and *lastly*, that in every axial correlation which, besides satisfying the six conditions in question, has an axis σ_1 passing through a given point $A_1 \equiv B_1$, the line corresponding to that point, being an *indeterminate line through a determinate point* of the associated axis σ_2, will not, in general, pass through *two* given points A_2, B_2, although each of these points, as indeed any point whatever of the plane, may properly be regarded as conjugate to $A_1 \equiv B_1$.

In illustration of the first of these theorems, it may be noted that the number of solutions [4000], as also the number [3100], is the same as the number [3020] = 1 (art. 42), because the system of correlations satisfying the six conditions (3000) does not contain any axial ones having an axis passing through an arbitrary point (art. 69). Again, the number [2200] is zero, or less by one than the number [2120] = 1, because amongst the correlations which satisfy the six conditions (2100) (art. 70) there is, clearly, an axial one whose axis passes through an arbitrary point of the second plane; it passes also through the given point in that plane, and has, for associate, the line joining the two given points in the first plane.

I may observe, in conclusion, that by means of the above two theorems we might, without difficulty, deduce successively from the well known result [4000] = 1, or from the still simpler one [2200] = 0, all those, contained in the table of art. 42, in which an even number of pairs of conjugate points or conjugate lines are involved.

On the Free Motion of a Solid through an Infinite Mass of Liquid.

By Prof. H. LAMB, M.A. (Adelaide).

[*Read May* 10, 1877.]

Let us suppose that we have a solid body of any form immersed in an infinite mass of perfect liquid, that motion is produced in this system from rest by the action of any set of impulsive forces applied to the solid, and that the system is then left to itself. The equations of motion of a body under these circumstances have been investigated independently by Thomson* and by Kirchhoff,† and completely integrated for certain special forms of the body. The object of this communication is, in the first place, to examine the various kinds of permanent or *steady* motion of which the body is capable, without making any restrictions as to its form or constitution; and, in the second, to

* *Phil. Mag.*, Nov. 1871.

† *Crelle*, Band 71 (1869—70). See also *Vorlesungen über Mechanik*, by the same Author, ch. xix.

show that, when the initiating impulses reduce to a couple only, the complete determination of the motion can be made to depend upon equations identical in form with Euler's well-known equations of motion of a perfectly free rigid body about its centre of inertia, although the interpretation of the solution is naturally more complex. It will be seen that a very liberal use has been made, throughout, of the ideas and the nomenclature of the Theory of Screws, as developed and established by Dr. Ball.

1. Whatever be the motion at any instant, we may suppose it to have been produced instantaneously from rest by a properly chosen system of impulsive forces applied to the body. This system when reduced, after the manner of Poinsot, to a force and a couple whose axis is in the direction of the force (or, in Dr. Ball's nomenclature, to an impulsive *wrench*), is called by Thomson* the "impulse" of the motion at the instant under consideration. The same writer has shewn, in a manner which admits of considerable simplification, that when no external impressed forces act, the impulse is constant in every respect throughout the motion. This is expressed analytically as follows:— Let us take a system of rectangular axes x, y, z fixed in the body, and let u, v, w be the velocities of the origin parallel to, p, q, r, the angular velocities of the body about, the instantaneous positions of these axes. Further, let ξ, η, ζ, λ, μ, ν be the components, relative to the same axes, of the force- and couple-resultants of the impulse. The following equations† express that these latter quantities only vary in consequence of the motion of the axes to which they are referred:

$$\left.\begin{aligned} \frac{d\xi}{dt} = r\eta - q\zeta, \quad \frac{d\eta}{dt} = p\zeta - r\xi, \quad \frac{d\zeta}{dt} = q\xi - p\eta, \\ \frac{d\lambda}{dt} = w\eta - v\zeta + r\mu - q\nu, \quad \frac{d\mu}{dt} = u\zeta - w\xi + p\nu - r\lambda, \\ \frac{d\nu}{dt} = v\xi - u\eta + q\lambda - p\mu \end{aligned}\right\} \quad \text{........ (1).}$$

The motion of the fluid being entirely due to that of the solid, the total energy, T, of the motion is a quadratic function of u, v, w, p, q, r, with constant coefficients; so that we may write

$$\begin{aligned} 2T = \; & Au^2 + Bv^2 + Cw^2 + 2A'vw + 2B'wu + 2C'uv \\ & + Pp^2 + Qq^2 + Rr^2 + 2P'qr + 2Q'rp + 2R'pq \\ & + 2p\,(Lu + Mv + Nw) \\ & + 2q\,(L'u + M'v + N'w) \\ & + 2r\,(L''u + M''v + N''w) \quad \text{..(2),} \end{aligned}$$

* *Edin. Trans.*, Vol. xxv., 1869.

