ON THE GEOMETRICAL INTERPRETATION OF APOLAR BINARY FORMS

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1. Binary algebraic forms are usually represented geometrically by sets of points; of the three methods explained in Grace and Young's *Algebra of Invariants*, according to which the symbolical expression a_x^n is represented by n points (i) in a straight line, (ii) on a conic, (iii) in the Argand diagram, the first two only are employed in this paper. It will be assumed that the coefficients in the binary forms considered are real, so that imaginary roots of the equation $a_x^n = 0$ will occur, if at all, in conjugate pairs. The second method of interpretation is thus always possible; the first is possible only when there are no imaginary roots.

It is well known that, if two harmonic (*i.e.*, apolar) quadratics are thus represented, the construction for obtaining the fourth point when three are given is linear. In this paper, I prove a similar property for any two apolar forms of the same order.

2. If the form a_x^n be represented by the points $A_1, A_2, \ldots, A_n$, and (xy) be the point P, then the first polar form $a_x^{n-1} a_y$ will be represented by $n-1$ points which I shall call *the polar* $(n-1)$*-points of* P *for* $A_1, A_2, \ldots, A_n$; since the number of the points is more important than the order of the polar form (the first). Similarly, $a_x^{n-2} a_y^2$ will be represented by the polar $(n-2)$-points of P; and so on. This nomenclature serves also to avoid confusion with the polar lines, conics, &c., of a point with respect to curves, which will sometimes occur in the work.

The principle on which the method of this paper depends is most easily expressed symbolically: if a_x^n and $b_x^{n-1} c_x$ are apolar n-ics, then $(ac)\, a_x^{n-1}$ and b_x^{n-1} are apolar $(n-1)$-ics. This exhibits the connection between polar and apolar forms.

3. Let a range of n points, representing a form a_x^n according to the first method, be regarded as the intersections of the line with a plane n-ic curve. Then, if P, another point on the line, be defined by (xy),

the line will meet the first, second, ..., polar curves of P in the $(n-1)$-points, $(n-2)$-points, ..., of P for the range.

This furnishes a method of constructing P_1, the polar 1-point of a point P for a given triad A_1, A_2, A_3. For, if we draw through A_1, A_2, A_3, the special cubic curve consisting of three straight lines which form a triangle LMN, the polar line of P with respect to this triangle will pass through P_1.

The problem of finding the polar 2-points, Q_1, Q_2, of P for the triad is really the converse of this. For P is the polar 1-point for the triad of each of the points Q_1, Q_2. If therefore we draw two or more lines through P and find the poles $B_1, B_2, \ldots$, of each of these lines with respect to the triangle LMN, all these points B must lie on the polar conic of P for the triangle, and hence the required points Q_1, Q_2 are the intersections of the line $A_1A_2A_3$ with a conic circumscribing LMN, of which any number of points can be obtained.

Now let P, Q, R be a triad of points which is apolar to the triad A_1, A_2, A_3, and let P, Q be given while R is to be constructed. We know that Q and R will be harmonically conjugate with respect to Q_1, Q_2; *i.e.*, the polar line of Q for the conic-locus of the points B will pass through R. And this polar line of Q can be linearly constructed, although the points Q_1, Q_2 cannot themselves be obtained except by a construction of the second degree.

4. Throughout the rest of this paper I use the second method of interpretation; and by the polar line of a point, or the pole of a straight line, I shall always mean the polar or pole with regard to the fundamental conic; whenever I have occasion to refer to the polar line of a point with respect to a triangle this will be specially stated.

Any quadratic form a_x^2, interpreted by two points on the conic, determines a straight line, *i.e.*, the line joining these points; and hence determines also a point in the plane of the conic, *i.e.*, the pole of this line. This single point can be linearly constructed, and upon it depends the construction in the harmonic case of two quadratics.

