"Extrait d'une Lettre adressée à M. Guccia," par M. H.-G. Zeuthen. ("Rendiconti del Circolo Matematico di Palermo.")

"Note sur les Huits Points d'Intersection de Trois Surfaces du Second Ordre," par H.-G. Zeuthen. ("Acta Mathematica.")

"American Journal of Mathematics," Vol. XII., No. 2; Baltimore.

On the Radial Vibrations of a Cylindrical Elastic Shell.

By A. B. Basset, M.A., F.R.S.

[*Read Dec.* 12*th*, 1889.]

The usual theory of thin plates and shells assumes that the three stresses R, S, T may be treated as zero; and in a previous paper* I pointed out, that this supposition is legitimate, only in the case in which these quantities are at least of the order of the square of the thickness of the plate or shell. I also showed that, by means of certain results given in a paper by Lord Rayleigh, it could be easily proved that this supposition is true in the case of a plane plate, which is not under the influence of forces applied to its surface.

The question whether R, S, T *may be treated as zero* (for they cannot actually be zero except in certain very special cases) is of the utmost importance in the theory of the free vibrations of thin curved shells. If these stresses may be treated as zero, it is possible to obtain the correct expression for the potential energy, and the correct equations of motion, as far as the terms involving the cube of the thickness; but, if they may not be treated as such, a satisfactory theory of thin shells seems almost hopeless. For, although the results obtained by considering (as Mr. Love has done) that part of the potential energy which is *solely* due to the extension of the middle surface, are correct as far as the terms involving the thickness are concerned; yet, inasmuch as there are reasons for thinking that the graver tones of a bell depend principally upon flexure rather than extension, it is essential to take the term in h^3 into consideration.

A perfectly rigorous solution of any question relating to the vibrations of cylindrical or spherical shells, might be obtained by means of the general equations of motion of an elastic solid, since these

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* Ante, p. 33.

equations contain within themselves the complete solution of every conceivable problem; but the mathematical difficulties of integrating these equations, except in special cases, are so great, that an investigation conducted by means of them would be very laborious and complicated. The radial vibrations of an infinitely long cylindrical shell can, however, be investigated by means of these equations without much difficulty as far as the term in the period which involves h^2, where $2h$ is the thickness of the shell; and, by comparing the result obtained by this method with the one furnished by the theory of thin plates, we shall obtain evidence respecting the legitimacy of the assumption on which the latter theory is based.

The vibrations which we are about to consider are exclusively normal, and the displacement w is a function of r alone, and satisfies the differential equation

$$(m+n)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}-\frac{w}{r^2}\right)=\rho\ddot{w} \quad\text{..................(1)},$$

as can easily be seen by putting $u=v=0$ in the general equations of motion of an elastic solid.

Putting $$w=W\epsilon^{\iota pt},\quad p^2\rho/(m+n)=a^2 \quad\text{.....................(2)},$$

(1) becomes $$\frac{d^2W}{dr^2}+\frac{1}{r}\frac{dW}{dr}+\left(a^2-\frac{1}{r^2}\right)W=0 \quad\text{..................(3)},$$

the solution of which is

$$W=AJ_1(ar)+BY_1(ar) \quad\text{.......................(4)},$$

where J and Y are the two kinds of Bessel's functions. Since the tangential stress parallel to the middle surface $r=a$ is obviously zero in this species of motion, the only surface condition is $R=0$, or

$$\frac{dW}{dr}+\frac{EW}{r}=0 \quad\text{.............................(5)},$$

where $$E=\frac{m-n}{m+n};$$

whence, putting $x=ar$, (5) becomes

$$A\left\{xJ_1'(x)+EJ_1(x)\right\}+B\left\{xY_1'(x)+EY_1(x)\right\}=0 \quad\text{.........(6)},$$

when $$r=a\pm h.$$

This equation enables us to determine the frequency, and we shall proceed to evaluate it as far as the term involving h^2.

Denoting the coefficients of A and B in (6) by $\phi(x)$ and $\chi(x)$, where x is now written for aa, the equation for the frequency is

$$\frac{\phi\,(x+ah)}{\phi\,(x-ah)} = \frac{\chi\,(x+ah)}{\chi\,(x-ah)},$$

whence, neglecting powers of h higher than h^2, we have

$$\phi\chi'-\chi\phi'+\tfrac{1}{2}a^2h^2\{\phi''\chi'-\chi''\phi'+\tfrac{1}{3}(\phi\chi'''-\chi\phi''')\} = 0 \quad \text{.........(7).}$$

In evaluating this equation the following properties of Bessel's functions will be useful, which can easily be proved by means of the fundamental equation which these quantities satisfy.