† See a paper by Mr. Hayward, *Camb. Trans.*, Vol. x.

where A, B, C, &c. are certain constants depending on the shape of the body and the distribution of matter in its interior, determinable by a transcendental analysis whose nature has been exhibited in a very elegant manner by Kirchhoff (*l. c.*).

The values of ξ, η, ζ, &c. in terms of u, v, w, &c. are then given by the formulæ

$$\xi, \eta, \zeta, \lambda, \mu, \nu = \frac{d\mathrm{T}}{du}, \frac{d\mathrm{T}}{dv}, \frac{d\mathrm{T}}{dw}, \frac{d\mathrm{T}}{dp}, \frac{d\mathrm{T}}{dq}, \frac{d\mathrm{T}}{dr} \quad \ldots\ldots\ldots\ldots (3),$$

as may be shewn by processes identical with those employed in Thomson and Tait's *Natural Philosophy*, Art. 313.

2. Kirchhoff has observed that the equations (1) are satisfied by u, v, w all constant, and p, q, r zero, provided we have

$$\frac{\frac{d\mathrm{T}}{du}}{u} = \frac{\frac{d\mathrm{T}}{dv}}{v} = \frac{\frac{d\mathrm{T}}{dw}}{w},$$

that is, provided the velocity of which u, v, w are the components be parallel to one of the principal axes of the ellipsoid

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{A}'yz + 2\mathrm{B}'zx + 2\mathrm{C}'xy = \text{const.} \quad \ldots\ldots\ldots (1a).$$

There exist, therefore, for every body three mutually perpendicular directions of permanent translation; that is to say, if the body be set in motion parallel to one of these directions, without rotation, it will continue to move in this manner. It is worthy of remark that these directions depend on the *shape* of the body, and not at all on the distribution of matter in its interior, or on the density of the surrounding fluid.

The above, although the simplest, are not the only permanent or steady motions of which the body is capable. In fact, the conditions that (1) should be satisfied by constant values of u, v, w, &c. are equivalent to only *four* independent relations between the five ratios $u : v : w : p : q : r$; viz., the first three of equations (1) give

$$\frac{\xi}{p} = \frac{\eta}{q} = \frac{\zeta}{r} \; [= h, \text{ say}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and the last three give, when combined with these,

$$\frac{\lambda - hu}{p} = \frac{\mu - hv}{q} = \frac{\nu - hw}{r} \; [= k, \text{ say}] \quad \ldots\ldots\ldots\ldots\ldots (5).$$

There exists, then, a simply-infinite system of possible steady motions. These motions consist each of a *twist* about a certain *screw*; and the axes of this system of screws lie on a certain skew surface, which is, of course, fixed relatively to the body.

The axis of the impulsive wrench necessary to produce motion according to one of these screws is the same with that of the screw. For (4) express that the axes of the screw and wrench are parallel, whilst the conditions that they should be coincident may be easily shewn to be

$$\frac{\lambda p-\xi u}{p^2}=\frac{\mu q-\eta v}{q^2}=\frac{\nu r-\zeta w}{r^2},$$

and we see, from (4) and (5), that each of these fractions is, in fact, equal to k. Indeed, it is otherwise obvious that the motion of the body about any assigned screw is steady when, and only when, it does not affect the configuration of the "impulse," fixed in space, relatively to the body; that is, when the axes of the screw and corresponding impulsive wrench coincide.

The *pitches*, however, of the screw and wrench are in general different. Calling κ the pitch of the screw, κ' that of the wrench, we have

$$\kappa'=\frac{\lambda\xi+\mu\eta+\nu\zeta}{\xi^2+\eta^2+\zeta^2}=\frac{pu+qv+rw}{p^2+q^2+r^2}+\frac{k}{h},$$

by (4) and (5), so that $k=h(\kappa'-\kappa)$ ....(6).

The expression for the kinetic energy may now be written

$$2\mathrm{T}=\xi u+\&c.+\lambda p+\&c.$$
$$=(\kappa+\kappa')\,h\omega^2,$$

where ω is the angular velocity $\sqrt{p^2+q^2+r^2}$.

3. If in (4) and (5) we substitute the values of ξ, η, ζ, &c. from (2) and (3), we get the following system of equations:

$$\left.\begin{array}{ll}(\mathrm{P}-k)\,p+\mathrm{R}'q+\mathrm{Q}'r+ & (\mathrm{L}-h)\,u+\mathrm{M}v+\mathrm{N}w=0,\\ \qquad\&c. & \qquad\&c.,\\ (\mathrm{L}-h)\,p+\mathrm{L}'q+\mathrm{L}''r+ & \mathrm{A}\,u+\mathrm{C}'v+\mathrm{B}'w=0,\\ \qquad\&c. & \qquad\&c.\end{array}\right\}\ \cdots\cdots\cdots (7),$$

whence, on elimination of the ratios $u:v:w:$ &c., we derive the following relation between h and k:

