On Cyclotomic Functions. *By* H. W. LLOYD TANNER, M.A., Professor of Mathematics in the University College of South Wales.

SECTION III.

The cyclctomics which belong to the f-nomial periods of the p^{th} *roots of unity, when p is a prime number.*

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Abstract. (Arts. 1–7.)

1. When p is an odd prime, $= 2e+1$ say, the binomial periods of the p^{th} roots of unity are

$$x+x^{-1},\quad x^2+x^{-2},\ \dots\ x^e+x^{-e},$$

where x is one of the roots of

$$x^{p-1}+x^{p-2}+\dots+x+1=0.$$

The periods are the roots of the well-known equation

$$\eta^e+\eta^{e-1}-(e-1)\,\eta^{e-2}-(e-2)\,\eta^{e-3} + \frac{(e-2)(e-3)}{1\,.\,2}\,\eta^{e-4}+\frac{(e-3)(e-4)}{1\,.\,2}\,\eta^{e-5}+\dots=0,$$

the expression on the left being continued to $e+1$ terms. The object of the present communication is to give the corresponding theorem for f-nomial periods of the p^{th} roots of unity when p, $= ef+1$, is a prime number. The difficulty is, that we have not for the f-nomial periods, as we have for the binomial periods, a form which can be written down without knowing p. For example, the leading trinomial periods of the 7^{th}, 13^{th}, 19^{th} roots of unity are

$$x+x^2+x^{-3},\quad x+x^3+x^{-4},\quad x+x^7+x^{-8},$$

respectively; and there is no general expression for these indices analogous to the expression ± 1 for the indices of the leading binomial period.

2. The result obtained is that the cyclotomic may be regarded as a product of three "factors." Each of these consists of an infinite number of terms with integral coefficients, the first coefficient being 1. The cyclotomic consists of the first $e+1$ terms of the product, and the remaining terms of the product are zeros as far as the term containing η^{-n}, if each of the three factors is calculated to this extent.

3. One of the factors, called the *asymptotic factor*, is the only one that appears in the binomial-period equation written above. For a given f, it is a series depending on e, and appropriate to every value of p, $= ef+1$. When f is a prime, the first f coefficients are independent of e; they are in fact identical with the first f coefficients of the expansion of $(1-fy)^{-1/f}$. The f coefficients which follow are linear functions of e, or more conveniently of ϵ, $= (f-1)!\,e$. In these linear functions the ϵ occurs multiplied by the first f coefficients of the asymptotic factor. The following sets of f coefficients are quadric, cubic, &c. functions of ϵ. For examples, see the tables for $f = 3, 5, 7$, appended to this paper.

When f is composite, each factor of f affects the form of the asymptotic factor; but the smallest factor has the most obvious influence. For instance, when f is even, the factor proceeds in pairs of terms, like the binomial period cyclotomic. Examples will be found in the tables for $f = 4, 6, 8, 9$.

The asymptotic factor presents itself as a product of other factors, one of which is $(1-fx)^{-1/f}$, and the others are "central" factors. The coefficients of these subsidiary factors are not all integral.

4. The second factor of the cyclotomic—called the *eccentric factor*—is not expressed in terms of e; so that it has to be calculated separately for each value of p. All its coefficients except the first are multiples of p; that is, it is of the form

$$1-p\,\{E_k y^k+E_{k+1}y^{k+1}+\dots\},\quad (y=\eta^{-1}),$$

where E_k, E_{k+1} are integers, and are positive at least as far as E_{2k}. This, like the asymptotic factor, naturally splits up into a product of other factors; but, unlike the asymptotic factor, all its factors have integral coefficients.

From the form of the eccentric factor it follows that the asymptotic factor is congruent to the cyclotomic, mod. p. There is a presumption in favour of the theorem, that by taking p sufficiently large, k may be made as large as we please: a theorem which would justify the epithet "asymptotic." But this is not proved. On the assumption that certain forms (for instance, the geometric series $1+a+\dots+a^{f-1}$ when f is prime) contain an infinite number of primes, the theorem can be proved; and, as the case of f prime requires very little space, I have discussed it. It did not seem worth while to extend the discussion: not only because it was based upon an unproved, though probable, assumption; but also because the inferior limit of k, determined in this way, seems to hold good for all values of p, and not merely for those values of p implied in the assumption.

5. To calculate the cyclotomic, it is sufficient to take the product of the asymptotic and eccentric factors. This product, however, differs from the cyclotomic. For instance, $c=3$, $e=2$, $p=7$; the coefficients of the product of the asymptotic and eccentric factors are

$$1, 1, 2, 0, 0, 0, 0, 1, 1, 2, 0, 0, \ldots\ldots$$

and to make this agree with cyclotomic

$$1, 1, 2,$$

a *third factor*, $\quad 1-\eta^{-p}+A\eta^{-2p}+\&c.,$

must be introduced. Although this factor is absolutely without influence on the calculation of the cyclotomic, yet it seems satisfactory to explain how such a factor arises; and this is done in the sequel.

The expression used for forming the cyclotomic for binomial periods is the asymptotic factor only. The eccentric factor in this case is $1+py^{p+2}+\ldots$, and the third factor is $1-py^p-\&c.$; so that neither of these influences the coefficients that are to be determined.

6. The arrangement of the work will now be indicated. The expression for log $\mathfrak{P}$, $\mathfrak{P}$ being the cyclotomic, is first formed. In the analysis of this expression, considerable use is made of a regular polygon of f sides, at the vertices of which are placed particles of various weights, all commensurable. According as the centre of gravity of these particles is or is not at the centre of the polygon, the system is termed a central or an eccentric system. The central factors of $\mathfrak{P}$ come from the central systems; and the eccentric factors from the eccentric systems. This weighted polygon promises to be useful in discussing complex numbers formed with roots of unity; but the application is hardly within the scope of the present communication.

7. It was necessary to find the conditions that a series

$$-s_1y-s_2y^2/2 - s_3y^3/3 - \&c.$$

should be the logarithm of a series

$$1+P_1y+P_2y^2+\ldots.$$

with integral coefficients. These conditions are obtained in the form

$$s_n+\Sigma s_\delta-\Sigma s_\epsilon \equiv 0, \text{ mod. } n,$$

where δ, ϵ are divisors of n, such that n/δ is a product of an even number of primes all different; and n/ϵ is a product of an odd number of primes all different. For instance,

$$s_6 - s_3 - s_2 + s_1 \equiv 0, \text{ mod. } 6,$$
$$s_{12} - s_6 - s_4 + s_2 \equiv 0, \text{ mod. } 12.$$

To prove this, the transformation

$$1 + P_1 y + P_2 y^2 + \dots = (1 - Q_1 y)(1 - Q_2 y^2)(1 - Q_3 y^3)\dots$$

is employed. The use of the coefficients Q turns out to be very labour-saving in passing to a series from its logarithm; and these coefficients appear to be of some significance in other respects. For instance, in the expansion of

$$(1 - fy)^{-1/f},$$

all the Q whose subscripts are prime to f are integral, and these Q remain unchanged in the asymptotic factor of the cyclotomic.