Let $$H_s = J_s' Y_s - Y_s' J_s,$$

then it can be proved that

$$\left.\begin{aligned} J_s'' Y_s - Y_s'' J_s &= -H_s/x, \\ J_s'' Y_s' - Y_s'' J_s' &= H_s\,(1-s^2/x^2), \\ J_s''' Y_s - Y_s''' J_s &= H_s\left(\frac{s^2+2}{x^2}-1\right), \\ J_s''' Y_s' - Y_s''' J_s' &= \frac{H_s}{x}\left(\frac{3s^2}{x^2}-1\right), \\ J_s^{\text{iv}} Y_s - Y_s^{\text{iv}} J_s &= -\frac{2H_s}{x}\left(\frac{3s^2+3}{x^2}-1\right), \\ J_s^{\text{iv}} Y_s' - Y_s^{\text{iv}} J_s' &= -H_r\left(\frac{s^4+11s^2}{x^4}-\frac{2s^2+3}{x^2}+1\right), \end{aligned}\right\} \quad \text{.........(8),}$$

to which may be joined

$$J_s''' Y_s'' - Y_s''' J_s'' = H_s\left(1-\frac{2s^2+1}{x^2}+\frac{s^4-s^2}{x^4}\right) \quad \text{............(9).}$$

It is also known that xH_s is independent of x.

Dropping the suffix 1, which is no longer necessary, we have

$$\phi = xJ' + EJ,$$

$$\phi' = xJ'' + (1+E)\,J',$$

$$\phi'' = xJ''' + (2+E)\,J'',$$

$$\phi''' = xJ^{\text{iv}} + (3+E)\,J''';$$

whence $$\phi\chi'-\chi\phi' = H\,(1-E^2-x^2) \quad \text{.....................(10).}$$

From this equation we see that, to a first approximation,

$$x^2 = 1-E^2 \qquad \text{.....(11)};$$

whence, restoring the values of x and E, we obtain

$$p^2 = \frac{4mn}{\rho a^2 (m+n)} \qquad \text{.....(12)},$$

which is a known result.

Again,

$$\phi''\chi' - \chi''\phi' = x^2 (J'''Y'' - Y'''J'') + x(1+E)(J'''Y' - Y'''J')$$
$$+ (1+E)(2+E)(J''Y' - Y''J')$$
$$= -H(3-x^2) + H(1+E)\left(\frac{3}{x^2} - 1\right) + H(1+E)(2+E)\left(1 - \frac{1}{x^2}\right)$$
$$= \frac{H}{x^2}\left[x^4 - 3x^2 + 1 - E^2 + (1+E)^2 x^2\right] \qquad \text{.....(13)}.$$

Also,

$$\phi\chi''' - \chi\phi''' = x^4 (Y^{iv}J' - J^{iv}Y') + xE(Y^{iv}J - J^{iv}Y)$$
$$+ x(3+E)(Y'''J' - J'''Y') + E(3+E)(Y'''J - J'''Y)$$
$$= \frac{H}{x^2}\left\{x^4 + x^2(E^2 + 2E - 2) + 3(1-E^2)\right\} \qquad \text{.....(14)}.$$

Substituting from (10), (13), (14), in (7), we obtain

$$x^2(1-E^2) - x^4 + a^2h^2\left\{\tfrac{2}{3}x^4 - 2x^2 + \tfrac{2}{3}(1+E)^2 x^2 + 1 - E^2\right\} = 0.$$

Now $x = a\alpha$; we may also, in the term involving h^2, substitute the approximate value of x from (11); we thus obtain

$$a^2(1-E^2) - a^4\alpha^2 + \tfrac{1}{3}h^2(1-E^2)(1+4E) = 0;$$

whence
$$p^2 = \frac{4mn}{\rho a^2 (m+n)}\left\{1 + \frac{h^2}{3a^2}(1+4E)\right\} \qquad \text{.....(15)},$$

from which it appears that the pitch rises as the thickness increases.

We shall now prove that the same result is given by the theory of thin plates, in which R is treated as zero.

Let
$$d\Omega = dz\,d\phi;$$

then, if T and W be the kinetic and potential energies,

$$T = \tfrac{1}{2}\rho \iiint_{-h}^{h} \dot{w}'^2 (a+h')\, dh'd\Omega,$$

where $a+h'$, ϕ are the coordinates of any point of the shell.