$$\begin{vmatrix}\mathrm{P}-k, & \mathrm{R}', & \mathrm{Q}', & \mathrm{L}-h, & \mathrm{M}, & \mathrm{N}\\ \mathrm{R}', & \mathrm{Q}-k, & \mathrm{P}', & \mathrm{L}', & \mathrm{M}'-h, & \mathrm{N}'\\ \mathrm{Q}', & \mathrm{P}', & \mathrm{R}-k, & \mathrm{L}'', & \mathrm{M}'', & \mathrm{N}''-h\\ \mathrm{L}-h, & \mathrm{L}', & \mathrm{L}'', & \mathrm{A}, & \mathrm{C}', & \mathrm{B}'\\ \mathrm{M}, & \mathrm{M}'-h, & \mathrm{M}'', & \mathrm{C}', & \mathrm{B}, & \mathrm{A}'\\ \mathrm{N}, & \mathrm{N}', & \mathrm{N}''-h, & \mathrm{B}', & \mathrm{A}', & \mathrm{C}\end{vmatrix}=0\ \ldots(8).$$

If we assign any arbitrary value whatever to h, this equation gives us three corresponding values of k; and, taking any one of these and substituting in (7), we get a system of linear equations to determine the values of the ratios $u:v:w:$ &c., and thence the axis and pitch of the corresponding permanent screw. The three values of k corresponding to any given value of h are all real; and the axes of the three corresponding screws are mutually at right angles, although they do not in general intersect. To prove these statements, let the axes of x, y, z (fixed in the body) be chosen parallel to the three directions of permanent translation, so that we have A′, B′, C′ all zero. The last three of equations (7) then give us at once u, v, w in terms of p, q, r; whence, substituting in the first three, we get to determine $p:q:r$ and k three equations of the form

$$\left.\begin{aligned} \mathfrak{P}p+\mathfrak{R}'q+\mathfrak{Q}'r &= kp \\ \mathfrak{R}'p+\mathfrak{Q}q+\mathfrak{P}'r &= kq \\ \mathfrak{Q}'p+\mathfrak{P}'q+\mathfrak{R}r &= kr \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9);$$

and the truth of the above statements then follows by known theorems. It also appears that two values of k corresponding to the same value of h cannot coincide unless h satisfy *two* independent conditions, which is in general impossible. This shews that the skew-surface which is the locus of the system of permanent screws consists of three distinct sheets.

If, on the other hand, in (8) we assign any arbitrary value to k, we get an equation of the sixth degree to determine h. The roots of this equation, when all real, determine by means of (7) a system of six *co-reciprocal* screws; that is to say, a wrench of the type of any one of the screws does no work on a body moving according to any other screw of the system. Let the suffixes $_1$ and $_2$ relate to two screws of the system; the condition that these should be reciprocal is

$$p_1u_2+q_1v_2+r_1w_2+p_2u_1+q_2v_1+r_2w_1=0 \quad\ldots\ldots\ldots\ldots(10).$$

Now in the six equations (7) suppose the suffix $_1$ attached to the symbols h, u, v, w, &c.; then, multiplying these equations by $p_2, q_2, r_2, u_2, v_2, w_2$ in order, and adding, we find

$$h_1\,(p_2u_1+q_2v_1+r_2w_1+p_1u_2+q_1v_2+r_1w_2) \quad\ldots\ldots\ldots\ldots(11),$$

equal to an expression symmetrical with respect to $_1$ and $_2$, viz. to

$$(\mathrm{P}-k)\,p_1p_2+\mathrm{P}'(q_1r_2+q_2r_1)+\text{\&c.},$$

which must therefore also be equal to

$$h_2\,(p_2u_1+q_2v_1+r_2w_1+p_1u_2+q_1v_2+r_1w_2) \quad\ldots\ldots\ldots\ldots(12).$$

But, h_1, h_2 being in general different, (11) and (12) cannot be identical unless (10) is satisfied.

4. Of the infinite number of groups of six co-reciprocal screws which may be chosen out of the system of permanent screws belonging to the body, there is one deserving of special notice. It is that in which each screw of the group has the same pitch as the impulsive wrench neces-

sary to produce motion about that screw only. The condition for this is, by (6), $k=0$, and there is no difficulty in showing that in this case the roots of (8) are all real. Consider, in fact, the determinant on the left-hand side of (8), and the subsidiary determinants obtained from it by erasing, first, the first row and column; next, the first two rows and columns; and, again, the first three rows and columns; and suppose k put $=0$ in each. We have thus a series of four determinants, each of which is symmetrical; and the argument employed by Salmon, *Higher Algebra*, Art. 44, shews that, when h is varied, no change in the number of variations of sign in the series can take place, except when h passes through a root of (8). Now, when $h=0$, the sign of each member of the series is $+$, by virtue of the well-known conditions necessary to ensure that the expression for 2T in (2) shall be positive whatever values be assigned to the variables u, v, w, &c. Again, when $h=\pm\infty$, the signs are $-+-+$; that is, there are three variations of sign in the series.