It will be seen that, though the object of this paper was to consider especially the case of a determinate f, yet some results have a bearing on the question of the e-section of a cyclotomic, when f is not determinate. But the paper had extended to such a length that it seemed discreet to postpone the development of this side.

Formation of log $\mathfrak{P}$. (Arts. 8–13.)

8. Let x be a root of the equation

$$x^{p-1} + x^{p-2} + \dots + x + 1 = 0,$$

where p is a prime number. Writing, as usual,

$$ef = p - 1,$$

where e, f are integers, there is one set of f-nomial periods of x. The set consists of e periods of which the leading period is

$$\eta = x + x^a + x^{a^2} + \dots + x^{a^{f-1}},$$

where a is a root of unity (mod. p) of order f, that is to say,

$$a^f \equiv 1, \text{ mod. } p,$$

but no lower power of a is congruent to unity.

These e periods are the roots of an equation

$$\eta^e + P_1 \eta^{e-1} + P_2 \eta^{e-2} + \dots + P_{e-1} \eta + P_e = 0,$$

where the coefficients P are integers. The expression on the left is the cyclotomic function discussed in this section. It is convenient to divide throughout by η^e, and then write y for η^{-1}, so that the cyclotomic becomes

$$\mathfrak{P} = 1 + P_1 y + P_2 y^2 + \ldots + P_e y^e.$$

To determine the coefficients, P, we make use of the logarithm of the cyclotomic, viz.,

$$\log \mathfrak{P} = -s_1 y - s_2 \frac{y^2}{2} - s_3 \frac{y^3}{3} - \ldots.$$

The series on the right is infinite, and the s are the power sums of the e periods.

9. Since $$\eta = x + x^a + x^{a^2} + \ldots + x^{a^{f-1}},$$

we have $$\eta^\sigma = (x + x^a + x^{a^2} + \ldots + x^{a^{f-1}})^\sigma$$
$$= A_\sigma + B_\sigma x + C_\sigma x^2 + \ldots + D_\sigma x^{p-1},$$

where A_σ represents the sum of the coefficients of x^p, x^{2p}, x^{3p}, &c. in the expansion of η^σ; B_σ means the sum of the coefficients of x, x^{p+1}, x^{2p+1}, &c. in the same expansion; and the like for the other letters. This transformation only postulates $x^p = 1$; and we may therefore put $x = 1$, which gives

$$A_\sigma + B_\sigma + C_\sigma + \ldots + D_\sigma = f^\sigma.$$

10. The expression for s_σ, the sum of the σ^{th} powers of the e periods, is at once formed from the value of η^σ, by taking account of the fact that s_σ is a symmetrical function of x, x^2, ... x^{p-1}. We have, namely,

$$s_\sigma = eA_\sigma + (B_\sigma + C_\sigma + \ldots + D_\sigma)(x + x^2 + \ldots + x^{p-1}) \frac{e}{p-1}$$
$$= eA_\sigma - (B_\sigma + C_\sigma + \ldots + D_\sigma) \frac{1}{f}$$

(since $x + x^2 + \ldots + x^{p-1} = -1$ and $p - 1 = ef$)

$$= \left(e + \frac{1}{f}\right) A_\sigma - (A_\sigma + B_\sigma + C_\sigma + \ldots + D_\sigma) \frac{1}{f}$$

$$= \frac{p}{f} A_\sigma - f^{\sigma-1}.$$

11. From this we obtain

$$\log \mathfrak{P} = \Sigma f^{\sigma-1} \frac{y^\sigma}{\sigma} - \frac{p}{f} \Sigma A_\sigma \frac{y^\sigma}{\sigma},$$

the summations extending to all positive integral values of σ.

The first sum may be written

$$\frac{1}{f}\,\Sigma f^{\sigma}y^{\sigma}/\sigma = -\frac{1}{f}\log(1-fy),$$

so that

$$(1-fy)^{-1/f}$$

is a "factor" of $\mathfrak{P}$.

12. To determine the value of A_σ, observe that it is the sum of the coefficients of x^p, x^{2p}, ... in the expansion of η^σ, so that

$$A_\sigma = \Sigma\frac{\sigma}{\lambda_0!\,\lambda_1!\,\dots\,\lambda_{f-1}!},$$

where

$$\lambda_0+\lambda_1+\dots+\lambda_{f-1} = \sigma,$$

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\dots+\lambda_{f-1}a^{f-1} \equiv 0, \text{ mod. } p,$$

and the summation includes every system of positive integers,

$$\lambda_0, \lambda_1, \dots \lambda_{f-1},$$

which satisfy this double condition.

13. Considering two systems

$$(\lambda_0, \lambda_1, \dots) \text{ and } (\mu_0, \mu_1, \dots),$$

it is obvious that, if each element of one system is equal to the corresponding element in the other, the two systems give the same term in A_σ; that is, in calculating A_σ, only one of them is counted. But, if any one of the equations $\lambda_0=\mu_0$, $\lambda_1=\mu_1$, ... is not satisfied, the two systems give different terms in A_σ, and each system contributes its full quota to A_σ.

14. It is clear that the congruence is the only effective condition, for the equation may be regarded as merely determining the rank of the A to which a system $(\lambda_0, \lambda_1, \dots)$ contributes. Accordingly, a considerable part of the sequel relates to the theory of the solutions of the congruence. We proceed to classify the solutions, *firstly* into recurring and non-recurring systems (Arts. 15, 16), and *secondly* into central and eccentric systems (Arts. 17–42).

Recurring and non-recurring systems. (Arts. 15, 16.)

15. From any system $\lambda_0, \lambda_1, \dots \lambda_{f-1}$,

which satisfies the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\dots+\lambda_{f-1}a^{f-1} \equiv 0, \text{ mod. } p,$$

we can by cyclic substitution derive others which also satisfy the congruence. For, whatever integer k may be,

$$\lambda_k, \lambda_{k+1}, \dots \lambda_{f-1}, \lambda_0, \lambda_1, \dots \lambda_{k-1}$$

is a solution, as is seen on multiplying both sides of the congruence into a^{f-k}. Every derived system gives to A_σ the same amount as the original system, so that, if the derived systems are distinct from each other, and from the original system (*cf.* Art. 13), the complete contribution from A_σ is

$$f\frac{\sigma!}{\lambda_0!\,\lambda_1!\,\dots\,\lambda_{f-1}!}.$$

On reference to the value of log $\mathfrak{P}$ it will be seen that the denominator of the fractional multiplier cancels out.

