X. *On the Real Nature of Symbolical Algebra.* *By* D. F. GREGORY, *B. A., Trin. Coll. Cambridge.*

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THE following attempt to investigate the real nature of Symbolical Algebra, as distinguished from the various branches of analysis which come under its dominion, took its rise from certain general considerations, to which I was led in following out the principle of the separation of symbols of operation from those of quantity. I cannot take it on me to say that these views are entirely new, but at least I am not aware that any one has yet exhibited them in the same form. At the same time, they appear to me to be important, as clearing up in a considerable degree the obscurity which still rests on several parts of the elements of symbolical algebra. Mr PEACOCK is, I believe, the only writer in this country who has attempted to write a system of algebra founded on a consideration of general principles, for the subject is not one which has much attraction for the generality of mathematicians. Much of what follows will be found to agree with what he has laid down, as well as with what has been written by the Abbé BUEE and Mr WARREN; but as I think that the view I have taken of the subject is more general than that which they have done, I hope that the following pages will be interesting to those who pay attention to such speculations.

The light, then, in which I would consider symbolical algebra, is, that it is the science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but by the laws of combination to which they are subject. And as many different kinds of operations may be included in a class defined in the manner I have mentioned, whatever can be proved of the class generally, is necessarily true of all the operations included under it. This, it may be remarked, does not arise from any analogy existing in the nature of the operations, which may be totally dissimilar, but merely from the fact that they are all subject to the same laws of combination. It is true that these laws have been in many cases suggested (as Mr PEACOCK has aptly termed it) by the laws of the known operations of number; but the step which is taken from arithmetical to symbolical algebra is, that, leaving out of view the nature of the operations which the symbols we use represent, we suppose the existence of classes of unknown operations subject to the same laws. We are thus able to prove certain relations between the different classes of operations, which, when expressed between the symbols, are called algebraical theorems. And if we can show that any operations in any science are subject to the same laws of combination as these classes, the theorems are true of these as included in the general case: Provided always, that the resulting combinations are all possible in the particular operation under consideration. For it may very well, and does actually happen, that, though each of two operations in a certain branch of science may be possible, the complex operation resulting from their combination is not equally possible. In such a case, the result is inapplicable to that branch of science. Hence we find, that one family of a class of operations may have a more general application than another family of the same class. To make my meaning more precise, I shall proceed to apply the principle I have been endeavouring to explain, by shewing what are the laws appropriate to the different classes of operations we are in the habit of using.

Let us take as usual F and f to represent any operations whatever, the natures of which are unknown, and let us prefix these symbols to any other symbols, on which we wish to indicate that the operation represented by F or f is to be performed.

I. We assume, then, the existence of two classes of operations F and f, connected together by the following laws.

$$(1.)\ \mathrm{F}\,\mathrm{F}(a) = \mathrm{F}(a). \qquad (2.)\ ff(a) = \mathrm{F}(a).$$
$$(3.)\ \mathrm{F}f(a) = f(a). \qquad (4.)\ f\mathrm{F}(a) = f(a).$$

Now, on looking into the operations employed in arithmetic, we find that there are two which are subject to the laws we have just laid down. These are the operations of addition and subtraction; and as to them the peculiar symbols of + and — have been affixed, it is convenient to retain these as the symbols of the general class of operations we have defined, and we shall therefore use them instead of F and f. As it is useful to have peculiar names attached to each class, I would propose to call this the class of *circulating* or *reproductive* operations, as their nature suggests.