In considering the cubic, I shall first suppose that the roots of the equation $a_x^3 = 0$ are $x_1/x_2 = 0,\ 1$, and ∞, so that

$$a_x^3 \equiv x_1^2 x_2 - x_1 x_2^2.$$

Let the points representing the form be P_1, P_2, P_3, and let the point Q represent any linear form (xy), for which the ratio $x_1/x_2 = \lambda$.

Then we may easily obtain the following results: for Q', the polar

1-point of Q for the triad, given by $a_x a_y^2 = 0$, the ratio

$$\frac{x_1}{x_2} = \frac{\lambda(2-\lambda)}{2\lambda-1};$$

for $\bar{Q}$, the harmonic conjugate of Q with respect to the Hessian points (for brevity, the H-points) of a_x^3, the ratio

$$\frac{x_1}{x_2} = \frac{\lambda-2}{2\lambda-1};$$

for q_2, the harmonic conjugate of Q with respect to P_1, P_3, the ratio

$$x_1/x_2 = -\lambda.$$

Hence we have

$$\{P_1 Q' P_3 \bar{Q}\} = \frac{\lambda(2-\lambda)}{2\lambda-1}\,\frac{2\lambda-1}{\lambda-2} = -\lambda = \{P_1 q_2 P_3 P_2\}; \qquad (1)$$

and, since any three points can be projected into any other three points, this result must hold for the general cubic.

Applying this result to the present case, we are concerned first with the H-points.

Let all the roots of the cubic be real, and let $a_x^3 \equiv \alpha_x \beta_x \gamma_x$, where α_x is represented by P_1, β_x by P_2, and γ_x by P_3.

Then the H-points are imaginary, and the pole I of the line joining them is within the conic.

To construct the point I [see *Algebra of Invariants*, Ex. (i.), p. 240], join P_1 to the pole of the line P_2P_3, P_2 to the pole of P_3P_1, and P_3 to the pole of P_1P_2: the joining lines will be concurrent in I.

Next join QI, and produce to meet the conic, obtaining the point $\bar{Q}$, and join Q to the pole of P_1P_3, the line meeting the conic again in q_2.

Then the result (1) furnishes the following construction:—Let $\bar{Q}q_2$ meet P_1P_3 in M; then P_2M will meet the conic again in Q', the required polar 1-point of Q.

There is thus obtained an easy line construction for the point Q' (see Fig. 1).

The point arrived at must be the same whether we take, in our construction, the points P_1, P_3, or either of the two other pairs of the triad, P_1, P_2 and P_2, P_3. We have thus the geometrical theorem that the line MP_2 and the two other lines LP_1, NP_3, obtained similarly, are concurrent in a point on the conic.

In the figure the point Q' is obtained in the three possible ways, the same position being found each time. It is therefore to be noticed that the actual process of finding Q' is very much simpler than appears from the complicated figure. It will be seen that for practical convenience it is desirable to select for the construction that pair of the points P_1, P_2, P_3 which are both on the same side of the line $QI\bar{Q}$.