16. It may, however, happen that the systems obtained by cyclic substitution are not all different; say

$$(\lambda_k, \lambda_{k+1}, \dots \lambda_0 \dots) = (\lambda_{k+h}, \lambda_{k+h+1}, \dots \lambda_h, \dots).$$

This means that every member of the first system is equal to the corresponding member of the second; that is to say,

$$\lambda_0 = \lambda_h = \lambda_{2h} = \dots, \quad \lambda_1 = \lambda_{h+1} = \lambda_{2h+1} = \dots, \ \&c.$$

The original system may therefore be written

$$\lambda_0, \lambda_1, \dots \lambda_{h-1}, \lambda_0, \lambda_1 \dots \lambda_{h-1}, \lambda_0, \dots\dots \lambda_{h-1},$$

and consists of f/h $(= g)$ cycles. Of the f systems only h are distinct from each other; viz., these are the systems which begin with $\lambda_0, \lambda_1, \lambda_2, \dots \lambda_{h-1}$ respectively. The contribution of the whole set to log $\mathfrak{P}$ is therefore

$$\frac{p}{f}\,h\frac{\sigma!}{(\lambda_0!\dots\lambda_{h-1}!)\,g} = \frac{p}{g}\,\frac{\sigma!}{(\lambda_0!\dots\lambda_{h-1}!)\,g}.$$

Herein g is any divisor of f, including 1 and f. When $g = 1$, the system consists of a single cycle; in other words, it is a non-recurring system, as in Art. 15. For every other value of g the system is recurring, and the total contribution to log $\mathfrak{P}$ is a fractional multiple of the multinomial coefficient.

Solutions independent of p. (Arts. 17, 18.)

17. A second classification of the systems

$$\lambda_0, \lambda_1, \dots \lambda_{f-1},$$

which satisfy the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1} a^{f-1} \equiv 0, \text{ mod. } p,$$

depends upon a more important property. There are some systems which are solutions of the congruence for all values of p of the form $ef+1$; while other systems are solutions for some only of these values.

For the quantity a which appears in this congruence, being a primitive f^{th} root of unity to modulus p, satisfies a congruence of degree τf,* say

$$Fa \equiv 0, \text{ mod. } p.$$

If, then,
$$\lambda_0+\lambda_1 a+\ldots+\lambda_{f-1} a^{f-1}$$
is a multiple of Fa, it is divisible by p.

Now the *equation* $\quad Fa = 0$

determines the primitive f^{th} roots of unity, which have nothing to do with p, so that the coefficients of Fa are also independent of p.

It follows that, if

$$\lambda_0+\lambda_1 a+\ldots+\lambda_{f-1} a^{f-1} = (\mu_0+\mu_1 a+\ldots)\, Fa,$$

where μ_0, μ_1 are any integers, then

$$\lambda_0, \lambda_1, \ldots \lambda_{f-1}$$

is a solution of the proposed congruence for all values of p of the form $ef+1$.

18. A more useful form of this result consists in the explicit statement of the relations between the λ which are necessary and sufficient to ensure that

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1} a^{f-1}$$

should be a multiple of Fa. If we make use of a function, Ga, defined by the equation

$$Fa \,.\, Ga = 1-a^f,$$

the conditions may be written

$$\lambda^k \,.\, G\lambda = 0,$$

where $k = 0, 1, 2, \ldots, \tau f-1$; and every power λ^i or λ^{f+i} is to be replaced by λ_i.

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* τf is Prof. Sylvester's symbol for the totient of f, that is to say, the number of numbers not greater than f and prime to f.

We have, in fact,

$$(\lambda_0+\lambda_1 a+\dots)\,/\,Fa = (\lambda_0+\lambda_1 a+\dots)\;Ga\,/\,(1-a^f).$$

Now, let $$Ga = g_0+g_1 a+g_2 a^2+\dots+g_{f-1}\,a^{f-1},$$

and $$(\lambda_0+\lambda_1 a+\dots)\;Ga = h_0+h_1 a+h_2 a^2+\dots .$$

This last expression is divisible by $1-a^f$, if for all values of k

$$h_k+h_{f+k} = 0.$$

But $$h_k = g_0\lambda_k+g_1\lambda_{k-1}+\dots+g_k\lambda_0,$$

and $$h_{f+k} = g_{k+1}\lambda_{f-1}+g_{k+2}\lambda_{f-2}+\dots+g_{f-1}\lambda_{k+1}.$$

Hence the condition of divisibility is

$$g_0\lambda_k+g_1\lambda_{k-1}+\dots+g_k\lambda_0+g_{k+1}\lambda_{f-1}+\dots+g_{f-1}\lambda_{k+1} = 0,$$

or, symbolically, $$\lambda^k.\,G(\lambda^{-1}) = 0.$$

Since $$Ga = -a^{-\gamma}.\,Ga^{-1},$$

where $\gamma, = f-\tau f$, is such as to make both sides of the same degree in a, the conditions may be written

$$\lambda^k.\,G\lambda = 0.$$

Graphic representation. (Arts. 19, 20.)

19. It is convenient to present the matter graphically. At the vertices of a regular convex polygon of f sides, suppose particles to be placed whose weights are $\lambda_0, \lambda_1, \dots \lambda_{f-1}$. Since the λ represent integers, the weights of particles must be commensurable. If the centroid of the particles is at the centre of the polygon, the system will be called a *central system;* if not, an *eccentric system.* We have, then, the theorem that every solution of the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\dots+\lambda_{f-1}\,a^{f-1} \equiv 0, \text{ mod. } p,$$

which is independent of p, is a central system; and conversely.

20. To prove this, take the centre of the polygon as origin, and the radius through λ_0 for the axis of x. Also take the length of this radius to be unity. Let the polar coordinates of the centroid be r, ϕ. Then

$$re^{\phi i}\,\Sigma\lambda = \lambda_0+\lambda_1 e^{\theta i}+\lambda_2 e^{2\theta i}+\dots+\lambda_{f-1}^{(f-1)\theta i},$$

where $\theta = 2\pi/f$.

Now $e^{\theta i}$ is a primitive f^{th} root of unity, so that

$$F(e^{\theta i}) = 0.$$

Hence it follows that, if $\lambda_0+\lambda_1 a+\lambda_2 a^2+\dots$ is divisible by Fa (that is to say, if $\lambda_0, \lambda_1, \lambda_2 \dots$ satisfies the congruence independently of p), then

$$r = 0,$$

and the system is a central system. Conversely, if the system is a central system,

$$\lambda_0+\lambda_1 e^{\theta i}+\lambda_2 e^{2\theta i}+\dots = 0,$$

and the expression on the left must be a multiple of $F(e^{\theta i})$; because $F(e^{\theta i})$ is irreducible.