Again, on looking into geometry, we find two operations which are subject to the same laws. The one corresponding to + is the turning of a line, or rather transferring of a point, through a circumference; the other corresponding to — is the transference of a point through a semicircumference. Consequently, whatever we are able to prove of the general symbols + and — from the laws to which they are subject, without considering the nature of the operations they indicate, is equally true of the arithmetical operations of addition and subtraction, and of the geometrical operations I have described. We see clearly from this, that there is no real analogy between the nature of the operations + and — in arithmetic and geometry, as is generally supposed to be the case, for the two operations cannot even be said to be opposed to each other in the latter science, as they are generally said to be. The relation which does exist is due not to any identity of their nature, but to the fact of their being combined by the same laws. Other operations might be found which could be classed under the general head we are considering. Mr PEACOCK and the Abbé BUEE consider the transference of property to be one of these; but as there is not much interest attached to it in a mathematical point of view, I shall proceed to the consideration of other operations.

II. Let us suppose the existence of operations subject to the following laws:

$$(1.)\ f_m(a).f_n(a) = f_{m+n}(a). \qquad (2.)\ f_m f_n(a) = f_{mn}(a).$$

Where f_m, f_n are different species of the same genus of operations, which may be conveniently named index-operations, as, if we define the form of f by making $f_1(a) = a$, and suppose m and n to be integer numbers, we have those operations which are represented in arithmetical algebra by a numerical index. For if m and n be integers, and the operation a^m be used to denote that the operation a has been repeated m times, then, as we know,

$$a^m . a^n = a^{m+n}. \qquad (a^m)^n = a^{mn}.$$

We have now to consider whether we can find any other actual operations besides that of repetition which shall be subject to the laws we have laid down. If we suppose that m and n are fractional instead of integer, we easily deduce from our definition that the notation $a^{\frac{p}{q}}$ is equivalent to the arithmetical operation of extracting the q^{th} root of the p^{th} power of a, or generally the finding of an operation, which being repeated q times, will give as a result the operation a^p. Thus we find, as might have been expected, a close analogy existing between the meanings of a^m when m is integer, and when it is fractional. Again, we might ask the meaning of the operation a^{-m}; and we find without difficulty, from the law of combination, that a^{-m} indicates the inverse operation of a^m, whatever the operation a may be. When, instead of supposing m to be a number integer or fractional, we suppose it to indicate any operation whatever, I do not know of any interpretation which can be given to the rotation, excepting in the case when it indicates the operation of differentiation, represented by the symbol d. For we know by TAYLOR'S theorem, that

$$\varepsilon^{h\frac{d}{dx}} f(x) = f(x+h)$$

Or,

$$a^{\frac{d}{dx}} f(x) = f(x + \log a).$$

In the case of negative indices, we have combined two different classes of operations in one manner, but we may likewise do it in another. What meaning, we may ask, is to be attached to such complex operations as $(+)^m$ or $(-)^m$? When m is an integer number, we see at once that the operation $(+)^m$ is the same as +, but $(-)^m$ becomes alternately the same as $+$ and as $-$, according as m is odd or even, whether they be the symbols of arithmetical or geometrical operations. So far there is no difficulty. But if it be fractional, what does $(+)^m$ or $(-)^m$ signify? In arithmetic, the first may be sometimes interpreted, as because $(+)^m = +$ when m is integer, $(+)^{\frac{1}{m}}$ also $= +$, and as $(-)^{2m} = +$, also $(+)^{\frac{1}{2m}} = -$: But the other symbol $(-)^m$ has, when m is a fraction with an even denominator, absolutely no meaning in arithmetic, or at least we do not know at present of any arithmetical operation which is subject to the same laws of combination as it is. On the other hand, geometry readily furnishes us with operations which may be represented by $(+)^{\frac{1}{m}}$ and $(-)^{\frac{1}{m}}$, and which are analogous to the operations represented by $+$ and $-$. The one is the turning of a line through an angle equal to $\frac{1}{m}$th of four right angles, the other is the turning of a line through an angle equal to $\frac{1}{m}$th of two right angles. Here we see that the geometrical family of operations admits of a more extended application than the arithmetical, exemplifying a general remark we had previously occasion to make. Whether when the index is any other operation, we can attach any meaning to the expression, has not yet been determined. For instance, we cannot tell what is the interpretation of such expressions as $(+)^{\frac{d}{dx}}$ or $(-)^{\frac{d}{dx}}$, or $(+)^{\log}$.