Now $$w' = w + h'\left(\frac{dw}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2w}{dr^2}\right),$$

where the brackets denote the values of the differential coefficients when $r=a$. Also, if we suppose that R is at most of the order of the square of the thickness, and neglect higher powers than h^3, in the expressions for T and W, we may write

$$\frac{dw}{dr} = \sigma_3 = -Ew/r,$$

$$\frac{d^2w}{dr^2} = E(1+E)\,w/r^2;$$

accordingly $$w' = \left\{1 - \frac{Eh'}{a} + \tfrac{1}{2}E(1+E)\frac{h'^2}{a^2}\right\} w;$$

whence $$T = \rho h \iint \left\{1 + \frac{h^2 E}{3a^2}(2E-1)\right\} \dot{w}^2 a\, d\Omega \quad \text{...........(16).}$$

Again, $$W = \iiint_{-h}^{h} \left\{\tfrac{1}{2}(m+n)\,\Delta'^2 - 2n\sigma_2\sigma_3\right\} r\, dh'd\Omega$$

$$= n(1+E)\iiint_{-h}^{h} \frac{w'^2}{r}\, dh'd\Omega$$

$$= 2nh(1+E)\iint \left\{1 + \frac{h^2}{3a^2}(1+E)(1+2E)\right\} \frac{w^2}{a^2} a\, d\Omega \text{..(17);}$$

whence

$$\rho a^2 p^2 \left\{1 + \frac{h^2 E}{3a^2}(2E-1)\right\} = 2n(1+E)\left\{1 + \frac{h^2}{3a^2}(1+E)(1+2E)\right\},$$

or $$p^2 = \frac{4mn}{\rho a^2 (m+n)}\left\{1 + \frac{h^2}{3a^2}(1+4E)\right\} \quad \text{..............(18),}$$

which agrees with our former result.

Although the agreement of the results obtained by the two theories in this particular case does not, of course, prove that R may be treated as zero in every conceivable case that may arise, yet the inference is that it may be treated as such, provided the surfaces of the shell are free from external pressures and tangential stresses, and provided also that the solution is not carried to a higher degree of approximation than terms in h^3 in the expressions for the kinetic and potential energies, and terms in h^2 in the period.

A similar method might be employed to investigate the radial vibrations of a complete spherical shell, for the equation of motion is

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\,\frac{d}{dr}+a^2-\frac{9}{4r^2}\right)(wr^{\frac{1}{2}})=0,$$

the solution of which is

$$w=r^{-\frac{1}{2}}\left\{AJ_{\frac{3}{2}}(ar)+BY_{\frac{3}{2}}(ar)\right\},$$

and evaluating by means of (8).

Note on a Quaternary Group of 51840 *Linear Substitutions.*

By Dr. Morrice.

[*Read December* 12*th*, 1889.]

Part I.

I give in Part I. a few fundamental data concerning the transformation of the double functions which will be useful to those who have not followed the development of the subject by Witting* and Maschke† in the *Mathematische Annalen.*

We have the periods

$$\begin{matrix}\omega_{11}, & \omega_{12}, & \omega_{13}, & \omega_{14},\\ \omega_{21}, & \omega_{22}, & \omega_{23}, & \omega_{24},\end{matrix}$$

of the hyper-elliptic integrals

$$u_1=\int\frac{\alpha+\beta z}{\sqrt{f(z)}}\,dz,\quad u_2=\int\frac{\gamma+\delta z}{\sqrt{f(z)}}\,dz,$$

and we transform by means of the relations

$$v_1=\frac{u_1\omega_{22}-u_2\omega_{12}}{p_{12}},\quad v_2=\frac{-u_1\omega_{21}+u_2\omega_{11}}{p_{12}},$$

where p_{ik} stands for $\omega_{1i}\omega_{2k}-\omega_{1k}\omega_{2i}$.

The new periods are

$$\begin{matrix}1, & 0, & \tau_{11}=\dfrac{p_{32}}{p_{12}}, & \tau_{12}=\dfrac{p_{42}}{p_{12}},\\ 0, & 1, & \tau_{21}=\dfrac{p_{13}}{p_{12}}, & \tau_{22}=\dfrac{p_{14}}{p_{12}},\end{matrix}$$

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* Witting, Inaugural Dissertation, Dresden, 1887, and *Math. Ann.*, Band 29.

† Maschke, *Math. Ann.*, Band 33, Heft 3.

and the relation, $$p_{13} = p_{42},$$
which exists between the old periods, becomes
$$\tau_{12} = \tau_{21}.$$

The double ϑ-function of the first order is

$$\vartheta_{\left|\begin{smallmatrix} g_1, g_2 \\ h_1, h_2 \end{smallmatrix}\right|} (v_1, v_2) = e^{\frac{1}{2} i\pi (g_1 h_1 + g_2 h_2)} \sum_{-\infty}^{+\infty} n_1 n_2 (-1)^{n_1 h_1 + n_2 h_2} e^{i\pi [(2n_1 + g_1) v_1 + (2n_2 + g_2) v_2} \times e^{\frac{1}{4} i\pi \phi (2n_1 + g_1, 2n_2 + g_2)},$$

where $$\phi (n_1, n_2) = \tau_{11} n_1^2 + 2\tau_{12} n_1 n_2 + \tau_{22} n_2^2.$$