Central Systems: Particular Cases. (Arts. 21–23.)

21. When f is a prime number,

$$G\lambda = 1-\lambda,$$

so that for a central system we have

$$\lambda_0 = \lambda_1 = \lambda_2 = \dots = \lambda_{f-1},$$

that is to say, all the particles must be of equal weight.

22. When f is the product of two different primes, say

$$f = a \, . \, b.$$

Supposing a to be the smaller factor, starting at any particle, form a clusters, each made up of a consecutive particles, the clusters being arranged symmetrically around the f-gon. All these clusters must have the same weight if the system is central. Starting at another point, we get another set of clusters which must be of equal weight, but not necessarily of the same weight as the clusters of the first set. For instance, in a hexagon the particles at the ends of any side must together weigh as much as the two particles at the ends of the opposite side.

All this is the interpretation of the equation

$$G\lambda = (1+\lambda+\lambda^2+\dots+\lambda^{a-1})(1-\lambda^b).$$

23. When f contains powers of primes, we may write

$$f = a, b \dots c \, . \, \pi \; (= f' . \, \pi \text{ say}),$$

where π is a product of powers of $a, b, \dots c$ (or some of them), but does not contain any other prime. Now, in this case, $F\lambda$, $G\lambda$ contain λ only in the powers λ^{π}, $\lambda^{2\pi}$, &c. That is to say, the relations for a central system involve

$$\lambda_h, \lambda_{h+\pi}, \lambda_{h+2\pi}, \ldots,$$

which are particles on an f'-gon. Hence it appears that, if a system on the f-gon is a central system, the particles on each f'-gon must form a central system. For example, if a dodecagon bears a central system, the particles on each of the regular hexagons in the figure must also form a central system.

This remark enables us to confine the discussion, where convenient, to the cases in which f has no square factor, without loss of generality.

Central System: General Case. (Arts. 24–33.)

24. We proceed to prove two properties of central systems when f is unrestricted in value. In expressing the first theorem it is convenient to speak of a central system on a regular polygon of a prime number of sides as a prime central system. The theorem may then be stated thus:—Every central system on an f-gon is identical with the sum of prime central systems belonging to the figure, or with the difference of two such sums.

Let the prime factors of f be $a, b, \ldots c$. Suppose the f-gon to be loaded with particles of weights $A_0, A_1, A_2, \ldots A_{f-1}$ so that the particles on each regular a-gon form a central system. This implies that the particles on each a-gon are of equal weight, so that

$$A_r = A_s \text{ if } r \equiv s, \text{ mod. } f/a.$$

Let $B, \ldots C$ have similar meanings with respect to the $b, \ldots, c$-gons. It is to be proved that, when

$$\lambda_0, \lambda_1, \ldots \lambda_{f-1}$$

form a central system, values of $A, B, \ldots C$ can be found to satisfy all the equations

$$\lambda_r = A_r + B_r + \ldots + C_r \; (r = 0, 1, 2, \ldots f-1),$$

where

$$A_r = A_s \text{ if } r \equiv s, \text{ mod. } f/a,$$
$$B_r = B_s \text{ if } r \equiv s, \text{ mod. } f/b,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$C_r = C_s \text{ if } r \equiv s, \text{ mod. } f/c.$$

Consider any linear function of $\lambda_0, \lambda_1, \lambda_2, \ldots$ which, when expressed in terms of the $A, B, \ldots C$, is free from A. Then for every λ_r in the function there must be another λ, say λ_s, with the opposite sign, and having $s \equiv r$, mod. a' (a' written for f/a). If, then, in this linear function of the λ we change subscripts into indices, the result will be divisible by $1-\lambda^{a'}$, because it is made up of pairs of terms each of which is so divisible. Similarly, any linear combination of λ_0, λ_1, &c. which is free from $A, B, \dots C$ must, when subscripts are changed to indices, be divisible by $1-\lambda^{a'}$, $1-\lambda^{b'}$, ... $1-\lambda^{c'}$, and thus it is divisible by the lowest common multiple of $1-\lambda^{a'}$, $1-\lambda^{b'}$, ... $1-\lambda^{c'}$, which is the function $G\lambda$ of Art. 18. The order of $G\lambda$ being γ, $=f-\tau f$, it follows that the γ equations

$$\lambda_r = A_r+B_r+\dots+C_r \; (r = 0, 1, 2, \dots \gamma-1)$$

imply no relation between the λ.

When $A, B, \dots C$ have been determined so as to satisfy these γ equations, λ_γ is determined by the condition (necessary to ensure that the system λ_0, λ_1, ... is central)

$$G\lambda = 0.$$

Now $$\lambda_\gamma = A_\gamma+B_\gamma+\dots+C_\gamma$$

satisfies this equation: for it is clear that an aggregate of central systems $A, B, \dots C$ must be central. And, since $G\lambda$ is linear in λ_γ, no other value of λ_γ is possible. Similarly, it follows that

$$\lambda_r = A_r+B_r+\dots+C_r \; (r = \gamma+1, \gamma+2, \dots f-1).$$

25. The determination of $A, B, \dots C$ may be effected in the following manner:—To determine the A, eliminate the $B, \dots C$ from the γ equations of the last article. The result of the elimination is written down by forming the lowest common multiple of $\lambda^{b'}-1, \dots \lambda^{c'}-1$, multiplying this by $\lambda^0, \lambda, \lambda^2, \dots$ in turn, until the power $\lambda^{\gamma-1}$ appears. In this expression change indices into subscripts, and equate the result to the corresponding function of the A. The system of equations thus formed will be satisfied if we assign arbitrary integral values to any $f/a-\tau(f/a)$ of the A, and determine the rest suitably. To determine the B, eliminate all the $A, B, \dots C$ except A, B. The A being known, it will be found that the equations for B are satisfied when arbitrary values are assigned to $f/b-\tau(f/b)-\tau(f/ab)$ of the B (but these may not be arbitrarily selected). Similarly all the rest of the weights $A, B, \dots C$ may be determined.

26. In spite of the number of arbitrary elements in the values of $A, B, \dots C$, it is not generally possible to make them all positive. An example will suffice to prove this. Take the pentagons in a 30-gon.

One of the equations for the A ($a = 5$) is

$$\lambda_{20}+\lambda_{15}-\lambda_5-\lambda_0 = A_2+A_3-A_5-A_0$$

(for $A_{20} = A_2$ and $A_{15} = A_3$). Now, if all the A, B, C are positive, we must have

$$A_2+A_3 \leqq \lambda_2+\lambda_3$$

(for $\lambda_2 = A_2+B_2+C_2$, $\lambda_3 = A_3+B_3+C_3$),

and

$$-A_5-A_0 \leqq 0,$$

Hence, if all the A, B, C are positive, we must have

$$\lambda_{20}+\lambda_{15}-\lambda_5-\lambda_0 \leqq \lambda_2+\lambda_3;$$

and, when this inequality does not hold, some of the A, B, C must be negative.