III. I now proceed to a very general class of operations, subject to the following laws:

$$(1.)\quad f(a) + f(b) = f(a+b).$$

$$(2.)\quad f_{,}f(a) = ff_{,}(a).$$

This class includes several of the most important operations which are considered in mathematics; such as the numerical operation usually represented by a, b, &c., indicating that any other operation to which these symbols are prefixed is taken a times, b times, &c.; or as the operation of differentiation indicated by the letter d, and the operation of taking the difference indicated by Δ. We therefore see what an important part this class of functions plays in analysis, since it can be at once divided into three families which are of such extensive use. This renders it advisable to comprehend these functions under a common name. Accordingly, SERVOIS, in a paper which does not seem to have received the attention it deserves, has called them, in respect of the first law of combination, *distributive* functions, and in respect of the second law, *commutative* functions. As these names express sufficiently the nature of the functions we are considering, I shall use them when I wish to speak of the general class of operations I have defined.

It is not necessary to enter at large here, into the demonstration that the symbols of differentiation and difference are subject to the same laws of combination as those of number. But it may not be amiss to say a few words on the effect of considering them in this light. Many theorems in the differential calculus, and that of finite differences, it was found might be conveniently expressed by separating the symbols of operation from those of quantity, and treating the former like ordinary algebraic symbols. Such is LAGRANGE'S elegant theorem, the first expressed in this manner, that

$$\Delta^n u_x = (\epsilon^{h \frac{d}{dx}} - 1)^n u_x;$$

or the theorem of LEIBNITZ, with many others. For a long time these were treated as mere analogies, and few seemed willing to trust themselves to a method, the principles of which did not appear to be very sound. Sir JOHN HERSCHEL was the person in this country who made the freest use of the method, chiefly, however, in finite differences. In France, SERVOIS was, I believe, the only mathematician who attempted to explain its principles, though BRISSON and CAUCHY sometimes employed and extended its application: and it was in pursuing this investigation that he was led to separate functions into distributive and commutative, which he perceived to be the properties which were the foundation of the method of the separation of the symbols, as it is called. This view, which, so far as it goes, coincides with that which it is the object of this paper to develope, at once fixes the principles of the method on a firm and secure basis. For, as these various operations are all subject to common laws of combination, whatever is proved to be true by means only of these laws, is necessarily equally true of all the operations. To this I may add, that when two distributive and commutative operations are such that the one does not act on the other, their combinations will be subject to the same laws as when they are taken separately; but when they are not independent, and one acts on another, this will no longer be true. Hence arises the increased difficulty of solving linear differential equations with variable coefficients; but for more detailed remarks on this, as well as for examples of a more extended use of the method of the separation of symbols than has hitherto been made, I refer to the Cambridge Mathematical Journal, Nos. 1, 2, and 3.

As we found geometrical operations which were subject to the laws of circulating operations, so there is a geometrical operation which is subject to the laws of distributive and permutative operations, and therefore may be represented by the same symbols. This is transference to a distance measured in a straight line. Thus if x represent a point, line, or any geometrical figure, $a(x)$ will represent the transference of this point or line; and it will be seen at once that

$$a(x) + a(y) = a(x + y);$$

or the operation a is distributive. What, then, will the compound operation $b\big(a(x)\big)$ represent? If x represent a point, $a(x)$, which is the transference of a point to a rectilinear distance, or the tracing out of a straight line, will stand for the result of the operation; and then $b\left(a(x)\right)$ will be the transferring of a line to a given distance from its original position. In order to effect this, the line must be moved parallel to itself, the effect of which will be the tracing out of a parallelogram. The effect will be the same if we suppose a to act on $b(x)$, since in this, as in the other case, the same parallelogram will be traced out: that is to say,

$$a\left(b(x)\right) = b\left(a(x)\right)$$

or a and b are commutative operations.