We have now to consider the group of linear transformations of the periods which leave the characteristic
$$\begin{vmatrix} 0, & 0 \\ 0, & 0 \end{vmatrix}$$
unaltered. Witting gives five from which any others may be generated by iteration and combination:

	M	N	P	Q	R
$\omega'_{i1} =$	ω_{i3}	ω_{i1}	$\omega_{i2} + \omega_{i3}$	ω_{i1}	$-\omega_{i1} + \omega_{i2}$
$\omega'_{i2} =$	ω_{i2}	$-\omega_{i1} - \omega_{i4}$	ω_{i2}	ω_{i4}	ω_i
$\omega'_{i3} =$	$-\omega_{i1}$	$\omega_{i1} + \omega_{i2} + \omega_{i3} + \omega_{i4}$	$-\omega_{i1} - \omega_{i2}$	ω_{i3}	$-\omega_{i3}$
$\omega'_{i4} =$	ω_{i4}	$-\omega_{i1} + \omega_{i2}$	$\omega_{i1} - \omega_{i2} - \omega_{i3} - \omega_{i4}$	$-\omega_{i2}$	$\omega_{i3} + \omega_{i4}$

$$i = 1, 2.$$

In connexion with the theory of transformation of order k, there have recently been established functions analogous to the sigma-functions for elliptic integrals; *i.e.*, we have a set of functions

$$X_{\alpha\beta} (v_1, v_2;\ \tau_{11}, \tau_{12}, \tau_{22})$$
$$= p_{12}^{\frac{1}{2}(k-1)} \frac{e^{ki\pi \Phi (v_1 v_2)}}{\vartheta (\tau_{11}, \tau_{12}, \tau_{22})^k} \sum_{-\infty}^{+\infty} e^{(i\pi/k) \phi (kn_1 + \alpha, kn + \beta) + 2i\pi [(kn_1 + \alpha) v_1 + (kn_2 + \beta) v_2]},$$
$$(\alpha, \beta = 0, 1, \dots k-1),$$

where $$\Phi (v_1, v_2) = A_{11} v_1^2 + 2A_{12} v_1 v_2 + A_{22} v_2^2,$$
$$A_{11} = -\frac{1}{5} \sum_1^{10} \frac{\partial \log \vartheta}{\partial \tau_{11}},$$
$$A_{12} = -\frac{1}{10} \sum_1^{10} \frac{\partial \log \vartheta}{\partial \tau_{12}},$$
$$A_{22} = -\frac{1}{5} \sum_1^{10} \frac{\partial \log \vartheta}{\partial \tau_{22}}.$$

For a transformation of the periods which leaves the characteristic

$$\begin{vmatrix} 0, & 0 \\ 0, & 0 \end{vmatrix}$$

unaltered, these $X_{\alpha\beta}$ functions are transformed into linear functions of themselves, with merely numerical coefficients. The particular case under consideration is that where $k = 3$.

We choose the four functions

$$\begin{aligned} z_1 &= X_{01} - X_{02}, \\ z_2 &= X_{10} - X_{20}, \\ z_3 &= X_{11} - X_{22}, \\ z_4 &= X_{12} - X_{21}, \end{aligned}$$

and establish the transformations which they undergo when the ω-periods undergo the transformations M, N, P, Q, R,

$$M \begin{cases} z_1' = \dfrac{i}{\sqrt{3}}(z_1 + z_3 - z_4), \\ z_2' = \dfrac{i}{\sqrt{3}}(\epsilon^2 - \epsilon)\, z_2, \\ z_3' = \dfrac{i}{\sqrt{3}}(z_1 + \epsilon^2 z_3 - \epsilon z_4), \\ z_4' = \dfrac{i}{\sqrt{3}}(-z_1 - \epsilon z_3 + \epsilon^2 z_4), \end{cases} \qquad N \begin{cases} z_1' = \dfrac{-i}{\sqrt{3}}(\epsilon - \epsilon^2)\, z_1, \\ z_2' = \dfrac{-i}{\sqrt{3}}(\epsilon z_2 + \epsilon^2 z_3 + z_4), \\ z_3' = \dfrac{-i}{\sqrt{3}}(\epsilon^2 z_2 + \epsilon z_3 + z_4), \\ z_4' = \dfrac{-i}{\sqrt{3}}(z_2 + z_3 + z_4), \end{cases}$$