It is this that makes necessary the alternative statement in the theorem enunciated, Art. 24.

27. The argument of Art. 24, proves that, when particles of any weights (λ_0, λ_1, λ_2, ...) have been placed at $f-rf$ consecutive vertices of an f-gon, there is one, and only one, central system which includes these particles. It should, however, be noted that this assumes that negative weights are admissible. The theorem cannot be extended to an arbitrarily selected set of $f-rf$ points (for instance, we cannot assign all the weights which actually appear in $G\lambda$); but there are some sets of $f-rf$ non-consecutive points which may be arbitrarily weighted, as we shall now show.

28. We assume that f contains no square factor, an assumption which does not really affect the generality of the result (Art. 23). Upon the f-gon select any vertex as the zero, and number the other vertices in order. A vertex whose number is a totitive of f (*i.e.*, prime to f), being called a totitive point, we have the theorem:—When arbitrary weights are placed at the non-totitive points, the totitive points can always be weighted, and that in one way only, so that the whole may be a central system. Negative weights will generally occur, but it will be shown how the difficulties thus arising may be set aside.

29. The proof of the theorem consists in writing down and verifying an expression for the weight, λ_t, at any totitive point t, in terms of the weights at the non-totitive points.

Let A be any divisor of f; then, since f contains no square factor, A

and f/A are prime to each other, and there is therefore one, and only one, positive integer, α, less than f which satisfies the congruences

$$\alpha \equiv 0, \text{ mod. } A, \equiv t, \text{ mod. } f/A.$$

Let α be determined for every divisor, A, of f, including 1 and f. Take the results as subscripts to λ and prefix to the λ the sign + or − according as the divisor A contains an even or an odd number of prime factors. It will now be proved that the aggregate of λ thus formed contains only one λ, viz. λ_t, with a totitive subscript; and that it vanishes when the λ form a central system. These two properties being proved, the theorem is established; for we have λ_t expressed in terms of the weights at the non-totitive points.

The first property comes at once from observing that α is a multiple of A, and therefore cannot be a totitive of f except when $A = 1$. In this excepted case $\alpha = t$.

The divisors of f may be arranged in two classes with respect to any prime factor, a, of f. One class contains the divisors A, B, C, ... which are not multiples of a; the other consists of Aa, Ba, Ca, ... Together these make up all the divisors of f, which by hypothesis does not contain the factor a^2. To determine the subscripts α, α' belonging to the pair of divisors A, Aa, we have

$$\alpha \equiv 0, \text{ mod. } A, \quad \equiv t, \text{ mod. } Ba,$$

$$\alpha' \equiv 0, \text{ mod. } Aa, \quad \equiv t, \text{ mod. } B,$$

where $B = f/Aa$.

By subtraction, $\alpha - \alpha' \equiv 0$, mod. A, $\equiv 0$, mod. B,

so that $\alpha - \alpha' \equiv 0$, mod. AB,

that is to say, $\alpha - \alpha' \equiv 0$, mod. f/a.

On the other hand, since

$$\alpha \equiv t, \text{ mod. } Ba, \text{ and } \alpha' \equiv 0, \text{ mod. } Aa,$$

it follows that $\alpha - \alpha' \equiv t$, mod. a;

so that α, α' cannot be equal, since t is not a multiple of a.

This pair α, α' gives to the aggregate of the λ a pair of terms

$$\lambda_\alpha - \lambda_{\alpha'}$$

with opposite signs, because Aa contains one prime factor more than

A; and it has been shown that

$$a - a' \equiv 0, \text{ mod. } f/a.$$

It follows at once that the aggregate of the λ, when indices are written instead of subscripts, is divisible by $1 - \lambda^{f/a}$. Similarly, treating the other prime factors of f, it appears that the aggregate is divisible by $G\lambda$ (as in Art. 24), and therefore vanishes for a central system.

30. An attempt to form a relation between the λ for non-totitive points leads to mere identities. Suppose, for instance, t, instead of being a totitive, were a multiple of a. Then the a, a' determined as above would be equal, and the λ-aggregate would consist of vanishing pairs $\lambda_a - \lambda_a$.

31. It is only necessary to find half of the a; for, if a correspond to the divisor A, then $f + t - a$ corresponds to the divisor f/A. Moreover, when one totitive λ, preferably λ_1, has been expressed in terms of the non-totitive λ, the values of the others can be found at once by multiplying the subscripts into the several totitives in turn. If a non-totitive multiplier be used, a mere identity results, as it should.

32. An example will make this clear. Let $f = 30$. Then for λ_1 we have

$A = 1,$	2,	3,	5,	6,	10,	15,	30
$a = 1,$	16,	21,	25,	6,	10,	15,	0.

The table is divided by vertical lines into compartments. In the first of these, A contains 0 factor; in the second it contains 1 factor; in the third, 2; and in the last, 3. Thus the λ-aggregate is

$$\lambda_1 - \lambda_{16} - \lambda_{21} - \lambda_{25} + \lambda_6 + \lambda_{10} + \lambda_{15} - \lambda_0 = 0.$$

If we multiply the subscripts by any totitive of 30, 7 for example, we get

$$\lambda_7 - \lambda_{22} - \lambda_{27} - \lambda_{25} + \lambda_{12} + \lambda_{10} + \lambda_{15} - \lambda_0 = 0,$$

which determines λ_7. But if we multiply by any non-totitives, such as 5, 6, we get the identities

$$\lambda_5 - \lambda_{20} - \lambda_{15} - \lambda_5 + \lambda_0 + \lambda_{20} + \lambda_{15} - \lambda_0 = 0,$$

$$\lambda_6 - \lambda_6 - \lambda_6 - \lambda_0 + \lambda_6 + \lambda_0 + \lambda_0 - \lambda_0 = 0.$$

33. It is an obvious corollary to the theorems of Arts. 24, 28, that if a polygon has $f - \tau f$ independent vertices (consecutive vertices, or

non-totitive vertices, for instance), but not all its vertices, unloaded, the system cannot be a central system.

Eccentric Systems. (Arts. 34–42.)

34. Every eccentric system is necessarily one of the non-recurring systems considered in Art. 15. For the centroids of any eccentric system, and those derived from it by cyclic substitutions, have f different positions; viz., one is in each of the sectors of the polygon. And it is plain that two systems cannot be identical if they have different centroids. It is not to be understood, and it is not a fact, that eccentric systems are the only non-recurring systems. For example: $f = 6$; 011102 is a central system which is non-recurring.

