On a Law of Combination of Operators bearing on the Theory of Continuous Transformation Groups. By J. E. CAMPBELL.

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Let y and x denote any operators which obey the distributive and associative laws, but not necessarily the commutative.

Let y_1 denote the operation $yx - xy$,

y_2 " " $y_1x - xy_1$,

and y_r " " $y_{r-1}x - xy_{r-1}$.

Let $[yx^r]$ denote the sum of the operations

$$yx^r + xyx^{r-1} + x^2yx^{r-2} + \ldots + x^ry.$$

The theorem to be proved is the following:—If

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots$$

is a series of numerical constants, of which a_1 is $\frac{1}{2}$, a_2 is $\frac{1}{12}$, a_3 is zero, $a_4 = -\frac{1}{720}$, and of which the law of formation is

$$(m+1)\,a_m = a_{m-1} - (a_1a_{m-1} + a_2a_{m-2} + \ldots + a_{m-1}a_1),$$

then
$$\frac{yx^r}{r!} = \left[y\frac{x^r}{(r+1)!}\right] + a_1\left[y_1\frac{x^{r-1}}{r!}\right] + a_2\left[y_2\frac{x^{r-2}}{(r-1)!}\right] + \ldots$$
$$\ldots + a_{r-1}\left[y_{r-1}\frac{x}{2!}\right] + a_ry_r.$$

From the fact that a_3 is zero, the law of formation at once shows that a_5, a_7, ..., the series of constants with odd suffixes, are all zero.

[This series of numbers was, I believe, discovered by Schur, in his investigation of the same problem which led me to consider them. A sketch of part of his work is given in Lie's *Transformationsgruppen*, III., § 144. I have consulted his writings in *Math. Annal.*, Bd. XXXV., § 161, which bear most closely on my results, but have not been able to consult the references given in Lie to *Leipz. Ber.*, 1889, § 229, and 1890, § 1. I had independently arrived at the law of their formation, but had not noticed that the odd numbers were all zero except a_1.]

Two lemmas are required for the proof, the first,

$$x^r y_m + [y_m x^{r-1}]\, x = [y_m x_r],$$

is obvious from the definition of the symbol [ ].

The second, $$y_m x^r = x^r y_m + [y_{m+1} x^{r-1}],$$

may be proved by induction.

When $r = 1$,

$$(1) \quad y_m x = x y_m + y_{m+1};$$

therefore $$y_m x^2 = x y_m x + y_{m+1} x;$$

therefore $$y_m x^2 = x(x y_m + y_{m+1}) + y_{m+1} x$$
$$= x^2 y_m + [y_{m+1} x].$$

Thus the result holds for $r = 2$.

Assume it holds for r;

$$y_m x^r = x^r y_m + [y_{m+1} x^{r-1}];$$

therefore $$y_m x^{r+1} = x^r y_m x + [y_{m+1} x^{r-1}]\, x;$$

therefore, from (1), $$= x^r (x y_m + y_{m+1}) + [y_{m+1} x^{r-1}]\, x$$
$$= x^{r+1} y_m + [y_{m+1} x^r],$$

by Lemma I., so that Lemma II. is established also.

Assume that for all integral values, up to and including r, it has been established that

$$y \frac{x^r}{r!} = \left[\frac{y x^r}{(r+1)!}\right] + a_1 \left[y_1 \frac{x^{r-1}}{r!}\right] + \ldots + a_r y_r.$$

[It would, of course, follow that we have also

$$y_1 \frac{x^r}{r!} = \left[y_1 \frac{x^r}{(r+1)!}\right] + a_1 \left[y_2 \frac{x^{r-1}}{r!}\right] + \ldots + a_r y_{r+1},$$

$$y_2 \frac{x^r}{r!} = \left[y_2 \frac{x^r}{(r+1)!}\right] + a_1 \left[y_3 \frac{x^{r-1}}{r!}\right] + \ldots + a_r y_{r+2},$$

&c.],

then it will be proved that the theorem holds also for the value $r+1$.

It is obvious that the theorem holds for $r = 1$. [I have also verified it for the cases $r = 2$, $r = 3$, and $r = 4$.]

