THE CANONICAL FORMS OF THE TERNARY SEXTIC AND QUATERNARY QUARTIC

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LET Q be a homogeneous sextic in x, y, z; it is known* that there exist linear expressions L_1, L_2, ... and constants A_1, A_2, ... such that Q is identically equal to

$$\sum_1^{10} A_r L_r^6,$$

and, further, that L_{10} may be arbitrarily chosen. I propose to shew how the reduction is to be actually carried out, and to find the number of ways in which it is possible.

Since a cubic can be described through nine points, there must be a contracubic, U, such that

$$\Delta^3 U L_r^6 = 0 \quad (r = 1, 2, \dots, 9),$$

and therefore

$$\Delta^3 UQ = A_{10} \Delta^3 U L_{10}^6,$$

a simple multiple of L_{10}^3. The ratios of the ten coefficients in U are uniquely determined by the fact that $\Delta^3 UQ$ is to be a constant multiple of L_{10}^3, and the value of A_{10} is thus also uniquely determined.

Hence the problem is to reduce the known sextic $Q - A_{10} L_{10}^6$, which is destroyed by the known cubic operator $\Delta^3 U$, to the form

$$\sum_1^9 A_r L_r^6.$$

Now, if we take 18 arbitrary tangents L_1', ..., L_{18}' to the contracubic U, we can express $Q - A_{10} L_{10}^6$ in the form

$$\sum_1^{18} A_r' L_r'^6.$$

* See, for instance, Palatini (*Rom. Acc. Lincei, Rendiconti*, Vol. XII.). For the notation see a paper on the the ternary quintic and septimic by Dr. Stuart and the writer (*ante*, p. 160).

With 19 arbitrary tangents or with 18 that touch a contrasextic there is a syzygy of the form $$\Sigma A_r L_r^6 = 0.$$

We shall discuss this syzygy and deduce a method for reducing

$$\sum_1^{18} A'_r L'^6_r$$

to an expression of the same form containing only nine terms.

Since L_r is a tangent to a given contracubic, we may, by a suitable linear transformation, put

$$L_r = (x - y\wp u_r - \wp' u_r) = \theta \,.\, \sigma(u_r - a)\sigma(u_r - b)\sigma(u_r + a + b)/\sigma^3 u_r,$$

where a, b, θ depend on x, y.

Now let $v_1, \ldots, v_{18}$ be any 18 arguments whose sum is 0 and let

$$u_1 + u_2 + \ldots + u_{19} = w;$$

the function $$\sigma(u-w)\prod_1^{18}\sigma(u-v_r)\Big/\prod_1^{19}\sigma(u-u_r)$$

is doubly periodic in u and its poles are $u_1, u_2, \ldots, u_{19}$. Hence the sum of the residues

$$\frac{\sigma(u_1 - w)\prod_1^{18}\sigma(u_1 - v_r)}{\prod_2^{19}\sigma(u_1 - u_r)} + \ldots = 0.$$

Putting $v_1 = v_2 = \ldots = v_6 = a, \quad v_7 = v_8 = \ldots = v_{12} = b,$

$$v_{13} = \ldots = v_{18} = -a - b,$$

we have here the syzygy $$\sum_1^{19} A_r L_r^6 = 0,$$

and it appears that

$$A_r = \sigma(u_r - w)(\sigma u_r)^{18}\Big/\prod_{\substack{s=1,2,\ldots,19\\ s\neq r}}\sigma(u_r - u_s).$$

The next thing is to find $u_1, u_2 \ldots, u_9$ when $u_{10}, \ldots, u_{19}$ and the ratios $A_{10} : A_{11} : \ldots : A_{19}$ are given. Let $u_1 + u_2 + \ldots + u_9 = \alpha$, $u_{10} + \ldots + u_{19} = \beta$, and consider the function

$$F(u) \equiv \sigma^2(u-\beta)\prod_1^9\sigma(u-u_r)\Big/\sigma(u-\alpha-\beta)\prod_{10}^{19}\sigma(u-u_r).$$

$F(u)$ is doubly periodic and its residue at the pole u_r ($r = 10, \ldots, 19$) is

$$B_r = \frac{1}{A_r}(\sigma u_r)^{18}\sigma^2(u_r - \beta)\Big/\prod_{\substack{s=10,\ldots,19\\ s\neq r}}\sigma^2(u_r - u_s).$$

It has one other pole, $a+\beta$, and thus

$$F(u) = \sum_{r=10}^{19} B_r \{ \zeta(u-u_r) - \zeta(u-a-\beta) \} + B,$$

where B is constant.