Hence the equation (8) (with $k=0$) has six real roots, three positive and three negative. The six determinate screws thus obtained (of which three are right-handed, three left-handed) may be called the six *principal screws of inertia* of the dynamical system. The name is due to Dr. Ball, who has pointed out the existence of such a set of screws for the case of a perfectly free rigid body.

5. Out of the infinite system of permanent screws we may form, besides the sets of co-reciprocal screws considered in Art. 3, sets of another kind, of at least equal importance. Two screws, such that the impulsive wrench necessary to produce motion according to one of them does no work on a body moving according to the other, may be called *conjugate*. We may obtain sets of six mutually conjugate permanent screws in the following manner:—Let us write, in (8), $k=\theta h$; this equation then gives, corresponding to any arbitrary value of θ, six values of h. When these are all real, we have in (7) a system of linear equations to determine the ratios $u : v : w :$ &c. in the several screws of the set. Let the suffixes $_1$ and $_2$ refer to any two screws of the system. The condition that these should be conjugate is that the expression

$$\xi_1 u_2+\eta_1 v_2+\zeta_1 w_2+\lambda_1 p_2+\mu_1 q_2+\nu_1 r_2 \quad\text{..................(13)}$$

should vanish. Now, by (4) and (5), we may write this expression in the form

$$h_1\{p_1u_2+q_1v_2+r_1w_2+p_2u_1+q_2v_1+r_2w_1 \\ +\theta(p_1p_2+q_1q_2+r_1r_2)\} \quad\text{.......................... (14).}$$

But when for ξ_1, η_1, ζ_1, &c. we substitute their values obtained from (3) and (2), (13) assumes a form which is, of course, symmetrical with respect to $_1$ and $_2$. Hence it must also be equal to an expression identical with (14), except that h_2 is written for h_1. Since h_1, h_2 are in general different, this requires that (14), and therefore (13), should be zero. The six screws corresponding to any given value of θ form, therefore, a mutually conjugate system.

Among these systems is included, of course, that of the six principal screws of inertia, obtained by making $\theta=0$.

The system which comes next in importance is found by making $\theta=\infty$. The equation (8) has then three of its roots infinite, and three zero. The infinite roots correspond to the three permanent translations of Art. 1; the zero roots give three screws such that, if the body be struck by an impulsive *couple* about the axis of one of these, it will be set in permanent motion about that screw.

6. The existence of this last set of three screws may be verified by observing that the equations (1) are satisfied by $\xi, \eta, \zeta = 0$, provided we have at the same time

$$\frac{\lambda}{p} = \frac{\mu}{q} = \frac{\nu}{r} \; [= k].$$

The directions of their axes, relatively to the three directions of permanent translation, are determined by the values of $p : q : r$, obtained from (9), where we now have

$$\mathfrak{P} = \mathrm{P} - \frac{\mathrm{L}^2}{\mathrm{A}} - \frac{\mathrm{M}^2}{\mathrm{B}} - \frac{\mathrm{N}^2}{\mathrm{C}},$$

$$\mathfrak{P}' = \mathrm{P}' - \frac{\mathrm{L'L''}}{\mathrm{A}} - \frac{\mathrm{M'M''}}{\mathrm{B}} - \frac{\mathrm{N'N''}}{\mathrm{C}},$$

&c. &c.

We infer that the axes are mutually at right angles, and that the three corresponding values of k, which I denote by k_1, k_2, k_3, are real. By putting $h=0$ in (8), we see that k_1, k_2, k_3 are, moreover, essentially positive, as is otherwise dynamically evident.

The interest of this set of screws rests on the fact that, when they are known, the motion of the body, when set in motion by any impulsive *couple* whatever, can be completely determined. In fact, let the initiating couple be resolved into three components F, G, H about the axes of the three screws, and let $\kappa_1, \kappa_2, \kappa_3$ be the pitches of the latter. Then the motion of the body at any subsequent instant is made up of motions s_1, s_2, s_3 about these screws; viz., s_1 denotes a rotation with angular velocity s_1 about the axis of the first screw, accompanied by a translation with velocity $\kappa_1 s_1$ parallel to this axis; and similarly for s_2, s_3. The initial values of these quantities are given by the formulæ

$$k_1 s_1 = \mathrm{F}, \quad k_2 s_2 = \mathrm{G}, \quad k_3 s_3 = \mathrm{H} \,;$$

and their values at any subsequent time t by the equations

$$\left.\begin{aligned} k_1\frac{ds_1}{dt} &= (k_2-k_3)\,s_2\,s_3 \\ k_2\frac{ds_2}{dt} &= (k_3-k_1)\,s_3\,s_1 \\ k_3\frac{ds_3}{dt} &= (k_1-k_2)\,s_1\,s_2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

These equations may be proved by putting $\xi, \eta, \zeta = 0$, and $p, q, r = s_1, s_2, s_3$, respectively, in (1); they express simply that the couple whose components are k_1s_1, k_2s_2, k_3s_3 is fixed in space. They are identical in form with Euler's equations of rotation of a perfectly free rigid body, which are, indeed, a particular case of (15), obtained by supposing the density of the fluid surrounding the solid to be zero. The solution of (15) may therefore be regarded as known; but its interpretation requires a knowledge of the properties of the *ternary screw-complex*, some of which are enumerated in the next Article.