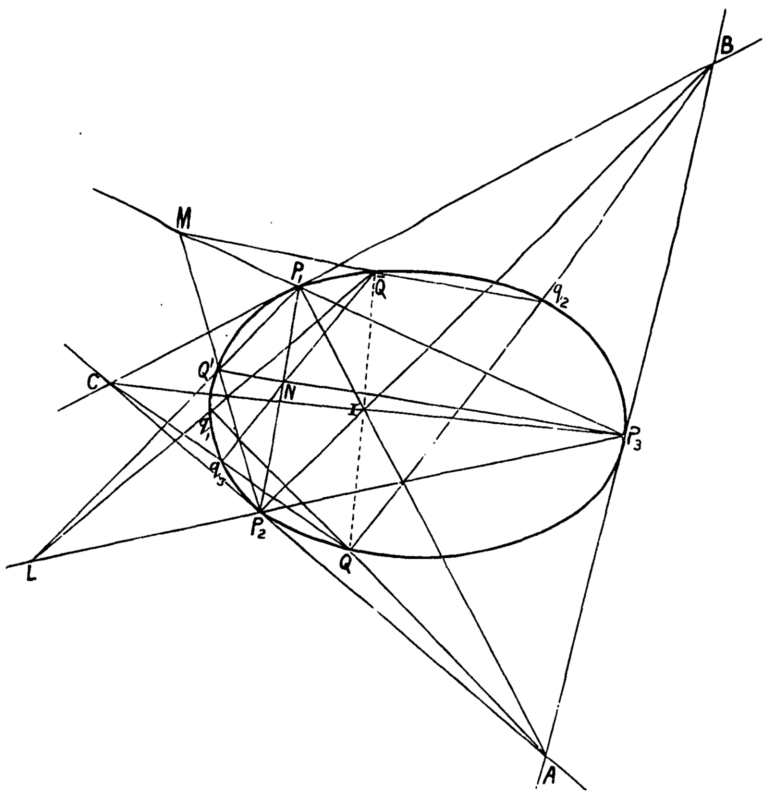

FIG. 1.

5. This geometrical theorem can be extended indefinitely. For the point q_1 is given by $\beta_y\gamma_x+\beta_x\gamma_y$, and therefore the triad q_1, q_2, q_3 is given by a cubic which is

$$= (a_y\beta_x\gamma_x+\ldots+\ldots)(a_x\beta_y\gamma_y+\ldots+\ldots)-a_x\beta_x\gamma_x a_y\beta_y\gamma_y$$

$$= 9a_x^2 a_y b_x b_y^2 - a_x^3 b_y^3.$$

The polar 1-point of Q with respect to this triad is given by $a_x a_y^2$, and is therefore the point Q'.

Hence in the figure we should arrive at the same point Q' if, instead of the triad P_1, P_2, P_3, we took the triad q_1, q_2, q_3; and therefore also if we took the triad obtained from q_1, q_2, q_3 in the same way as the triad q_1, q_2, q_3 is obtained from P_1, P_2, P_3; and so on.

6. If Q_1, Q_2 are the polar 2-points of Q for the triad P_1, P_2, P_3, we know that Q_1, Q_2 are harmonically conjugate to Q, Q'; and they are so also with respect to the H-points.

Therefore the line TI, where T is the pole of QQ', will meet the conic in Q_1 and Q_2; and these points can therefore be constructed.

The pole K of the line Q_1Q_2 is the intersection of the line QQ' with the Hessian line; and, if any line whatever be drawn through K, meeting the conic in R and S, then the triad Q, R, S is apolar to the triad P_1, P_2, P_3.

If the points Q, R are given, and S is to be found, we therefore find K as above, and join KR: it will meet the conic in S. Another way of proceeding, which is really equivalent to the last, is this: Join QR, and produce to meet the Hessian line in K'. The polar of K' will meet the conic in the polar 2-points of S. It will be seen in § 7 that from this we are able to find S.

The solution of the problem is therefore complete.

This method of constructing Q', and then Q_1 and Q_2, is, however, somewhat artificial. I proceed to give two alternative methods of finding the points Q_1, Q_2, both of which are direct and do not depend on Q' being first obtained. The one is of interest chiefly on account of the simplicity of the result; the other because it can be extended to the general problem of two n-ics.

7. The first of these admits of very easy proof; it will therefore suffice to state the result, which is as follows:

The line joining Q_1, Q_2, the polar 2-points of Q, is the polar line of Q with respect to the triangle $P_1P_2P_3$.

By means of this result we can find the point whose polar 2-points are given (see § 6).

[We have seen that the line Q_1Q_2 always passes through I for all positions of Q; that is, for all points on the fundamental conic the polar line with respect to the triangle $P_1P_2P_3$ passes through I. Hence the fundamental conic is the polar conic of I with respect to the triangle.]

When the points Q_1, Q_2 have been constructed in this way, Q', the polar 1-point of Q, may be obtained by a much simpler process than that explained above, by finding the harmonic conjugate of Q with respect to Q_1, Q_2.