We proceed to consider the properties of systems which have their centroids at the same point, and those which have centroids on the same radius of the polygon. It is convenient to take separately the cases in which f is, and those in which f is not, a prime.

Concentric Systems: f prime.

35. Let $(\mu_0, \mu_1, \ldots)$ and $(\nu_0, \nu_1 \ldots)$ be two systems on the same f-gon which have a common centroid whose coordinates are r, ϕ. Then

$$re^{\phi i}\Sigma\mu = \Sigma\mu_k e^{k\theta i}, \quad (\theta = 2\pi/f),$$

$$re^{\phi i}\Sigma\nu = \Sigma\nu_k e^{k\theta i}.$$

Eliminating r,

$$\Sigma\,(\mu_k\Sigma\nu - \nu_k\Sigma\mu)\,e^{k\theta i} = 0$$

Therefore particles of weight $\mu_k\Sigma\nu - \nu_k\Sigma\nu$ form a central system, so that

$$\mu_0\Sigma\nu - \nu_0\Sigma\mu = \ldots = \mu_k\Sigma\nu - \nu_k\Sigma\mu = \ldots = \mu_{f-1}\Sigma\nu - \nu_{f-1}\Sigma\mu$$

$$= \frac{1}{f}\,(\Sigma\mu\,.\,\Sigma\nu - \Sigma\nu\,.\,\Sigma\mu) = 0,$$

whence $$\mu_0 : \nu_0 = \ldots = \mu_k : \nu_k = \ldots;$$

or, what is equivalent, $(\mu_0, \mu_1, \ldots)$ and $(\nu_0, \nu_1, \ldots)$ are both multiples of a third system $(\lambda_0, \lambda_1 \ldots)$ which has its centroid at the same place; and it may be taken that

$$\lambda_0, \lambda_1, \ldots$$

have no common factor but unity. It is obvious that all multiples of $(\lambda_0, \lambda_1, \ldots)$ will be concentric with it; and it has just been proved that there are no other systems concentric with it.

36. It is to be noted that, if $(\mu_0, \mu_1, \ldots)$ satisfy the congruence of Art. 12, viz.,

$$\mu_0 + \mu_1 a + \mu_2 a^2 + \ldots \equiv 0, \text{ mod. } p,$$

the same is true of $(\lambda_0, \lambda_1, \ldots)$, save only when all the μ are multiples of p, a case which will be discussed hereafter (Art. 60).

Co-radial centroids. (Arts. 37–39.)

37. Next consider two systems whose centroids, G_1, G_2, are in a common radius OG_1G_2; and suppose $OG_2 > OG_1$. If OG_1, OG_2 are commensurable, the system G_1 can be derived from the system G_2 by combining a proper multiple of the set at G_2 with a central system. It follows from Art. 35 a system formed in this way so as to have its centroid at G_1 must be identical with the given system, or they must both be multiples of a system centred at G_1. It is more useful, however, to proceed outwards from G_2; by subtracting central systems from the given system G_2. This may be continued until one of the reduced weights vanishes. The centroid of the system thus obtained is an extreme point for that particular radius; and all the systems co-radial with it may be derived by adding central systems to it. It is noticeable that between O and the centroid G of the extreme system there are an infinite number of centroids, viz., a centroid at every point G_1, such that OG_1 is commensurable with OG; but beyond G there is not any centroid whose distance from O is commensurable with OG. The extreme system has an advantage in that it is instantly distinguished from a central system in which all the weights must vanish when one of them vanishes.

38. Whether there can be upon one radius two centroids G_1, G_2, such that OG_1, OG_2 are incommensurable, I cannot say. But, if so, the two sets of centroids would be as distinct as if they were in different radii.

39. When f is composite, a centroid does not belong exclusively to one system and its multiples. For, if $(\mu_0, \mu_1, \ldots)$ and $(\nu_0, \nu_1, \ldots)$ be two central systems such that $\Sigma\mu = \Sigma\nu$; and $(\lambda_0, \lambda_1 \ldots)$ be any eccentric system, then $(\lambda_0 - \mu_0 + \nu_0, \lambda_1 - \mu_1 + \nu_1, \ldots)$ is concentric with $(\lambda_0, \lambda_1, \ldots)$, and has the same total weight; but clearly it is not identical with it. And the same sort of thing happens with co-radial systems. But, in

reference to the latter, there is a remark of some interest relating to extreme systems.

Extreme systems with all weights positive. (Arts. 40–42.)

40. Let there be an eccentric system E, upon an f-gon, the vertices of which are numbered 0, 1, 2, ... in order. Let C be a central system the particles of which at the non-totitive points are of the same weight as the corresponding particles of E. Then the system $E-C$ will have its $f-rf$ non-totitive vertices unloaded, and its centroid will lie on the same radius as that of E. In general, however, at some of the totitive points $E-C$ will comprise negative weights, and so be unavailable. We shall now explain how this may be modified; and for this purpose we re-state, graphically, the results already communicated in Art. 47 of the first section of this memoir. To fix the ideas, the particular case in which $f = 15$ is discussed; but the method is quite general.

41. The totitives of 15 form a group which may be expressed, and that in one way only, as a product of two simple groups, one belonging to the factor 5, and the other to the factor 3, of 15. The decomposed form is

$$(1\,.\,7\,.\,4\,.\,13)\,(1\,.\,11).$$

The totitives may therefore be considered as distributed on two pentagons of the 15-gon, four points of each being occupied. *Cf.* Fig. 1, in which the continuous lines give the deficient pentagon,

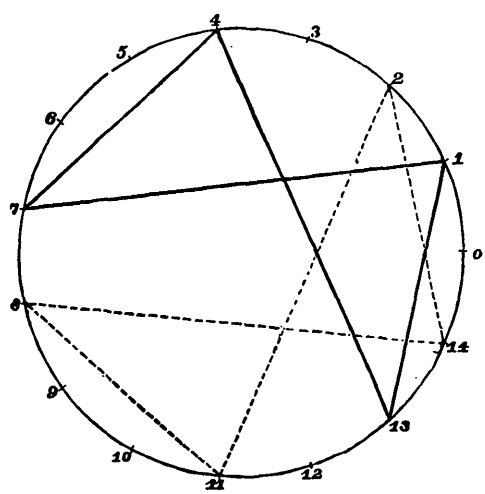

Fig. 1.

1, 7, 4, 13, and the dotted lines mark the multiple, 11, 2, 14, 8. Or, they may be considered as distributed on four equilateral triangles (two points of each being occupied). These are marked in the second figure, the continuous line indicating the group 1, 11, while the dotted lines show the multiples (7, 2), (4, 14), (13, 8).