7. The screws about which motion can be produced by the action of an impulsive couple only form a doubly-infinite system, *coordinate* (to use Dr. Ball's expression) with the three permanent screws above-mentioned. The properties of this system, which are the same as those of the general *ternary screw-complex*, may be obtained as follows:—The three permanent screws may be considered as lying along alternate edges of a rectangular parallelepiped; let us take as axes of coordinates lines drawn through the centre of this parallelepiped parallel to its edges. The equations to the axes of the three fundamental screws are then of the forms

$$\begin{aligned} y &= -g, & z &= h; \\ z &= -h, & x &= f; \\ x &= -f, & y &= g \end{aligned}$$

respectively; and if $\kappa_1, \kappa_2, \kappa_3$ be the pitches of these screws, a unit motion according to any other screw of the coordinate system (whose direction-cosines are, say, l, m, n) is equivalent to rotations l, m, n about the axes of x, y, z, and to a translation of the origin whose components, parallel to the same axes, are

$$\kappa_1 l - mh - ng, \quad \kappa_2 m - nf - lh, \quad \kappa_3 n - lg - mf$$

respectively. This translation is, therefore, perpendicular to the plane conjugate, with respect to the quadric

$$\kappa_1x^2+\kappa_2y^2+\kappa_3z^2-2fyz-2gzx-2hxy = \text{const.} \quad\ldots\ldots\ldots\ldots (16),$$

to the direction (l, m, n) of the axis of the screw, and the pitch κ of the latter is given by the formula

$$\kappa = \kappa_1l^2+\kappa_2m^2+\kappa_3n^2-2fmn-2gnl-2hlm;$$

i. e., it is inversely proportional to the square of the diameter of (16) drawn in the same direction. The principal axes of (16) coincide, therefore, with the axes of three screws of the system. The properties of the latter are then best investigated by taking the three axes last-mentioned as axes of coordinates. If a, b, c be the pitches of the corresponding screws, the equations to the axis of the coordinate screw whose components are l, m, n, are

$$\frac{al+mz-ny}{l} = \frac{-zl+bm+nx}{m} = \frac{yl-xm+cn}{n} \quad \text{.........} \quad (17),$$

and its pitch is $\kappa = al^2+bm^2+cn^2$.

Through any point (x, y, z) pass the axes of three screws of the system; for we have, to determine $l : m : n$ and κ, the equations

$$\left.\begin{array}{r} (a-\kappa)\,l + zm - yn = 0 \\ -zl + (b-\kappa)\,m + xn = 0 \\ yl - xm + (c-\kappa)\,n = 0 \end{array}\right\} \quad \text{..............} (18).$$

Hence κ is given by the cubic

$$(a-\kappa)(b-\kappa)(c-\kappa) + (a-\kappa)\,x^2 + (b-\kappa)\,y^2 + (c-\kappa)\,z^2 = 0 \;....(19).$$

The three values of κ hence derived, of which two are imaginary when (x, y, z) lies beyond a certain region in the neighbourhood of the origin, give, on substitution in (18), three determinate sets of values of the ratios $l : m : n$. The form of (18) shews that all the screws of the system having a given pitch lie on a certain hyperboloid. The axes of all the screws of the complex may therefore be obtained by taking one system of generators of each of the series of hyperboloids obtained by giving κ, in (19), all values intermediate between the algebraically greatest and least of the quantities a, b, c. They may also be found in the following manner :—Construct the quadric

$$ax^2+by^2+cz^2 = e^3 \quad \text{..........................} (20),$$

e being any arbitrary quantity; and from the origin O draw any radius vector OP meeting (20), or the conjugate surface (obtained by changing the sign of e), in P. The pitch κ of that screw of the complex which is parallel to OP is $\kappa = \dfrac{e^3}{\mathrm{OP}^2}$. Now let OM be the perpendicular from O on the tangent plane at P, and along the normal through O to the plane POM measure off a length ON $= \kappa$ tan POM. The *sense* in which this line is to be measured is that in which a rotation of the same sign as κ about OP would tend to carry the point M. A line through N, parallel to OP, will be the axis of a screw of the complex, of pitch κ.

It is easily seen that the point whose coordinates are $(c-b)\,mn$, $(a-c)\,nl$, $(b-a)\,lm$ lies on the straight line (17); in fact, that it is the point N of the above construction, the rest of the proof of which then follows without difficulty.