We also have $F(\beta) = 0$, $F'(\beta) = 0$. From the second of these

$$\sum_{10}^{19} B_r \{ \wp(\beta - u_r) - \wp a \} = 0,$$

which gives $\wp a$ uniquely, since β and the ratios $B_{10} : B_{11} : \ldots : B_{19}$ are known. There are two distinct values of a; taking either of them, we find B from the condition

$$F(\beta) = 0,$$

and then $u_1, u_2, \ldots, u_9$ are the zeros of $F(u)$ other than β.

Hence it follows that the sum of ten terms of the form AL^6 can be reduced to nine in two ways. The two sets of nine must be tangents to the same contrasextic; this follows either from the syzygy * or from the fact that the two values of a are equal and opposite.

The sum of 18 terms can be reduced successively to 17, 16, ..., and lastly to 9. Hence the original problem is solved and the number of solutions is two.

The reduction from 18 to 9 may be accomplished directly as follows :—

If $n > 18$, there will be $n-18$, say m, syzygies

$$\sum_1^n A_r L_r^6 = 0.$$

To exhibit them, take the doubly periodic function

$$\prod_1^m \sigma(u-w_r) \prod_1^{18} \sigma(u-v_r) \Big/ \prod_1^n \sigma(u-u_r),$$

where $\Sigma v_r = 0$ as before, $\Sigma w_r = \Sigma u_r$. The sum of the residues is again zero, and we deduce, as before, that

$$A_r = \prod_{s=1}^m \sigma(u_r - w_s)(\sigma u_r)^{18} \Big/ \prod_{\substack{s=1,2,\ldots,n \\ s \neq r}} \sigma(u_r - u_s).$$

Here $w_1, w_2, \ldots, w_{m-1}$ are arbitrary.

To make the reduction we must take $n = 27$, suppose $u_{10}, u_{11}, \ldots, u_n$ and $A_{10} : A_{11} : \ldots : A_n$ given, and find the values of $u_1, u_2, \ldots, u_9$, $w_1, w_2, \ldots, w_m$.

* It could be seen from the beginning that not more than two solutions were possible; for, if we had $\sum_1^9 AL^6 = \sum_1^9 A'L'^6 = \sum_1^9 A''L''^6$, the lines L would be residual to L' and also to L'', and the lines L' to L''. Hence each set of nine would be residual to itself and would not be of full generality.

Consider the function

$$F(u) \equiv \prod_1^{2m} \sigma(u-\beta_r) \prod_1^{9} \sigma(u-u_r) \Big/ \prod_1^{m} \sigma(u-w_r) \prod_{10}^{n} \sigma(u-u_r),$$

where $\beta_1, \beta_2, \ldots, \beta_{2m}$ are any fixed arguments whose sum is $2(u_{10}+\ldots+u_n)$. We have

$$F(u) = \sum_{10}^{n} B_r \zeta(u-u_r) + \sum_1^{m} C_r \zeta(u-w_r) + C,$$

where $$A_r B_r = (\sigma u_r)^{18} \prod_{s=1}^{2m} \sigma(u_r-\beta_s) \Big/ \prod_{\substack{s=10,\ldots,n \\ s \neq r}} \sigma^2(u_r-u_s),$$

so that the ratios of $B_{10}, B_{11}, \ldots, B_n$ are known, while $C, C_1, C_2, \ldots$ are to be determined.

