for Legendre functions we have

$$\lim_{n=\infty}\left[n^{-m}\mathrm{P}_n^m\left(1-\frac{x^2}{2n^2}\right)\right]=\mathrm{J}_m(x).$$

We can establish a similar property for certain two-variable polynomials.

Let us consider the two-variable polynomials discussed by Appell,* and defined by

$$\mathrm{A}_{m,n}(x,\,y)=x^{1-\gamma}y^{1-\gamma'}(1-x-y)^{\gamma+\gamma'-\delta+m+n}\frac{\partial^{m+n}}{\partial x^m\partial y^n}[x^{m+\gamma-1}y^{n+\gamma'-1}(1-x-y)^{\delta-\gamma-\gamma'}].$$

As shown by Appell himself, they can be written under the form

$$\mathrm{A}_{m,n}=(\gamma,\,m)(\gamma',\,n)(1-x-y)^{m+n}\mathrm{F}_2\left(\gamma+\gamma'-\delta\,;\ -m,\ -n\,;\ \gamma,\gamma'\,;\ \frac{x}{x+y-1},\ \frac{x}{x+y-1}\right).$$

* *Archiv Math. Phys.*, lxvi, 1881, p. 238.

Let us now divide x by m and y by n, causing m and n to tend simultaneously to infinity: we observe then that

$$\lim_{m,\, n=\infty} \frac{A_{m,\, n}\left(\frac{x}{m}, \frac{y}{n}\right)}{(\gamma, m)(\gamma', n)} = e^{-2(x+y)}\Psi_2(\gamma + \gamma' - \delta\,;\ \gamma, \gamma'\,;\ x, y),$$

showing a connection between $A_{m,\, n}$ and Ψ_2 of the same nature as the connection between Legendre and Bessel functions.

2. *The Two-variable Polynomials $H_{m,\, n}$ of Hermite.*—These functions, introduced by Hermite,* arise from the derivation of an exponential where the exponent is a quadratic form of x and y; their definition is

$$H_{m,\, n}(x, y) = (-1)^{m+n} e^{\frac{1}{2}\phi(x,\, y)} \frac{\partial^{m+n}}{\partial x^m \partial y^n} e^{-\frac{1}{2}\phi(x,\, y)}$$

where

$$\phi(x, y) = ax^2 + 2bxy + cy^2.$$

It may be shown that this polynomial depends essentially on the function

$$W_{\frac{m+n-1}{2},\, -\frac{1}{4},\, -\frac{1}{4}}.$$

3. *The Two-variable Bessel Function of order Zero.*—Several results have been published lately on the subject of new functions of two variables possessing certain properties analogous to Bessel functions.† These two-variable Bessel functions are defined by the expansion

$$e^{\frac{x}{2}\left(u-\frac{1}{u}\right)+\frac{y}{2}\left(u^2-\frac{1}{u^2}\right)} = \sum_n J_n(x, y)u^n,$$

or by the integral

$$J_n(x, y) = \frac{1}{\pi}\int_0^\pi \cos(nu - x\sin u - y\sin 2u)du.$$

It may be shown that the simplest of these functions, $J_0(x, y)$, satisfies the same differential equation as our solution

$$e^{-iy}\Phi_3(\tfrac{1}{2}, 1, 2iy, -\tfrac{1}{4}x^2).$$

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* *Œuvres*, ii, p. 293.

† *Cf.* a paper by Jekhowsky, with a bibliography of the subject, in *Bull. Astron.*, t. xxxv, 1918.