We see that the axes of the various screws of the complex all pass within a certain finite distance of O, so that at a great distance from O they present the appearance of a system of straight lines radiating from a point.

The properties of the complex are determined by the quadric (20). To find the form and position of this surface, we may proceed as follows:—Starting from the general expression for 2T in Art. 1, we first change the directions of the axes of coordinates so as to coincide with the principal axes of (1a), so that we now have $A', B', C' = 0$. Next, transfer the origin to the point whose coordinates are $\frac{1}{2}\left(\frac{M''}{B}-\frac{N'}{C}\right)$, $\frac{1}{2}\left(\frac{N}{C}-\frac{L''}{A}\right)$, $\frac{1}{2}\left(\frac{L'}{A}-\frac{M}{B}\right)$; the new values of L', L'', &c. then satisfy the following conditions:—

$$\frac{M''}{B}=\frac{N'}{C},\quad \frac{N}{C}=\frac{L''}{A},\quad \frac{L'}{A}=\frac{M}{B}.$$

The new origin is the point O of the present Article.

The equation to the quadric (20), referred to the new axes, is

$$\frac{L}{A}x^2+\frac{M'}{B}y^2+\frac{N''}{C}z^2+\left(\frac{M''}{B}+\frac{N'}{C}\right)yz+\left(\frac{N}{C}+\frac{L''}{A}\right)zx+\left(\frac{L'}{A}+\frac{M}{B}\right)xy = -e^3 \ldots\ldots (21).$$

The theorems of the present Article are not, I believe, all new;* but I am not aware to what extent the above investigations of them are so.

8. We are now prepared to construct the solution of (15). The motion of the body may be considered as made up of two parts, the motion of the point O above defined, and the motion relative to O. The latter is given at once by Poinsot's well-known construction; viz., if we construct the ellipsoid (fixed in the body), whose equation, referred to axes through O parallel to the three fundamental screws, is

$$k_1x^2+k_2y^2+k_3z^2 = \mathfrak{m}e'^4,$$

(where $\mathfrak{m}$, e', are two quantities of the nature of a mass and a line, respectively), and the plane (fixed in space) whose equation relative to the *initial* positions of the above axes is

$$Fx+Gy+Hz=\sqrt{2T\,.\,\mathfrak{m}}\,.\,e'^2,$$

* See the Postscript to Dr. Ball's paper in the *Phil. Trans.* for 1874; also Art. 31 of a paper by Prof. Everett, "On a New Method in Statics and Kinematics," in the *Messenger of Mathematics*, 1874. I may perhaps be allowed to mention that my knowledge of the Theory of Screws is derived almost entirely from the two sources here indicated; I have not seen, and have not access to, Dr. Ball's papers in the *Trans. R. I. Academy*.

the *angular* motion of the body is obtained by making the ellipsoid roll on the plane with angular velocity $\sqrt{\frac{2T}{m}} \cdot \frac{OI}{e'^3}$ about the radius vector drawn to the point of contact I.

The representation of the actual motion may then be completed by impressing on the whole system of rotating ellipsoid and plane a motion of translation equal to that of O, obtained as follows. Construct the quadric (20) or (21), and let OI meet it in P. The motion of O at any instant, is in the direction of the normal OM to the tangent plane to this quadric at P, and its velocity is $\sqrt{\frac{2T}{m}} \cdot \frac{e^3 . OI}{e'^3 . OP . OM}$.

Suppose the body to be set in motion according to one of the three fundamental screws of this and the preceding Articles, and then slightly disturbed, say by an infinitely small impulse. It is plain, from the analogy with the theory of the rotation of a perfectly free solid about its centre of inertia, that if the disturbing impulse reduce to a couple only, the motion is stable about either of the two screws for which k is respectively greatest and least, and unstable about the remaining screw. It is not difficult, with the help of equations (1), to extend these conclusions to the case when the disturbing impulse is any whatever.

9. For particular varieties of the moving solid, the nature and relative configuration of the various screws considered in the preceding theory become, of course, greatly simplified. For instance, suppose the body to have a plane of symmetry, as regards both its form and the distribution of matter in its interior, and let us take this as the plane of xy. It is easy to see that the energy of the motion must be unaltered if we change the signs of w, p, q. This requires that A′, B′, P′, Q′, L, M, L′, M′, N″ should all vanish. The axis of z is then one of the axes of permanent translation; the other two are at right angles in the plane xy. The pitch of each of the three fundamental screws of Art. 6 is zero; the axis of one of them is parallel to z, and those of the other two are in the plane xy, but do not in general intersect the first axis. The quadric (21) reduces to a hyperbolic cylinder.