Since

$$y\frac{x^r}{r!}=\left[y\frac{x^r}{(r+1)!}\right]+a_1\left[y_1\frac{x^{r-1}}{r!}\right]+\ldots,$$

therefore

$$y\frac{x^{r+1}}{r!}=\left[y\frac{x^r}{(r+1)!}\right]x+a_1\left[y_1\frac{x^{r-1}}{r!}\right]x+\ldots$$

$$=\left[y\frac{x^{r+1}}{(r+1)!}\right]+a_1\left[y_1\frac{x^r}{r!}\right]+\ldots-\frac{x^{r+1}}{(r+1)!}y-a_1\frac{x^r}{r!}y_1-\ldots,$$

by Lemma I.; therefore

$$y\frac{x^{r+1}}{r!}+y\frac{x^{r+1}}{(r+1)!}+a_1y_1\frac{x^r}{r!}+a_2y_2\frac{x^{r-1}}{(r-1)!}+\ldots+a_ry_rx$$

$$\equiv\left[y\frac{x^{r+1}}{(r+1)!}\right]+a_1\left[y_1\frac{x^r}{r!}\right]+\ldots+a_r[y_rx]$$

$$+\left[y_1\frac{x^r}{(r+1)!}\right]+a_1\left[y_2\frac{x^{r-1}}{r!}\right]+\ldots+a_ry_{r+1},$$

by Lemma II.

Expanding all the terms

$$y_1\frac{x^r}{r!},\quad y_2\frac{x^{r-1}}{(r-1)!},\ \ldots\ y_rx,$$

by the theorem which has been assumed to hold for all integral values up to and including r, and subtracting from the two sides of the equation, we see that

$$y\frac{x^{r+1}}{r!}+y\frac{x^{r+1}}{(r+1)!}\equiv\left[y\frac{x^{r+1}}{(r+1)!}\right]+\sum_{m=1}^{m=r+1}b_m\left[y_m\frac{x^{r-m+1}}{(r-m+2)!}\right];$$

where

$$b_m=a_{m-1}+(r-m+2)\,a_m-\left[a_1a_{m-1}+a_2a_{m-2}+\ldots+a_{m-1}a_1\right]-a_m$$

$$=(r+2)\,a_m,$$

by the law of formation of the coëfficients.

Dividing each side of the identity by $r+2$, we see that

$$y\frac{x^{r+1}}{(r+1)!}=\left[y\frac{x^{r+1}}{(r+2)!}\right]+a_1\left[y_1\frac{x^r}{(r+1)!}\right]+\ldots+a_r\left[y_r\frac{x}{2!}\right]+a_{r+1}y_{r+1};$$

that is, the theorem also holds for $r+1$, and therefore holds universally, since it obviously holds for $r=1$.

Since y and x are symbols obeying the distributive and associative law,

$$(x+\mu y)^r = x^r + \mu\left[yx^{r-1}\right] + \text{terms involving higher powers of } \mu;$$

so that, if we take μ a constant so small that its square and higher powers may be neglected,

$$(x+\mu y)^r = x^r + \mu\left[yx^{r-1}\right].$$

Let $$z \equiv y + a_1 y_1 + a_2 y_2 + \dots;$$

then, as above, $$(x+\mu z)^r = x^r + \mu\left[zx^{r-1}\right] + \dots.$$

From the theorem we have established, we have the following equations:—

$$\begin{aligned}
y &= y, \\
yx &= \tfrac{1}{2}[yx] + a_1 y_1, \\
y\frac{x^2}{2!} &= \left[y\frac{x^2}{3!}\right] + a_1\left[y_1\frac{x}{2!}\right] + a_2 y_2, \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
y\frac{x^r}{r!} &= \left[y\frac{x^r}{(r+1)!}\right] + a_1\left[y_1\frac{x^{r-1}}{r!}\right] + \dots, \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Adding these expressions, we get

$$y\epsilon^x = z + \left[z\frac{x}{2!}\right] + \left[z\frac{x^2}{3!}\right] + \dots \text{ to infinity};$$

therefore

$$(1+\mu y)\,\epsilon^x = 1 + x + \mu z + \frac{x^2}{2!} + \mu\left[z\frac{x}{2!}\right] + \dots + \frac{x^r}{r!} + \mu\left[z\frac{x^{r-1}}{(r-1)!}\right] + \dots;$$

therefore

$$(1+\mu y)\,\epsilon^x = 1 + x + \mu z + \frac{(x+\mu z)^2}{2!} + \dots + \frac{(x+\mu z)^r}{r!} + \dots,$$

if μ is a small constant whose square and higher powers may be neglected.