The problem of two apolar triads is then completed as before; and the result obtained may be thus stated:

If the triads P_1, P_2, P_3 and Q, R, S are apolar, then the polar triangle *of QRS with respect to the conic is inscribed in the polar triangle of QRS with respect to the triangle* $P_1P_2P_3$.

8. To obtain the third construction for Q_1, Q_2 referred to above, we notice that $a_x^2 a_y$ can be written in the form

$$\alpha_y \beta_x \gamma_x + \alpha_x (\beta_y \gamma_x + \beta_x \gamma_y);$$

and therefore $a_x^2 a_y$ is apolar to the pair of points which is apolar both to β_x, γ_x and to α_x, $\beta_y\gamma_x + \beta_x\gamma_y$. That is, if q_1 is the harmonic conjugate of Q with respect to P_2, P_3 and P_1q_1 meets P_2P_3 in L', then the line Q_1Q_2 passes through L'.

Hence, if we construct similarly the points M' and N', these points L', M', N' must be collinear, the line of collinearity being the line Q_1Q_2 and so passing through I.

9. We have seen that if the points Q, R, S form a triad apolar to P_1, P_2, P_3, then R and S may be taken as the intersections of the conic with any line drawn through the pole of Q_1Q_2. If, therefore, the line chosen does not (geometrically) meet the conic at all, so that the points R and S are imaginary, the triad Q, R, S is still apolar to P_1, P_2, P_3.

Also, if Q, R, given points (while S is to be found), are imaginary points lying on a line which does not meet the conic, we can still construct S by the second method given in § 6.

But, if in the original triad P_1, P_2, P_3 two points, say P_2 and P_3, are imaginary, the constructions which have been given above are no longer immediately possible.

If K be the point of intersection of P_1A and P_2P_3 in Fig. 1, the range P_1, I, K, A is harmonic. From this we can, in the present case, construct I and afterwards the H-points, which are now real. Also q_1 and L', but not q_2, q_3, M', N', can be obtained; and then the line $L'I$ will meet the conic in Q_1 and Q_2, which may be either real or imaginary.

Or it is still possible to use the method of § 4; for L can be obtained, but not M and N, and then the line P_1L gives the point Q'.

10. If P_1, P_2, P_3 and Q, R, S be two apolar triads, and RI, SI meet the conic again in $\bar{R}$ and $\bar{S}$ respectively, it is known that the line $\bar{R}\bar{S}$ meets RS on the polar of I; therefore Q, $\bar{R}$, $\bar{S}$ is also an apolar triad to P_1, P_2, P_3.

There are thus three triads associated with Q, R, S, each of which is apolar to P_1, P_2, P_3; they are Q, $\bar{R}$, $\bar{S}$; $\bar{Q}$, R, $\bar{S}$; and $\bar{Q}$, $\bar{R}$, S.

And the members of the doubly-infinite system of triads apolar to P_1, P_2, P_3 are associated in sets of four, those in one such set being derivable from any one of the four.

11. Before passing on to consider the geometry of the quartic, I add a short note on the covariant points of two different cubics a_x^3, b_x^3 given by $(ab)^2 a_x b_x$. When the cubics coincide these points become, of course, the H-points; and when the cubics are apolar it will be seen that they are the double points of a certain involution.

Let F_1, F_2 be the points to be interpreted; then, since

$$((ab)^2 a_x b_x,\ a_x'^3) = \tfrac{1}{2}((aa')^2 a_x a_x',\ b_x^3),$$

we have that the point which with F_1, F_2 makes a triad apolar to a_x^3 is also the point which with the H-points of a_x^3 makes a triad apolar to b_x^3.

Hence the following construction:—Find the point which with the H-points of a_x^3 makes an apolar triad to b_x^3. Construct its polar 2-points for a_x^3 and find A_1, the pole of the line joining them. Similarly find B_1. Then $A_1 B_1$ meets the conic in the required points F_1, F_2.

Let us now take any point on the conic and find its polar 2-points with respect to the cubic a_x^3; then find the point which with these two points makes an apolar triad to b_x^3. This last point will be given by $(ab)^2 a_y b_x$.