Again, let us suppose that the body has a second plane of symmetry, at right angles to the former one; and let us take this as the plane of zx. We find, in the same manner as before, that the coefficients C′, R′, N, L″ must now also vanish. The three directions of permanent translation are then parallel to the axes of coordinates. The axis of x is the axis of one of the fundamental screws, (now pure rotations,) and the axes of the other two are perpendicular to this, one in each plane of symmetry, but do not in general intersect it in the same point. The quadric (21) is a hyperbolic cylinder asymptotic to the two planes of symmetry.

Finally, if the body has three planes of symmetry, mutually at right angles, the expression for 2T reduces to the form

$$2T = Au^2 + Bv^2 + Cw^2 + Pp^2 + Qq^2 + Rr^2.$$

In this case the axes of coordinates are the directions of the permanent translations, and also the axes of the three permanent rotations which could be produced by the action of impulsive couples only.

Let us next consider another class of cases, where the body has no plane of symmetry, but has a sort of skew symmetry with respect to a certain axis; viz., it is identical with itself turned through two right angles about that axis. An instance is the screw-propeller of a ship. Let us take the axis above mentioned as axis of x; it is plain that the energy of the motion must be unchanged if we reverse the signs of v, w, q, and r. Hence the coefficients B', C', Q', R', M, N, L', L'' must all vanish. The axis of x is then one of the axes of permanent translation; it is also the axis of one of the fundamental screws of Art. 6, the pitch being $-\dfrac{L}{A}$.

If the body be identical with itself turned through *one* right angle about the same axis (as, *e.g.*, in the case of a four-bladed screw-propeller), the expression (2) must be unchanged when we write v for u and $-u$ for v. This gives $B = C$, $Q = R$, $M' = N''$, $M'' = -N'$; and we see further, that by changing the origin to the point O, defined in Art. 7, we make M'', N' both vanish; so that (2) now assumes the form

$$\begin{aligned} 2T = Au^2 + B\,(v^2+w^2) + Pp^2 + Q\,(q^2+r^2) \\ + 2Lpu + 2M'\,(vq+wr). \end{aligned}$$

Any direction in the plane yz is a direction of permanent translation; and any line through O in that plane is the axis of a permanent screw motion (due to an impulsive couple only) of pitch $-\dfrac{M'}{B}$.

If the body be, moreover, identical with itself turned through a right angle about any axis in the plane yz, we see that (2) must become

$$\begin{aligned} 2T = A\,(u^2+v^2+w^2) + P\,(p^2+q^2+r^2) \\ + 2L\,(pu+qv+rw). \end{aligned}$$

Any direction whatever is then a direction of permanent translation; and any screw whose axis passes through O is a permanent screw. The equations (1) are in this case at once integrable, as has been pointed out by Thomson,* who has also shewn how to construct a body possessing the properties defined by (22).

Postscript.—I give here the equations of motion of a solid, under the circumstances considered in this paper, referred to the six principal

* *Phil. Mag.*, Nov. 1871.

screws of inertia whose existence and properties have been proved in Art. 4. I have not made any application of these equations, but their *form* seems of itself worthy of notice.

Let the coordinates of the six principal screws, referred to axes fixed in the body, be $(a_1, b_1, c_1, l_1, m_1, n_1)$ ...... $(a_6, b_6, c_6, l_6, m_6, n_6)$; viz., l_1, m_1, n_1 are the direction-cosines of the axis of the first screw, a_1, b_1, c_1 the velocities of the origin, parallel to x, y, z, produced by unit of twist-velocity about this axis, and similarly for the other screws of the system. These quantities are connected by 15 relations of the form

$$a_1 l_2 + b_1 m_2 + c_1 n_2 + a_2 l_1 + b_2 m_1 + c_2 n_1 = 0 \quad \ldots\ldots\ldots\ldots (23).$$

The pitches $\kappa_1, \kappa_2, \ldots \kappa_6$ of the principal screws are given by the formulæ

$$\kappa_1 = a_1 l_1 + b_1 m_1 + c_1 n_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

&c. &c.

Let us suppose now that the motion of the body at any instant is made up of twist-velocities $s_1, s_2, \ldots s_6$ about these screws, and let $\sigma_1, \sigma_2, \ldots \sigma_6$ be the force-components of the corresponding impulsive wrenches. We have then, with the notation of Art. 1,

$$\left.\begin{aligned} u &= a_1 s_1 + a_2 s_2 + \ldots + a_6 s_6, \\ v &= b_1 s_1 + \ldots, \\ w &= c_1 s_1 + \ldots, \\ p &= l_1 s_1 + l_2 s_2 + \ldots + l_6 s_6, \\ q &= m_1 s_1 + \ldots, \\ r &= n_1 s_1 + \ldots, \end{aligned}\right\} \ldots (25), \quad \text{and} \quad \left.\begin{aligned} \xi &= l_1 \sigma_1 + l_2 \sigma_2 + \ldots + l_6 \sigma_6, \\ \eta &= m_1 \sigma_1 + \ldots, \\ \zeta &= n_1 \sigma_1 + \ldots, \\ \lambda &= a_1 \sigma_1 + a_2 \sigma_2 + \ldots + a_6 \sigma_6, \\ \mu &= b_1 \sigma_1 + \ldots, \\ \nu &= c_1 \sigma_1 + \ldots, \end{aligned}\right\} \ldots (26).$$