[It might be objected that we are dealing with an infinite series of operations, and that, for instance, the coefficient of μ^2 is an operation the result of which R when applied to any function might not be a convergent series; and hence $\mu^2 R$ could not be neglected in comparison with μ.

The limitation to be placed upon the subject of the operation is that, when operated upon by

$$1+(\lambda x+\mu y)+\frac{1}{2!}(\lambda x+\mu y)^2+\ldots \text{ to infinity}$$

(where λ and μ are constants), it will give a convergent series.

The coefficients of the different powers and products of λ, μ will then give convergent series.

The limitation has been implicitly assumed in the proof of Lie's theorem for

$$x_i'=1+(\lambda_1 X_1+\ldots)\,x_i+\frac{1}{2!}(\lambda_1 X_1+\ldots)^2 x_i+\ldots,$$

and it is assumed that x_i' is finite and definite.]

Let X denote the linear operator

$$\sum_{i=1}^{i=n}\xi_i\,(x_1, x_2, \ldots x_n)\,\frac{\partial}{\partial x_i};$$

and Y the operator $\displaystyle\sum_{i=1}^{i=n}\eta_i\,(x_1, x_2, \ldots x_n)\,\frac{\partial}{\partial x_i}$;

let Y' denote the linear operator

$$\sum_{i=1}^{i=n}\eta_i\,(x_1', x_2', \ldots x_n')\,\frac{\partial}{\partial x_i'},$$

obtained from Y by writing for x_i, x_i', where

$$x_i'=\left(1+tX+t^2\frac{X^2}{2!}+\ldots\right)x_i,$$

t being a constant; then it will be proved that

$$Y'\equiv Y-tY_1+\frac{t^2}{2!}Y_2-\frac{t^3}{3!}Y_3+\ldots,$$

where

$$Y_1\equiv YX-XY,$$
$$Y_2\equiv Y_1X-XY_1,$$
$$Y_r\equiv Y_{r-1}X-XY_{r-1},$$

all of the Y's being linear operators.

[This is the completion of the theorem given by Lie, *Transformationsgruppen*, I., p. 141 where he neglects powers of t above the first.]

Now $$Y'\equiv Y'(x_1)\frac{\partial}{\partial x_1}+Y'(x_2)\frac{\partial}{\partial x_2}+\ldots,$$

this being a general property of all linear differential operators;

and $$x_i \equiv \left(1 - tX' + \frac{t^2}{2!} X'^2 - \dots\right) x'_i \quad (a)$$ [Lie, I., p. 53 (7a)];

therefore $$Y'(x_i) \equiv Y'\left(1 - tX' + \frac{t^2}{2!} X'^2 - \dots\right) x'_i;$$

therefore since the right-hand member is now a function $x'_1, x'_2, \dots,$ we have [Lie, I., p. 52 (7)]

$$(\beta) \quad Y'(x_i) \equiv \left(1 + tX + \frac{t^2}{2!} X^2 + \dots\right) Y \left(1 - tX + \frac{t^2}{2!} X^2 - \dots\right) x_i.$$

[For convenience of reference, I give a proof of these theorems only slightly modified from Lie's proof.

Since $$x'_i = \left(1 + tX + t^2 \frac{X^2}{2!} + \dots\right) x_i,$$

therefore $$\frac{\partial x'_i}{\partial t} = X\left(1 + tX + t^2 \frac{X^2}{2!} + \dots\right) x_i$$

$$= Xx'_i;$$

therefore $$\frac{\partial f(x'_1, x'_2, \dots x'_n)}{\partial t} \equiv \frac{\partial f}{\partial x'_1} Xx'_1 + \dots$$

$$\equiv Xf(x'_1 \dots x'_n).$$

Similarly, $$\frac{\partial^r f'}{\partial t^r} \equiv X^r f',$$

writing f' for $f(x'_1 \dots x'_n)$.

Now, by Taylor's theorem,

$$f' = (f')_{t=0} + t\left(\frac{\partial f'}{\partial t}\right)_{t=0} + \dots.$$

Now, $$(f')_{t=0} \equiv f \text{ and } (X^r f')_{t=0} \equiv X^r f;$$

therefore $$f' = f + tXf + \frac{t^2 X^2}{2!} f + \dots.$$

This is the theorem of which (β) is a particular case. Again, $X'^r f'$ is a function of $x' \dots x'_n$, and therefore, by the above theorem,

$$X'^r f' \equiv \left(1 + tX + \frac{t^2 X^2}{2!} + \dots\right) X^r f;$$

therefore $\left(1-tX'+\frac{t^2X'^2}{2!}-\dots\right)f'$

$$\equiv\left(1+tX+\frac{t^2X^2}{2!}+\dots\right)\left(1-tX+\frac{t^2X^2}{2!}-\dots\right)f\equiv f$$

(proof exactly the same as proof that $\epsilon^x\epsilon^{-x}=1$).