$$P \begin{cases} z_1' = \dfrac{-1}{\sqrt{3}}(z_1 + \epsilon z_3 - \epsilon^2 z_4), \\ z_2' = \dfrac{-1}{\sqrt{3}}(\epsilon^2 - \epsilon)\, z_2, \\ z_3' = \dfrac{-1}{\sqrt{3}}(\epsilon^2 z_1 + \epsilon^2 z_3 - \epsilon^2 z_4), \\ z_4' = \dfrac{-1}{\sqrt{3}}(-\epsilon z_1 - z_3 + \epsilon^2 z_4), \end{cases} \qquad Q \begin{cases} z_1' = \dfrac{-1}{\sqrt{3}}(\epsilon^2 - \epsilon)\, z_1, \\ z_2' = \dfrac{-1}{\sqrt{3}}(z_2 + z_3 + z_4), \\ z_3' = \dfrac{-1}{\sqrt{3}}(z_2 + \epsilon^2 z_3 + \epsilon z_4), \\ z_4' = \dfrac{-1}{\sqrt{3}}(z_2 + \epsilon z_3 + \epsilon^2 z_4), \end{cases}$$

$$R \begin{cases} z_1' = -z_3, \\ z_2' = z_2, \\ z_3' = -z_1, \\ z_4' = -z_4, \end{cases} \qquad (\epsilon = e^{\frac{2}{3}i\pi}).$$

PART II.

It appears natural to inquire what transformations are common to the periods, on the one hand, and the four Jacobian functions z_1, z_2, z_3, z_4, on the other. Is it possible to find a sub-group of transformations common to the two groups, such that to the transformations of the ω's correspond the same transformations taken in a different order?

I know of no general method for answering these questions, and after searching through some hundreds of the matrices, I only found three common to the two groups; but these three have a simple property which appears worthy of observation.

To the matrix (which occurs in both groups)

$$S(\omega)\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

in the ω-group, corresponds the matrix

$$S(z)\begin{vmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{vmatrix}$$

in the z-group.

To the matrix (which occurs in both groups)

$$T(\omega)\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

in the ω-group, corresponds the matrix

$$T(z)\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix}$$

in the z-group; for the sake of simplicity certain numerical factors of the matrices are disregarded.

Now when the ω's are subjected to the matrix

$$S.T(\omega)\begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

the z's are transformed by the same matrix, *i.e.*, are covariant for this matrix.

Moreover, we find by actual composition

$$S(\omega)\,T(z) = T(\omega)\,S(z) = \{T.S(\omega)\}^4 = \{T.S(z)\}^4,$$

$$S(z)\,T(\omega) = T(z)\,S(\omega) = \{S.T(\omega)\}^4 = \{S.T(\omega)\}^4,$$

$$S(\omega)\,S(z)\,S^{-1}(\omega) = T(z).$$

In terms of Witting's matrices,

$$S = Q^3M(\omega),$$

$$T = M^2(QMRM^2)^3(\omega).$$

Herr Burkhardt's *résumé* of Klein's lectures in *Mathematische Annalen*, Band xxv., Heft 2, suggests that there may be some geometrical explanation of the curious simplicity of the sub-group in question.

Notes on the Plane Cubic and a Conic. *By* R. A. Roberts.

[*Read Dec.* 12*th*, 1889.]

I commence by showing that a plane cubic and a conic can be reduced to the forms

$$U = \alpha x^3 + \beta y^3 + \gamma z^3 + \delta u^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$V = ax^2 + by^2 + cz^2 + du^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

in a single way, where x, y, z, u are four lines in the plane, and α, β,

γ, δ, a, b, c, d are constants. These may thus be considered as canonical forms for a cubic and a conic.

To prove this, I introduce the consideration of another conic Σ touching the lines x, y, z, u. This conic possesses the property that if we substitute differential symbols in its tangential equation for λ, μ, ν, respectively, and operate on both the equations of the cubic and conic, the results will vanish. For, if we operate in this way on the power of any line, L^n say, the result will be proportional to L^{n-2} multiplied by the condition that L and Σ should touch. Consequently, since the cubic and conic only involve powers of x, y, z, u, and the conic Σ touches these lines, the result just stated must have place. Now, if a conic be such that the result of substituting differential symbols in its tangential equation and operating on a cubic vanishes, it must satisfy three invariant relations with the cubic; for such a result must be linear in x, y, z, $Ax+By+Cz$, say; and, in order that this should vanish identically, we should have

$$A = B = C = 0.$$

Hence the conic Σ satisfies three invariant relations with the cubic U, and one with the conic V. But these four conditions are geometrically expressed in the fact that it touches the four lines x, y, z, u, when U and V are written in the forms (1), (2). Thus, if U and V are written in any other forms, and we express that Σ satisfies certain three conditions with U, and one condition with V, and find then that these four conditions, taken together, are equivalent to expressing that Σ touches four fixed lines, we infer that these four lines are the lines x, y, z, u, when U and V are written in the forms (1), (2); and that this is so I proceed to prove.