Hence $$2T = \xi u + \&c. + \lambda p + \&c.$$
$$= 2\kappa_1 \sigma_1 s_1 + 2\kappa_2 \sigma_2 s_2 + \ldots + 2\kappa_6 \sigma_6 s_6,$$

by virtue of (25), (26), and of the 15 relations of the type (23). Again, multiplying (25) by $l_1, m_1, n_1, a_1, b_1, c_1$ in order, and adding, we find

$$2\kappa_1 s_1 = l_1 u + m_1 v + n_1 w + a_1 p + b_1 q + c_1 r,$$

and similarly from (26) we derive

$$2\kappa_1 \sigma_1 = a_1 \xi + b_1 \eta + c_1 \zeta + l_1 \lambda + m_1 \mu + n_1 \nu.$$

To form the equations of motion in terms of s_1, s_2, &c., we must substitute for $\dot{\xi}, \dot{\eta}, \dot{\zeta}$, &c. from (1), in the equation

$$2\kappa_1 \dot{\sigma}_1 = a_1 \dot{\xi} + \&c. + l_1 \dot{\lambda} + \&c.,$$

and, in the result, for ξ, η, ζ, &c. from (26). The final result is rather long, but it is easy to verify that the coefficients of $\sigma_1 s_1, \sigma_2 s_2, \sigma_3 s_3$, &c. are all zero, as are also those of $\sigma_1 s_2, \sigma_2 s_1$, &c. The coefficient of $\sigma_2 s_3$ is found to be

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ l_3 & m_3 & n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 & c_3 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

which denote by [123]. We have, obviously, [123] = [231] = [312] = −[132] = −[213] = −[321], so that the number of these coefficients is, irrespective of sign, 20.* The equations of motion then become

$$2\kappa_1 \dot{\sigma}_1 = [123]\, \sigma_2 s_3 + [124]\, \sigma_2 s_4 + \&c.$$
$$+ [132]\, \sigma_3 s_2 + [142]\, \sigma_4 s_2 + \&c. \qquad (20 \text{ terms})$$

with five similar equations. If $h_1, h_2, \ldots h_6$ be the six roots of (8), (with $k=0$,) we have $\sigma_1 = h_1 s_1$, $\sigma_2 = h_2 s_2$, &c., so that the above equations may also be written

$$2\kappa_1 h_1 \dot{s}_1 = [123]\,(h_2 - h_3)\, s_2 s_3 + [124]\,(h_2 - h_4)\, s_2 s_4 + \&c.$$
$$\&c. \qquad \&c. \qquad (10 \text{ terms})$$

We verify at once that these have the particular integrals

$$\kappa_1 h_1 s_1^2 + \kappa_2 h_2 s_2^2 + \ldots + \kappa_6 h_6 s_6^2 = \text{const.},$$
$$\kappa_1 h_1^2 s_1^2 + \kappa_2 h_2^2 s_2^2 + \ldots + \kappa_6 h_6^2 s_6^2 = \text{const.}$$

On some Cases of Parallel Motion. *By* Mr. Harry Hart, M.A., Mathematical Instructor at the Royal Military Academy, Woolwich.

[*Read April 12th*, 1877.]

1. Let ABCD (Fig. 1)† be a contra-parallelogram, where AB = CD = $2ae$, AC = BD = $2a$, and O, O′ the middle points of AB, CD. Then P, the intersection of AC, BD, describes an ellipse relatively to AB or DC, the major axis being $2a$, the eccentricity e. Let $2b$ = minor axis, so that $b^2 = a^2(1-e^2)$.

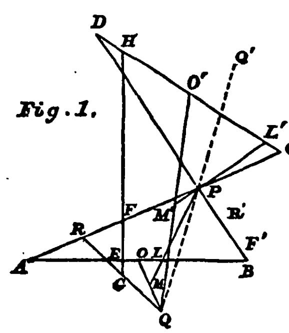

Fig. 1.

On the normal QPQ′ let two points Q, Q′ be taken such that PQ = PQ′ = semi-diameter conjugate to P. It can easily be shown that OQ = O′Q′ = $a-b$, OQ′ = O′Q = $a+b$, and that AB, DC bisect the angles QOQ′, QO′Q′ respectively.

---

* All theoretically expressible, however, in terms of the six pitches $\kappa_1, \kappa_2, \ldots \kappa_6$ and of three other independent quantities.

† To render the figure as simple as possible, some of the lines described in the text have been omitted.