The binomial theorem, the most important in symbolical algebra, is a theorem expressing a relation between distributive and commutative operations, index operations, and circulating operations. It takes cognizance of nothing in these operations except the six laws of combination we have laid down, and, as we shall presently shew, it holds only of functions subject to these laws. It is consequently true of all operations which can be shewn to be commutative and distributive, though apparently, from its proof, only true of the operations of number. The difficulties attending the general proof of this theorem are well known, and much thought has been bestowed on the best mode of avoiding them. The principles I have been endeavouring to exhibit appear to me to shew in a very clear light the correctness of EULER's very beautiful demonstration. Starting with the theorem as proved for integer indices, which he uses as a suggestive form, he assumes the existence of a series of the same form when the index is fractional or negative, which may be represented by $f_m(x)$. He then considers what will be the form of the product $f_m(x) \times f_n(x)$. This form must depend only on the laws of combination to which the different operations in the expression are subject. When x is a distributive and commutative function, and m and n integer numbers, we know that $f_m(x) \times f_m(x) = f_{m+n}(x)$. Now integer numbers are one of the families of the general class of distributive and permutative functions; and if we actually multiplied the expressions $f_m(x)$ and $f_n(x)$ together, we should, even in the case of integers, make use only of the distributive and permutative properties. But these properties hold true also of fractional and negative quantities. Therefore, in their case, the form of the product must be the same as when the indices are integer numbers. Hence $f_m(x) \times f_n(x) = f_{m+n}(x)$ whether m and n be integer or fractional, positive or negative, or generally if m and n be distributive and permutative functions.

The remainder of the proof follows very readily after this step, which is the key-stone of the whole, so that I need not dwell on it longer. I will only say, that this mode of considering the subject shews clearly, that not only must the quantities under the vinculum be distributive and commutative functions, but also the index must be of the same class,—a limitation which I do not remember to have seen any where introduced. Therefore the binomial theorem does not apply to such expressions as $(1+a)^{\log}$ or $(1+a)^{\sin}$; and, though it does apply to $(1+a)^{\frac{d}{dx}}$, since both a and $\frac{d}{dx}$ are distributive and commutative operations, it does not apply to $\left(1+f(x)\right)^{\frac{d}{dx}}$, as $f(x)$ and $\frac{d}{dx}$ are not relatively commutative.

Closely connected with the binomial theorem is the exponential theorem, and the same remarks will apply equally to both. So that, in order that the relation

$$\epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

may subsist, it is necessary, and it suffices, that x should be a distributive and commutative function. On this depends the propriety of the abbreviated notation for Taylor's theorem

$$f(x+h) = \epsilon^{h\frac{d}{dx}} f(x).$$

Properly speaking, however, the symbol ϵ ought not to be used, as it implies an arithmetical relation, and instead, we ought to employ the more general symbol of $\log^{-1}$. But this depends on the existence of a class of operations on which I may say a few words.

IV. If we define a class of operations by the law

$$f(x) + f(y) = f(xy),$$

we see that, when x and y are numbers, the operation is identical with the arithmetical logarithm. But when x and y are any thing else, the function will have a different meaning. But so long as they are distributive and commutative functions, the general theorems such as

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

being proved solely from laws we have laid down, are true of all symbols subject to those laws. It happens that we are not generally able to assign any known operation to which the series is equivalent when x is any thing but a number, and we therefore say that $\log(1+x)$ is an abbreviated expression for the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$ But there may be distinct meanings for such expressions as $\log\left(1+\frac{d}{dx}\right)$ or $\log\left(\frac{d}{dx}\right)$, as there are for $\epsilon^{h\frac{d}{dx}}$, that is $\log^{-1}\left(\frac{d}{dx}\right)$. In the case of another operation, Δ, we know that $\log(1+\Delta) = \frac{d}{dx}$.