Taking the triads in the reverse order, we get the point $(ab)^2 a_x b_y$. We have thus defined a (1, 1) correspondence of points on the conic of which the united points are F_1, F_2.

But
$$(ab)^2 a_y b_x - (ab)^2 a_x b_y = -(ab)^3 (xy),$$

and therefore when the triads are apolar the correspondence becomes an involution with F_1, F_2 as its double points.*

Another property of F_1 and F_2 is that the two lines joining the polar 2-points of either of them with respect to the two triads are conjugate.

The problem of finding a triad apolar to two given triads occurs above, and the method is obvious. That of finding the unique triad apolar to three given triads arises naturally in considering the case of the quartic. Its discussion is postponed to § 13.†

* Apolar forms of different orders are connected by a property somewhat similar to this. Let there be any two n-ics of points. Then a (1, 1) correspondence is defined by adding to each a single point in order to get two apolar $(n+1)$-ics. If the n-ics are themselves apolar, this correspondence becomes an involution. When n is 3 this involution is the same as that above, having F_1, F_2 for double points; but the correspondence (when the cubics are not apolar) is not the same as the above correspondence.

† Another method of interpreting binary cubics geometrically is by means of points on a rational twisted cubic curve (see *Algebra of Invariants*, § 194). If the curve is given by $\xi = a_x^3$, $\eta = b_x^3$, $\zeta = c_x^3$, $\varpi = d_x^3$, then every binary cubic form defines three points of the curve, and therefore corresponds uniquely to a plane. If two cubic forms are apolar, the point of intersection of the osculating planes of the three points lying in one of the corresponding apolar planes is itself in the other.

12. *The Quartic.*—Let P_1, P_2, P_3, P_4 represent the form

$$a_x^4, \quad \equiv \alpha_x \beta_x \gamma_x \delta_x ;$$

and let Q be any other point (xy).

Then the polar 1-point of Q for the four points may be thus obtained:

Find q_1, the harmonic conjugate of Q for any two of the points, say P_1 and P_2; and find q_2, the harmonic conjugate of Q for the other two, P_3 and P_4.

Then find Q', the harmonic conjugate of Q for q_1 and q_2; it will be the point required, and is therefore the same in whatever way we divide the quartic into pairs of points.

To construct the polar 2-points of Q, choose any three of the four points, say P_2, P_3, P_4; and let q' be the polar 1-point and q'_1, q'_2 the polar 2-points of Q for this triad.

Let $q'_1 q'_2$ and $q'P_1$ meet in Q'_1.

By selecting other sets of three points, we obtain similarly the points Q'_2, Q'_3, Q'_4.

Then these points Q'_1, Q'_2, Q'_3, Q'_4 are collinear, and their line of collinearity meets the conic in the required points Q_1, Q_2.

For Q_1 and Q_2 are given by $a_x^2 a_y^2$, which can be written

$$\alpha_x(\beta_x\gamma_y\delta_y+\beta_y\gamma_x\delta_y+\beta_y\gamma_y\delta_x)+\alpha_y(\beta_y\gamma_x\delta_x+\beta_x\gamma_y\delta_x+\beta_x\gamma_x\delta_y),$$

and in this expression the coefficient of α_x is the form giving q', and the coefficient of α_y is the form giving q'_1 and q'_2.

Therefore Q_1 and Q_2 are apolar to the pair of points which is apolar both to q', P_1 and to q'_1, q'_2. Hence the line $Q_1 Q_2$ passes through Q'_1; similarly, it passes through Q'_2, Q'_3, and Q'_4.

The polar 2-points of Q having been obtained in this way, the polar 1-point, Q', may be constructed by finding the harmonic conjugate of Q with respect to Q_1 and Q_2.

The polar 3-points of Q are given by

$$(\alpha_y\beta_x+\alpha_x\beta_y)\gamma_x\delta_x+\alpha_x\beta_x(\gamma_y\delta_x+\gamma_x\delta_y),$$

and therefore the triad which they form is apolar to any triad which is itself apolar both to P_3, P_4, q_1 and to P_1, P_2, q_2. Such a triad can easily be constructed.