If the cubic be written in one of its own canonical forms, namely,

$$U \equiv x^3+y^3+z^3+6mxyz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and the conic V is then at the same time

$$V = (a, b, c, f, g, h)(x, y, z)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

the results of substituting differential symbols in the tangential equation of a conic Σ $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2$ and operating on the cubic U and the conic V are, respectively,

$$(A+2mF)\,x+(B+2mG)\,y+(C+2mH)\,z,$$

$$Aa+Bb+Cc+2Ff+2Gg+2Hh\,;$$

and, in order that these should identically vanish, we must have

$$A+2mF=0,\quad B+2mG=0,\quad C+2mH=0,$$

$$Aa+Bb+Cc+2Ff+2Gg+2Hh=0,$$

showing that Σ must be of the form

$$A\,(m\lambda^2-\mu\nu)+B\,(m\mu^2-\nu\lambda)+C\,(m\nu^2-\lambda\mu)=0 \quad\text{.........}(5),$$

where $$A\,(ma-f)+B\,(mb-g)+C\,(mc-h)=0 \quad\text{..............}(6).$$

Now, subject to these conditions, Σ evidently touches the four lines determined by the equations

$$\frac{m\lambda-\mu\nu}{ma-f}=\frac{m\mu^2-\nu\lambda}{mb-g}=\frac{m\nu^2-\lambda\mu}{mc-h} \quad\text{..................}(7),$$

and these must therefore be the four lines involved in the canonical forms (1), (2). Their equation may be obtained by taking the equation of Σ in x, y, z coordinates, which may be written

$$A^2\,(4myz-x^2)+B^2\,(4mzx-y^2)+C^2\,(4mxy-z^2),$$

and $$2BC\,(yz+4m^2x^2)+2CA\,(zx+4m^2y^2)+2AB\,(xy+4m^2z^2)=0\text{...}(8),$$

and forming its envelope subject to the condition

$$A\,(ma-f)+B\,(mb-g)+C\,(mc-h)=0.$$

This gives

$$\begin{aligned}&(ma-f)^2\,mx\,(3mxyz-y^3-z^3-m^3x^3)\\&+(mb-g)^2\,my\,(3mxyz-z^3-x^3-m^3y^3)\\&+(mc-h)^2\,mz\,(3mxyz-x^3-y^3-m^3z^3)\\&+(mb-g)(mc-h)\,\{2\,(m^4-m)\,y^2z^2+x^2yz\,(1-4m^3)+m^2x\,(x^3+y^3+z^3)\}\\&+(mc-h)(ma-f)\,\{2\,(m^4-m)\,z^2x^2+y^2zx\,(1-4m^3)+m^2y\,(x^3+y^3+z^3)\}\\&+(ma-f)(mb-g)\,\{2\,(m^4-m)\,x^2y^2+z^2xy\,(1-4m^3)+m^2z\,(x^3+y^3+z^3)\}\\&\qquad=0 \quad\text{..................................}(9).\end{aligned}$$

This equation, it may be observed, is of the second degree in the coefficients of the conic, and of the sixth in those of the cubic.

We may notice that, if polar conics of U be described to have double contact with V, the chords of contact corresponding to the four solutions are the lines x, y, z, u in the forms (1), (2). For a polar conic of U is

$$\alpha x_1x^2+\beta y_1y^2+\gamma z_1z^2+\delta u_1u^2=0,$$

and this will have double contact with V at points lying on $u=0$, if we take

$$\alpha x_1 = a, \quad \beta y_1 = b, \quad \gamma z_1 = c,$$

and similarly in the case of the other lines x, y, z.

The coefficients $\alpha, \beta, \gamma, \delta$ can be found as the roots of a biquadratic equation as follows. The diagonals of the quadrilateral x, y, z, u can, it is easy to see, be written in the form

$$J \equiv x^3+y^3+z^3+u^3 = 0 \quad \text{.......................(10)},$$

on the supposition $\quad x+y+z+u = 0,$

so that $U-\lambda J$ is

$$(\alpha-\lambda)\,x^3+(\beta-\lambda)\,y^3+(\gamma-\lambda)\,z^3+(\delta-\lambda)\,u^3 = 0,$$

and the invariant S of this cubic is proportional to

$$(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda)$$

(see Salmon's *Higher Plane Curves*, Art. 239, 3rd ed.), which thus being equated to zero will give the required biquadratic. In order to find the equation of J when the cubic and conic are written in the forms (3), (4), we observe that the diagonals are the common self-conjugate triangle of all the conics inscribed in the quadrilateral. Hence, according to the theory of conics, their equation will be the Jacobian of three conics of the system; for instance, if we take the three conics to be those whose tangential equations are

$$\left.\begin{aligned}(ma-f)(m\mu^2-\nu\lambda)-(mb-g)(m\lambda^2-\mu\nu) &= 0,\\ (mb-g)(m\nu^2-\lambda\mu)-(mc-h)(m\mu^2-\nu\lambda) &= 0,\\ (mc-h)(m\lambda^2-\mu\nu)-(ma-f)(m\nu^2-\lambda\mu) &= 0,\end{aligned}\right\} \quad \text{.........(11)},$$