Let $$\sum_{10}^{n} B_r \{\zeta(u-u_r)-\zeta u\} = G(u),$$

$$\sum_1^{m} C_r \{\zeta(u-w_r)-\zeta u\} + C = H(u),$$

so that $$G(u)+H(u) = F(u).$$

Then $G(u)$ is a known function, and $H(u)$ is a doubly periodic function, of which we know the following facts :—

(1) $H(\beta_r) = -G(\beta_r) \quad (r = 1, 2, \ldots, 2m)$;

(2) $H(0)$ is infinite, the residue being the known quantity ΣB_r $(= -\Sigma C_r)$;

(3) $H(u)$ has $m+1$ poles in all.

Hence we may put

$$H(u) = \frac{M+M_0\wp u+M_1\wp' u+\ldots+M_m\wp^{(m)}u}{N+N_0\wp u+N_1\wp' u+\ldots+N_m\wp^{(m)}u};$$

from (2) we have $$N_m = 0$$

and $$(m+1)M_m+\Sigma B_r N_{m-1} = 0;$$

from (1) we have $2m$ other equations linear in the $2m+4$ unknowns $M, N, \ldots$. Thus all the coefficients are determined in terms of two of them, say M, N and $H(u) = \{MP_1(u)+NP_2(u)\}/\{MQ_1(u)+NQ_2(u)\}$, where P_1, P_2, Q_1, Q_2 are known functions with poles at 0 of orders $m+2$, $m+2$, $m+1$, $m+1$. To satisfy (3) we must make the numerator and denominator of $H(u)$ have a common zero ; this will be a zero of

$$P_1Q_2-P_2Q_1.$$

Now $P_1Q_2-P_2Q_1$ has the $2m$ known zeros $\beta_1, \beta_2, \ldots, \beta_{2m}$, and its pole 0 is only of order $2m+2$. Hence it has two other zeros, say γ_1, γ_2.

Taking either of these, γ, we can complete the determination of $H(u)$. Then $u_1, u_2, \ldots, u_9$ are fixed as the unknown roots of the equation

$$G(u)+H(u) = 0,$$

and when they are determined the problem is solved: $w_1, w_2, \ldots, w_m$ are the zeros, other than γ, of the denominator of $H(u)$. The two values of Σw are $-\gamma_1$, $-\gamma_2$, whose sum is $\Sigma\beta$ or $2\sum_{10}^{n} u_r$. Hence, as before, the two values of $\sum_1^9 u_r$ are equal and opposite.

The use of elliptic functions in the discussion is convenient, but not essential. As a matter of actual working, the process of reduction is algebraic.

The Quaternary Quartic.

Let Q be a homogeneous quartic in x, y, z, w. The problem is to reduce it to the form $\sum_1^{10} A_r L_r^4$, where L_r is linear (see Reye, *Crelle's Journal*, Vol. LXXVIII., pp. 123–9). Here, again, L_{10} may be chosen at will, and the quadratic operator $\Delta^2 U$ (notation as in the ternary case) which destroys L_r^n $(r = 1, 2, \ldots, 9)$ is determined uniquely by the fact that it reduces Q to a multiple of L_{10}^2; A_{10} is also determined. Then L_9 can be chosen at will among the tangent planes to $U = 0$; the quadratic operator $\Delta^2 V$ which reduces Q to a multiple of L_9^2 can be found, as also A_9, and again uniquely.

We now have a known quartic $Q - A_9 L_9^4 - A_{10} L_{10}^4$ destroyed by two known quadratic operators $\Delta^2 U$, $\Delta^2 V$. Such a quartic can be put in the form $\sum_1^{16} A'_r L'^4_r$ where $L'_1, \ldots, L'_{16}$ are arbitrary common tangent planes to $U = 0$, $V = 0$. There would be a syzygy $\Sigma A L^4 = 0$ with 17 such planes, or with 16 if they all touched a contraquartic surface.

Here we have $L_r \propto (z - x\wp u_r - y\wp' u_r - \wp'' u_r)$ after a suitable linear transformation. The syzygy may be discussed as before, and we again conclude that Reye's reduction is possible in two ways.