This is the theorem of which (a) is a particular case.]

Expanding the right-hand member of β, we see that it is equal to

$$\{Y+t(XY-YX)+\dots\}\,x_i.$$

It is at once seen that the coefficient of t^r in the bracket is

$$\frac{X^rY}{r!}-\frac{X^{r-1}Y}{(r-1)!}\frac{X}{1!}+\frac{X^{r-2}Y}{(r-2)!}\frac{X^2}{2!}-\dots;$$

and it is to be shown that this is

$$(-1)^r\frac{Y_r}{r!}.$$

Assume $\quad (-1)^{r-1}\dfrac{Y_{r-1}}{(r-1)!}=\dfrac{X^{r-1}Y}{(r-1)!}-\dfrac{X^{r-2}Y}{(r-2)!}\dfrac{X}{1!}+\dots;$

then $\quad (-1)^{r-1}\dfrac{\{Y_{r-1}X-XY_{r-1}\}}{(r-1)!}$

$$=-\frac{X^rY}{(r-1)!}+r\frac{X^{r-1}Y}{(r-1)!}\frac{X^1}{1!}-r\frac{X^{r-2}Y}{(r-2)!}\frac{X^2}{2!}+\dots;$$

therefore $\quad \dfrac{(-1)^rY_r}{r!}=\dfrac{X^rY}{r!}-\dfrac{X^{r-1}Y}{(r-1)!}\dfrac{X}{1!}+\dots,$

so that the theorem, being true when r is 1, is true universally.

Therefore $\quad Y'(x_i)\equiv\left(Y-tY_1+\dfrac{t^2}{2!}Y_2-\dots\right)x_i,$

but $\quad Y'\equiv Y'(x_1)\dfrac{\partial}{\partial x_1}+Y'(x_2)\dfrac{\partial}{\partial x_2}+\dots,$

$$Y_1\equiv Y_1(x_1)\frac{\partial}{\partial x_1}+Y_1(x_2)\frac{\partial}{\partial x_2}+\dots,$$

with similar expressions for Y_2, Y_3, ... ;

therefore $\quad Y'\equiv Y-tY_1+\dfrac{t^2}{2!}Y_2-\dfrac{t^3}{3!}Y_3+\dots.$

I propose to employ these results to prove the theorem given

(*Transformationsgruppen*, I., p. 158) by Lie, and forming the foundation of his theory.

That theorem might be thus stated:—If $x_1', x_2', \dots x_n'$ is a point obtained from the point $x_1, x_2, \dots x_n$ by the operation

$$1+X+\frac{X^2}{2!}+\dots;$$

and $x_1'', x_2'', \dots x_n''$ is a point obtained from the point $x_1', x_2', \dots x_n'$ by the operation

$$1+Y'+\frac{Y'^2}{2!}+\dots,$$

where $$X \equiv \lambda_1 X_1 + \dots \lambda_r X_r,$$

and $$Y \equiv \mu_1 X_1 + \dots \mu_r X_r,$$

X_k denoting the linear operator

$$\sum_{i=1}^{i=n} \xi_{ki}(x_1, x_2, \dots x_n) \frac{\partial}{\partial x_i},$$

then $x_1'', x_2'', \dots x_n''$ can be directly derived from the point $x_1, x_2, \dots x_n$ by the operation

$$1+Z+\frac{Z^2}{2!}+\dots,$$

where $$Z \equiv \nu_1 X_1 + \dots \nu_r X_r,$$

provided that, for all values of k, j,

$$X_k X_j - X_j X_k \equiv \sum_{s=1}^{s=r} c_{kjs} X_s.$$

In Lie's theorem the sets λ, μ, ν and c are all constants; I shall prove that the same result holds if they are any functions of the variables.