J will be the Jacobian of

$$\left.\begin{aligned}\{(mb-g)\,x+(ma-f)\,y\}^2+4m^2\,(ma-f)\,(mb-g)\,z^2&\\ -4mz\,\{(ma-f)^2\,x+(mb-g)^2\,y\} &= 0,\\ \{(mc-h)\,y+(mb-g)\,z\}^2+4m^2\,(mb-g)\,(mc-h)\,x^2&\\ -4mx\,\{(mb-g)^2\,y+(mc-h)^2\,z\} &= 0,\\ \{(ma-f)\,z+(mc-h)\,x\}^2+4m^2\,(ma-f)\,(mc-h)\,y^2&\\ -4my\,\{(ma-f)^2\,x+(mc-h)^2\,z\} &= 0,\end{aligned}\right\} \quad \text{......(12)}.$$

When a conic is written in the form (2) it is easily seen that the extremities of each of the three diagonals of the quadrilateral x, y, z, u are conjugate with respect to the curve, so that, if it breaks up into right lines, one line can be assumed arbitrarily, and then the other will be completely determined, namely, will pass through the three fourth harmonics on each of the diagonals. Hence, if one line is the line at infinity, the other will pass through the middle points of the diagonals. Now, when a cubic is written in the form (1), its Hessian is

$$\beta\gamma\delta yzu+\gamma\delta\alpha zux+\delta\alpha\beta uxy+\alpha\beta\gamma xyz=0,$$

that is, passes through the six points of intersection of the lines x, y, z, u. Hence we see that the problem, to describe a quadrilateral to have its six intersections of sides on a cubic so that the middle points of the diagonals should lie on a given line, admits of three solutions; for there are three cubics which have a given cubic for a Hessian.

When a cubic and a conic are written in the forms (1), (2), a certain covariant conic is also expressible linearly in terms of the squares of x, y, z, u. This covariant may be found as follows. Writing a line in the form

$$\lambda x+\mu y+\nu z+\rho u=0,$$

its polar conic with regard to the cubic, namely, the envelope of the polars with regard to the cubic of the points on the line, is found to be

$$\Sigma\gamma\delta zu(\lambda-\mu)^2=0 \quad\text{.........................(13).}$$

Now the tangential equation of V is

$$\Sigma cd(\lambda-\mu)^2=0 \quad\text{.........................(14).}$$

Hence, if we substitute differential symbols $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{d\nu}, \frac{d}{d\rho}$ for x, y, z, u in the former equation, and operate on the latter, we get

$$\Sigma ab\gamma\delta(\lambda-\mu)^2=0 \quad\text{.........................(15),}$$

which is therefore a contravariant conic.

But this latter is the tangential equation of

$$abcdx^2+\beta cday^2+\gamma dabz^2+\delta abcu^2=0 \quad\text{..............(16).}$$

If this conic be written

$$a'x^2+b'y^2+c'z^2+d'u^2=0,$$

we have $$aa':bb':cc':dd'=\alpha:\beta:\gamma:\delta,$$

which shows that V and the covariant (16) are reciprocally related with regard to the cubic. This will be readily seen when the cubic and conic are written in the forms (3), (4). If the tangential equation of V is then

$$(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0,$$

the tangential equation of the contravariant is

$$(m^2A-F,\ m^2B-G,\ m^2C-H,\ mA-m^2F, mB-m^2G, mC-m^2H)(\lambda, \mu, \nu)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

From these equations it is readily seen that the conic V and the contravariant will coincide if either is of the form

$$A\,(\lambda^2+2\theta\mu\nu)+B\,(\mu^2+2\theta\nu\lambda)+C\,(\nu^2+2\theta\lambda\mu) = 0,$$

where $$\theta^2-2m^2\theta+m = 0,$$

that is, is a derived conic of either of the curves of the third class

$$\lambda^3+\mu^3+\nu^3+6\theta\lambda\mu\nu = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(18).$$

If one of the coefficients of V in (2) vanish, d say, the contravariant becomes, from (15), after dividing by δ,

$$bca\,(\mu-\nu)^2+ca\beta\,(\nu-\lambda)^2+ab\gamma\,(\lambda-\mu)^2 = 0,$$

that is, it breaks up into factors, the two points corresponding to which lie on the line $u = 0$. Now x, y, z form a triangle inscribed in the Hessian, so that the points where the sides meet the curve again lie on a line; hence, if it is possible that such a triangle should be self-conjugate with regard to a conic, the invariant relation connecting the cubic and the conic is found by taking the discriminant of (17). This relation is therefore of the sixth degree in the coefficients of the cubic, and the third degree in the tangential coefficients of the conic.