V. The last class of operations I shall consider is that involving two operations connected by the conditions

(1) $$a\,\mathrm{F}(x+y) = \mathrm{F}(x)f(y) + f(x)\,\mathrm{F}(y)$$

and (2) $$a f(x+y) = f(x)f(y) - c\,\mathrm{F}(x)\,\mathrm{F}(y).$$

These are laws suggested by the known relation between certain functions of elliptic sectors; and when a and c both become unity, they are the laws of the combinations of ordinary sines and cosines, which may be considered in geometry as certain functions of angles or circular sectors, but in algebra we only know of them as abbreviated expressions for certain complicated relations between the first three classes of operations we have considered. These relations are,

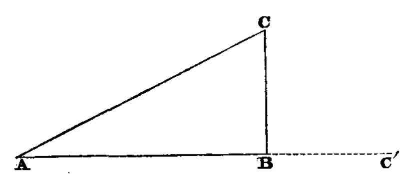
$$\mathrm{Sin}\,x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}\ \&\mathrm{c.}$$

$$\mathrm{Cos}\,x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4}\ \&\mathrm{c.}$$

The most important theorem proved of this class of functions is that of DE-MOIVRE, that

$$(\cos x + (-)^{\frac{1}{2}} \sin x)^n = \cos nx + (-)^{\frac{1}{2}} \sin nx.$$

It is easy to see that, in arithmetical algebra, the expression $\cos x + (-)^{\frac{1}{2}} \sin x$ can receive no interpretation, as it involves the operation $(-)^{\frac{1}{2}}$. In geometry, on the contrary, it has a very distinct meaning. For if a represent a line, and $a \cos x$ represent a line bearing a certain relation in magnitude to a, and $a \sin x$ a line bearing another relation in magnitude to a, then $a\left(\cos x + (-)^{\frac{1}{2}} \sin x\right)$ will imply, that we have to measure a line $a \cos x$, and from the extremity of it we are to measure another line $a \sin x$; but in consequence of the sign of operation $(-)^{\frac{1}{2}}$, this new line is to be measured, not in the same direction as $a \cos x$, but turned through a right angle. As, for instance, if $\mathrm{AB} = a \cos x$, and $\mathrm{BC'} = a \sin x$, we must not measure it in the prolongation of AB, but turn it round to the position BC; and thus, geometrically, we arrive at the point C. Also, from the relation between $\sin x$ and $\cos x$, we know



that the line AC will be equal to a, and thus the expression $a\left(\cos x + (-)^{\frac{1}{2}} \sin x\right)$ is an operation expressing that the line whose length is a, is turned through an angle x. Hence, the operation indicated by $\cos \frac{2\pi}{n} + (-)^{\frac{1}{2}} \sin \frac{2\pi}{n}$ is the same as that indicated by $(+)^{\frac{1}{n}}$, the difference being, that, in the former, we refer to rectangular, in the latter to polar co-ordinates. Mr PEACOCK has made use of the expression $\cos x + (-)^{\frac{1}{2}} \sin x$ to represent direction, while Mr WARREN has employed one which, though disguised under an inconvenient and arbitrary notation, is the same as $(+)^{\frac{1}{n}}$. The connection between these expressions is so intimate, that, being subject to the same laws, they may be used indifferently the one for the other. This has been the case most particularly in the theory of equations. The most general form of the root is usually expressed by $a\left(\cos\theta + (-)^{\frac{1}{2}}\sin\theta\right)$, while the more correct symbolical form would be $(+)^{\frac{p}{q}}a$, since the expression

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \&c. + P_n = 0$$

does not involve any sine or cosine, but may be considered as much a function of + as of x, so that the former symbol may be easily supposed to be involved in the root. Hence, instead of the theorem that every equation must have a root, I would say every equation must have a root of the form $(+)^{\frac{p}{q}}a$, p and q being numbers, and a a distributive and commutative function.