It has been proved [Lie I., § 13] that* every transformation of the simple group

$$1+tY+\frac{t^2}{2!}Y^2+\dots$$

can be obtained through repeated operations with the infinitesimal transformation $1+\partial t Y$; it will therefore be sufficient to prove the

* [Just as in ordinary algebra, we see that

$$\left(1+\frac{tY}{n}\right)^n \equiv 1+tY+\frac{t^2}{2!}Y^2+\dots,$$

when n is taken a very large integer.]

theorem for the case when $x_1'', x_2'', \dots x_n''$ is indefinitely near to $x_1', x_2', \dots x_n'$.

We have to prove therefore that, $\mu_1, \mu_2, \dots \mu_r$ being so small that their squares may be neglected,

$$(1+\mu_1 X_1'+\mu_2 X_2'+\mu_r X_r')\, x_i' = \left(1+Z+\frac{Z^2}{2!}+\frac{Z^3}{3!}+\dots\right) x_i$$

X' denoting the result of substituting x' for x in X.

Now we have proved that

$$Y' \equiv Y - tY_1 + \frac{t^2}{2!} Y_2 - \dots;$$

and, by our hypothesis, Y_1 must belong to the family

$$\rho_1 X_1 + \rho_2 X_2 + \dots + \rho_r X_r,$$

where $\rho_1, \rho_2, \dots$ are some functions of the variables $x_1, x_2, \dots x_n$; and, since Y_1 belongs to the family, so also must Y_2, and, by parity of reasoning, $Y_3, Y_4, \dots$. That is, Y' belongs to the family

$$\rho_1 X_1 + \rho_2 X_2 + \dots + \rho_r X_r;$$

the theorem required will then be proved if we can prove that

$$(1+k_1 X_1 + k_2 X_2 + \dots + k_r X_r)\, x_i' \equiv \left(1+Z+\frac{Z^2}{2!}+\dots\right) x_i,$$

where k_1, k_2, k_r are small; or, remembering that

$$x_i' = \left(1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\dots\right) x_i,$$

we have to prove that

$$(1+kY)\left(1+X+\frac{X^2}{2!}+\dots\right) \equiv \left(1+Z+\frac{Z^2}{2!}+\dots\right),$$

where Y and X belong to the family

$$\rho_1 X_1 + \dots + \rho_r X_r,$$

and when k is now a constant so small that its square may be neglected.

Now, we have already proved that

$$(1+kY)\, \epsilon^X \equiv 1 + (X+k\overline{Y}) + \frac{(X+k\overline{Y})^2}{2!} + \dots + \frac{(X+k\overline{Y})^p}{p!} + \dots,$$

where $$\overline{Y} \equiv Y + a_1 Y_1 + a_2 Y_2 + \dots;$$

Y, Y_1, Y_2 being each members of the family, and therefore $\overline{Y}$ also being a member; that is,

$$(1+kY)\,\epsilon^X \equiv \epsilon^Z,$$

where Z is a member of the family. This proves the generalization of Lie's theorem.

It might appear that, in taking k a constant, the proof of the generalization was vitiated, but this is not so; the variables come in through k_1, k_2, ..., and k is merely introduced to make kY small, *e.g.*, we might take $k = \dfrac{1}{m}$ where m is any large integer.

Some Notes on Symmetric Functions. By William H. Metzler.

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1. In this paper I wish to state three laws by means of which certain symmetric functions are immediately obtained from those already known.

Let $g_1, g_2, g_3, \ldots g_n$ represent the n roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0, \tag{1}$$

and let $a_1, a_2, a_3, \ldots a_r$ and $b_1, b_2, b_3, \ldots b_r$ represent the

$$\lambda = n_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

products of those roots r and $n-r$ at a time respectively.

The coefficient p_i may be said to be complementary to p_{n-i} with respect to n.

The first law in question may be stated as follows:—

If we have given the value, in terms of the coefficients, of the symmetric functions $$\Sigma g_1^\alpha g_2^\beta g_3^\gamma \ldots g_n^\kappa, \tag{a}$$

where $\alpha \geqq \beta \geqq \gamma \geqq \ldots \geqq \kappa \geqq 0$ and $\alpha+\beta+\gamma+\ldots+\kappa = n$,

we can immediately write down all those terms, involving the coefficients $(p_1, p_2, \ldots p_n)$ only in the value of

$$\Sigma g_1^{\alpha'-\kappa} \ldots g_{n-2}^{\alpha'-\gamma} g_{n-1}^{\alpha'-\beta} g_n^{\alpha'-\alpha}, \tag{b}$$