We now proceed to show how to obtain the conditions that a conic should be circumscribed about or inscribed in a triangle which is inscribed in the Hessian, so that the points where the sides meet the curve again are collinear. Suppose that the cubic is

$$ax^3+\beta y^3+\gamma z^3+\delta u^3 = 0,$$

and if a conic circumscribes the triangle xyz, we may write it

$$fyz+gzx+hxy = 0.$$

Substituting the differential symbols $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{d\nu}$ for x, y, z, respectively, in the latter equation, and operating on the tangential equation of the polar conic of a point, namely,

$$\Sigma \gamma\delta zu\,(\lambda-\mu)^2 = 0,$$

we get a covariant conic which is found to be

$$\delta u\,(fax+g\beta y+h\gamma z) = 0,$$

that is, it breaks up into factors in the case under consideration. When the cubic and conic are written in the forms (3), (4), the covariant is

$$a\,(yz-m^2x^2)+b\,(zx-m^2y^2)+c\,(xy-m^2z^2)+2f\,(m^2yz-mx^2)$$
$$+2g\,(m^2zx-my^2)+2h\,(m^2xy-mz^2) = 0 \qquad (19).$$

It is thus linear in the coefficients of the conic, and of the second degree in those of the cubic, so that its discriminant, or the condition required, is of the third degree in the coefficients of the conic and of the sixth degree in those of the cubic.

For example, if the circumscribing conic is a circle which cuts a given circle orthogonally, the foregoing relation shows that its centre lies on a given curve of the third degree. The conic and the covariant (19), it is easy to see, are reciprocally related with regard to the cubic; in fact, one passes through the points on the Hessian corresponding to those in which the other intersects it. This follows from the fact that, for points on the Hessian, the polar conics break up into pairs of lines intersecting at the corresponding points. Hence we see that the result given above follows at once; for, when three points on the Hessian are collinear, the corresponding points form a triangle whose sides meet the curve again in three points on a line.

It may be noticed from this that, if a conic touch the Hessian in some point, its covariant (19) will touch the Hessian in the corresponding point.

We now proceed to find the conditions that a cubic and a conic can be reduced to the forms

$$U \equiv ax^3+\beta y^3+\gamma z^3+\delta u^3 = 0,$$

$$V \equiv l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy = 0.$$

Now, the tangential equation of V being

$$l\mu\nu+m\nu\lambda+n\lambda\mu = 0,$$

and the polar conic of any point on the line u being

$$\alpha x'x^2+\beta y'y^2+\gamma z'z^2=0,$$

the result of operating with one upon the other vanishes. Again, the polar conic of u with respect to U is

$$\beta\gamma yz+\gamma\alpha zx+\alpha\beta xy=0,$$

which circumscribes a triangle circumscribed about V; therefore the invariant relation $\Theta^2=4\Delta\Theta'$ is satisfied between these two conics (see Salmon's *Conics*, Art. 376).

Now, in the forms (3), (4), let the line u be

$$\lambda x+\mu y+\nu z=0;$$

then, since the result of operating on the polar conic of x, y, z is

$$(A+2mF)\,x+(B+2mG)\,y+(C+2mH)\,z=0,$$

we must have $A+2mF:B+2mG:C+2mH=\lambda:\mu:\nu$.

We then form the invariant relation

$$\Theta^2-4\Delta\Theta'=0$$

between the conic and the polar conic of

$$\lambda x+\mu y+\nu z=0,$$

namely, $$\lambda^2(yz-m^2x^2)+\mu^2(zx-m^2y^2)+\nu^2(xy-m^2z^2)$$
$$+2\mu\nu(m^2yz-mx^2)+2\nu\lambda(m^2zx-my^2)+2\lambda\mu(m^2xy-mz^2)=0,$$

and find $$m^4(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\,\lambda\mu\nu$$
$$+2(m-m^4)\left\{F(m\lambda^2-\mu\nu)+G(m\mu^2-\nu\lambda)+H(m\nu^2-\lambda\mu)\right\}=0,$$

or $$m(\lambda^3+\mu^3+\nu^3)+(1-4m^3)\,\lambda\mu\nu$$
$$-(m-m^4)(A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu)=0\;\ldots(20).$$