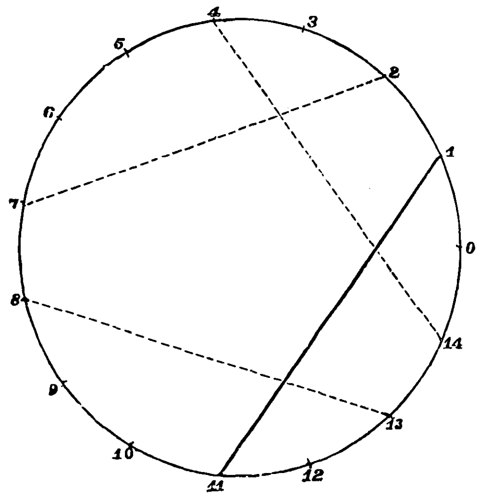

FIG. 2.

42. Suppose now that the system $E-C$ has negative weights at some of the totitive points; say it contains $-\lambda_2$, $-\lambda_4$, $-\lambda_8$, and let $\lambda_2 > \lambda_8$. If now λ_4 be added to each vertex of the pentagon 1, 4, 7, 10, 13, and λ_2 to each vertex of the pentagon 2, 5, 8, 11, 14 (these additions being two central systems), an eccentric system is obtained whose centroid is co-radial with that of E, and which is positively weighted at the points 1, 5, 7, 8, 10, 11, 13, 14.

Or, we may place a particle of weight λ_2 at each vertex of the triangle, 2, 7, 12; λ_4 at each vertex 4, 9, 14; and λ_8 at each vertex 3, 8, 13. We then get an eccentric system co-radial with E, positively weighted at the eight points 1, 3, 7, 9, 11, 12, 13, 14, and comprising no other particles.

These systems with $f - rf$ unloaded points, and the others positively weighted, are analogous to the extreme systems described in Art. 37; but the relations between the different forms require further examination.

General method of solving the fundamental congruence. (Art. 43.)

43. The method which gives numerical solutions of the congruence

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-1} a^{f-1} \equiv 0, \text{ mod. } p,$$

and the means by which it is ensured that no suitable solution is excluded, will now be explained. For the calculation of the cyclotomic, we require only those solutions

$$\lambda_0, \lambda_1, \lambda_2, \ldots \lambda_{f-1},$$

such that $$\sigma, = \lambda_0+\lambda_1+\lambda_2+\ldots+\lambda_{f-1},$$

does not exceed e; though, for a check on the work, it is desirable to extend the limit to $e+1$. Hence solutions in small positive integers are especially required. To obtain these we select the smallest value of a, and then express p, $2p$, $3p$, ... as numbers in the scale of a. The "digits" in any one of these numbers form a solution of the congruence. Supposing the multiplication table formed so far as to include all multiples of p which are less than a^f, all the solutions will be obtained in which the λ are less than a. If a is less than e, this condition may possibly exclude appropriate solutions; but the solutions excluded thus may easily be formed by replacing any consecutive λ, say λ_k, λ_{k+1} (including the case $k=f-1$, $k+1=0$) by λ_k+a, $\lambda_{k+1}-1$. This increases σ by $a-1$: and therefore the number of times the operation may be repeated is fixed beforehand. Of course, when a solution may be amplified in this way, every consecutive pair of λ must be modified, giving f new solutions.

Abbreviated methods. (Arts. 44–46.)

44. The process described enables us to determine the solutions to any proposed extent, and with no risk of omission. But, unless f and a are small, the work is very great, and only a small percentage of the solutions obtained are suitable (*i.e.*, are such that $\sigma < e+1$). As it generally happens, so in this case, many artifices serve to reduce the labour to a manageable amount. Some of these will now be noticed.

It is convenient to calculate the central systems separately. When f is a prime, these are merely 111 ..., 222 ..., 333 ..., &c. When f is a power of a prime, then the central systems are combinations of the central systems for the prime, sandwiched together. For instance, $f=9$; the central systems for $f=3$ being 000; 111; 222, &c., the central systems for $f=9$ are such as 012012012 or 021021021. When f is a product of different primes, the central systems may be formed by using the theorem of Art. 24. Remembering that some of the A, B, C may be negative, it is necessary to examine the solutions formed by subtraction as well as by addition; and probably it will be found best to make up a special rule for each case. In the case $f = 6$, the only case I have worked with, it appears that all central systems can be formed by addition merely—and a rule is at once suggested by which the central systems can be written down in regular succession by an almost mechanical process.

45. The central systems being found as far as necessary, the multiplication table can at once be safely reduced to a fraction of its original extent. For instance, when f is a prime the table need not extend beyond a^{f-1}, since all eccentric systems can be derived by adding central systems to eccentric systems which contain at least one 0. When f is not a prime, the reduction is considerably more important.

46. One other contrivance may be mentioned. Take any two solutions whose total weights are σ_1, σ_2, respectively. If these, or any cyclic permutations of them, be added together, the result is a new solution whose sum is $\sigma_1+\sigma_2$; but if any "carrying" has been done in the addition, then for each "carrying" the new σ is reduced by $a-1$. As an example of this, the case of $p = 71$, $f = 7$, $a = 20$ may be quoted. A multiple of p is found to be 0000 . 1 . 19 . 1 (which means $a^2+19a+1$). Adding this to 000.1.19.1.0, the solution 0002101 is obtained. Another solution 0001.0.1.3 occurs early in the table: and from these and the central system all the suitable solutions may be compounded.

Of course, in any such short cut, there is considerable risk of omitting solutions; but such omissions are detected by trying whether the value of P_{e+1}, a coefficient of the cyclotomic, vanishes as it will do if the work has been complete.

Minimum of σ for eccentric systems. (Arts. 47, 48.)

47. When f is a prime, it sometimes happens—for instance, when $p = 31$, $f = 5$, $a = 2$—that the value of p, when expressed in the scale of a, is 111 ... 1. If in such a number two consecutive 1's are replaced by 0, $a+1$, the resulting system is an eccentric system. Hence there is an eccentric system for which

$$\sigma = a + f - 1.$$

Probably this is the minimum value of σ. But, however that may be, it is easy to show that, in this case, the minimum value of σ is greater than a. It is clear that a system for which σ is a minimum must contain at least one zero. For a system which contains no zero can be made by adding a central system to the extreme system coradial with it. This implies that in the addition there is no "carrying": for carrying alters the radius on which the centroid lies. Suppose the system cyclically transformed so that λ_{f-1} is a zero. Then

$$\lambda_0+\lambda_1 a+\lambda_2 a^2+\ldots+\lambda_{f-2} a^{f-2}$$

is a multiple of p. But, that this may be possible, at least one of the λ must be greater than a; for, if all of them be made equal to a, the sum

$$= a+a^2+\ldots+a^{f-1}=p-1$$

by hypothesis, and therefore cannot be a multiple of p. Since, then, one of the λ is greater than a, it follows *a fortiori* that

$$\sigma > a.$$

Since $$a^f > p,$$

it follows further that $$\sigma > \sqrt[f]{p}.$$

48. There is a presumption in favour of the theorem that, when f is prime, $1+a+a^2+\ldots+a^{f-1}$ is a prime number for an infinite number of values of a; and that for composite values of f, Fa contains an infinite number of primes. If this were so, it would follow that, by taking p sufficiently large, the minimum value of σ may be made as large as we please. But the examination of a considerable number of examples seems to indicate that the inequality, $\sigma > \sqrt[f]{p}$, holds good for all values of p, and not merely for those which have the special property assumed in the last article. If this guess is right, the inequality ought to be capable of proof without making the assumption; but I have not succeeded in finding such a proof.