By dividing the quartic into pairs of points in different ways, we can obtain five other triads apolar to the triad required. Any three of these six triads are sufficient to define the polar 3-points of Q; and the problem is therefore reduced to that of finding the unique triad apolar to three given triads, referred to at the end of § 11.

13. I have been unable to obtain a solution by means of a linear construction; the following is the simplest solution which I have been able to find.

If the points X, Y, Z form the triad apolar to three given ones, then Y, Z are harmonically conjugate with the polar 2-points of X for each of the three triads.

Therefore the three lines joining these three pairs of polar 2-points of X must be concurrent, and the poles of these lines must be collinear.

Now, as a point P on the conic moves, the pole of the line joining its polar 2-points for the first triad moves along the Hessian line of that triad. And thus the three poles referred to above move along three straight lines, viz., the Hessian lines of the three triads.

The position of any one of these three poles determines P and the other two poles uniquely.

We have thus a (1, 1, 1) correspondence of points on three straight lines, and we require the three sets of corresponding points which are collinear.

Let the three Hessian lines be called A, B, C.

Then the lines joining corresponding points on A and B envelop a conic which touches A and B. And the lines joining corresponding points on B and C envelop a conic which touches B and C.

These two conics have four common tangents, real or imaginary. One of these is the real line B; the other three are the straight lines required.

Now, by taking different positions of P on the conic, we may linearly construct as many sets of corresponding points on A, B, and C as we desire.

Thus we can construct as many tangents to the above two conics as we please.

Hence the lines required are the three unknown common tangents to two conics, each of which touches a given straight line and is defined by its tangents.

If we obtain a conic similarly from the correspondence on A and C, it will also be touched by the three straight lines just found. Its fourth common tangent with the first of the above conics will be A; that with the second will be B.

These three straight lines are actually the lines joining two at a time the points X, Y, Z. They therefore meet in pairs on the fundamental conic, and form the triangle defined by the required apolar triad.

14. Hence we can obtain (though not by a line construction) the polar 3-points of a point Q with respect to a quartic P_1, P_2, P_3, P_4.

The solution of the problem: *Given a quartic and a triad of points, to construct the single point which with the triad forms an apolar quartic*, can now be stated thus:

Find the polar 3-points of one of the given triad of points for the quartic.

Find the polar 2-points of another point of the triad for these three points.

Find the polar 1-point of the third point of the triad for these two points. This will be the point required, whatever be the order in which the points of the given triad are taken.

In order to give a complete linear solution, I go on to shew that the second of the above three steps can be linearly performed without actually constructing the polar 3-points at all.

Let Q be the point (xy) and R the point (xz).

The polar 3-points of Q are given by

$$(a_y\beta_x+a_x\beta_y)\,\gamma_x\delta_x+(\gamma_y\delta_x+\gamma_x\delta_y)\,a_x\beta_x,$$

and therefore, as has been seen already, the polar 3-points of Q form a triad apolar to every triad which is itself apolar both to P_3, P_4, q_1 and to P_1, P_2, q_2. The two points which with R form such a triad are those on the line joining the poles of the polar lines of R with respect to the two triangles $P_3P_4q_1$, $P_1P_2q_2$.

Thus, by the six methods of dividing the quartic into pairs of points [and also from the four similar cases obtained by writing the polar 3-points of Q in the form

$$a_y\beta_x\gamma_x\delta_x+a_x(\beta_y\gamma_x\delta_x+\beta_x\gamma_y\delta_x+\beta_x\gamma_x\delta_y)],$$

we get six [and four] pairs of points which with R form apolar triads to the polar 3-points of Q.

The lines joining these different pairs of points must all be concurrent in the pole of the line which joins the polar 2-points of R for the triad consisting of the polar 3-points of Q.

These polar 2-points are given by

$$a_x\beta_x(\gamma_y\delta_z+\gamma_z\delta_y)+\text{five similar terms},$$

or $a_x^2a_ya_z$; and thus these points (which I shall call the mixed polar 2-points of y and z) can be linearly constructed (since the pole of the line joining them can be found) without first constructing the polar 3-points of Q.

This completes the linear solution of the problem for the case of the quartic.

Although we cannot construct linearly the polar 3-points of Q, we can at least construct the H-points of the triad which they form. For taking two positions, R_1 and R_2, of the point R, the lines obtained from them as above will intersect in the pole of the required Hessian line.

15. *General n-ic form.*—Let the form considered be $a_x a_x^{n-1}$.

The polar 1-point and 2-points of any point y with respect to a_x^{n-1} are given by $a_x a_y^{n-2}$ and $a_x^2 a_y^{n-3}$.

Then it is easy to prove that the intersection of the line joining the point $a_x a_y^{n-2}$ to the point a_x with the line joining the points $a_x^2 a_y^{n-3}$ lies on the line joining the polar 2-points of the point y for the n-ic.

If therefore we are able to construct linearly the polar 1-point and 2-points of y for an $(n-1)$-ic, we can obtain n different points all lying on the line joining the polar 2-points of y for any n-ic.

The polar 1-point of y for the n-ic is then obtained by constructing the harmonic conjugate of y with respect to the polar 2-points already found.

And it is then possible to proceed to the $(n+1)$-ic.

Hence starting from the cubic and quartic, we can construct the polar 1-point and 2-points of a point y for any binary form by this means.

16. There remains to be considered the problem: *Given n points on the conic, and n—1 other points, to find the point required to make two apolar n-ics.*

Here, as in the last paragraph, the method is that of proceeding from the case of a form of lower degree to that of one of degree higher by unity.

The solutions for the cubic and the quartic have been given: it will therefore be a sufficient explanation of the method for the general form if the case of the quintic is considered.

Let the quintic be $a_x\beta_x\gamma_x\delta_x\epsilon_x$, and let the four other given points be Q, R, S, and T, Q being the point y.

The polar 4-points of Q for the quintic are given by

$$(a_y\beta_x+a_x\beta_y)\,\gamma_x\delta_x\epsilon_x+a_x\beta_x\,(\gamma_y\delta_x\epsilon_x+\gamma_x\delta_y\epsilon_x+\gamma_x\delta_x\epsilon_y). \qquad (2)$$

This form is here written as the sum of two parts, and each part separately gives a quartic of points which can be easily constructed.

Now, by the known case of $n = 4$, we can construct linearly the lines joining the mixed polar 2-points of R and S for each of these quartics.

But it is obvious that the line joining the mixed polar 2-points of Q, R, and S for the quintic passes through the point of intersection of these lines.

And, by dividing the expression (2) in different ways, we can construct many other points, all lying on the line joining the mixed polar 2-points of Q, R, and S.

The line itself can therefore be constructed, and the mixed polar 2-points of Q, R, and S obtained. Then the harmonic conjugate of T with respect to these two points is the point required.

The most convenient way of dividing the expression (2), which gives the polar $(n-1)$-points of Q, is into 3 terms and $n-3$ terms. The first part then gives an $(n-1)$-ic of points which can be constructed. The second part gives an $(n-1)$-ic of points which cannot themselves be constructed if n is > 6, but the mixed polar 2-points of R, S, ... for them can be obtained by means of a process similar to that just used, viz., this second part must be sub-divided into 3 terms and $n-6$ terms, in two or more different ways, and the same applies to the sub-part of $n-6$ terms, and so on. Also at each point the results obtained for the previously considered case of the $(n-1)$-ic form are assumed.

The process is thus exceedingly complicated, and is, as might have been expected, of little practical use for forms of degree higher than the sixth or seventh. It is, nevertheless, theoretically complete.

It is scarcely necessary to point out that, since the forms (2), &c., may be divided into parts and sub-parts in different ways, this theory leads to an immense number of geometrical propositions concerning the collinearity of points and the concurrence of lines.