Determination of $\mathfrak{P}$. (Art. 49.)

49. The value of log $\mathfrak{P}$ having been determined, the value of $\mathfrak{P}$ can be found by the rules given in text-books on the theory of equations. It is unnecessary to dwell upon this further than to remark that the use of the Q form introduced in Art. 50 appears to give the reduction with less trouble than the ordinary process.

Conditions that a function given by its logarithm should have all its coefficients integral. (Arts. 50–53.)

50. For the purposes of the present paper it is essential to determine the conditions that a function, given by its logarithm,

should have all its coefficients integral, the first coefficient being 1. The function will be represented by

$$1+P_1y+P_2y^2+\dots\dots\dots\dots$$

Now let

$$1+P_1y+P_2y^2+\dots=(1-Q_1y)(1-Q_2y^2)(1-Q_3y^3)\ \dots,$$

where the $=$ is meant to express that the values of P_1, P_2, ... and of Q_1, Q_2 ... up to any assigned extent, are such that the two expressions are identical up to that extent: so that there is no question of convergence. In other words, the equation is an abbreviation of the several statements,

$$\begin{aligned}
P_1 &= -Q_1 \\
P_2 &= -Q_2 \\
P_3 &= -Q_3+Q_1Q_2 \qquad\qquad (P,\ Q), \\
&\dots\dots\dots\dots \\
P_6 &= -Q_6+Q_1Q_5+Q_2Q_4-Q_1Q_2Q_3, \\
&\&c.
\end{aligned}$$

It follows that

$$\begin{array}{llllllll}
\log(1+P_1y+P_2y^2+\dots) & & & & & & & \\
=\log(1-Q_1y) & =-Q_1y & -Q_1^2\frac{y^2}{2} & -Q_1^3\frac{y^3}{3} & -Q_1^4\frac{y^4}{4} & -Q_1^5\frac{y^5}{5} & -Q_1^6\frac{y^6}{6} & -\&c. \\
+\log(1-Q_2y^2) & & -Q_2y^2 & & -Q_2^2\frac{y^4}{2} & & -Q_2^3\frac{y^6}{3}\dots & \\
+\log(1-Q_3y^3) & & & -Q_3y^3 & & & -Q_3^2\frac{y^6}{2}\dots & \\
+\log(1-Q_4y^4) & & & & -Q_4y^4 & & & \\
+\log(1-Q_5y^5) & & & & & -Q_5y^5 & \dots & \\
+\log(1-Q_6y^6) & & & & & & -Q_6y^6\dots & \\
+\ \&c. & & & & & & & -\&c.
\end{array}$$

On the other hand we have

$$\log(1+P_1y+P_2y^2+\dots)=-s_1y-s_2\frac{y^2}{2}-s_3\frac{y^3}{3}-\&c.$$

Hence
$$\begin{aligned}
s_1 &= Q_1, \\
s_2 &= 2Q_2+Q_1^2, \\
s_3 &= 3Q_3+Q_1,
\end{aligned}$$

$$s_4 = 4Q_4 + 2Q_2^2 + Q_1^4, \qquad (s, Q),$$

$$s_5 = 5Q_5 + Q_1^5,$$

$$s_6 = 6Q_6 + 3Q_3^2 + 2Q_2^3 + Q_1^6,$$

and, generally, $$s_n = nQ_n + \ldots + \delta Q_\delta^{\delta'} + \ldots + Q_1^n,$$

where δ, δ' are integers, such that $\delta\delta' = n$, and the expression includes all values of δ.

51. From the equations (P, Q) it is seen that, if $Q_1, Q_2 \ldots Q_n$ are integers, then $P_1, P_2 \ldots P_n$ are likewise integers. From the equations (s, Q) it appears that, for $Q_1 \ldots Q_n$ to be integers, it is necessary (but not sufficient) that $s_1 \ldots s_n$ should be integers. Suppose that $Q_1 \ldots Q_{n-1}$ are all integral; then, in order that Q_n may be integral, it is necessary that s_n when divided by n should leave a certain remainder which is determined by the values of some of the Q of lower rank. This remainder is, however, more simply expressed in terms of the s of lower rank; and it comes out that the relations necessary and sufficient to ensure that the Q are integers are such as

$$s_{ab} - s_a - s_b + s_1 \equiv 0, \text{ mod. } ab,$$

$$s_{a^2b} - s_{ab} - s_{a^2} + s_a \equiv 0, \text{ mod. } a^2b,$$

where a, b are primes. The rule for forming the critical expression for s_n is this. Divide n by each of its prime factors in turn; use the quotients as subscripts to s; and denote the aggregate of the s thus found by Σ_1. Again, divide n by the product of every pair of different prime factors. The aggregate of the s whose subscripts are the quotients may be called Σ_2. The process is to be continued as far as possible. Then the critical expression is

$$s_n - \Sigma_1 + \Sigma_2 - \Sigma_3 + \ldots,$$

and, if this is divisible by n, Q_n is integral (presuming that the preceding Q are all integral).

The analogy of this rule to that for forming the function Fa is too striking to escape notice.

52. The proof comes from considering the value of the critical expression when written in terms of Q. For instance,

$$s_{ab} - s_a - s_b + s_1,$$

in terms of Q, is

$$ab \,.\, Q_{ab} + a\,(Q_a^b - Q_a) + b\,(Q_b^a - Q_b) + Q_1^{ab} - Q_1^b - Q_1^a + Q_1.$$

Each of the first three terms is seen to be divisible by ab. The part containing Q_1 may be written

$$(Q_1^b)^a - Q_1^b - \{Q_1^a - Q_1\},$$
$$(Q_1^a)^b - Q_1^a - \{Q_1^b - Q_1\},$$

so that it is divisible by a, and by b, and therefore by ab. If, then, Q_{ab}, Q_a, Q_b, Q_1 are integral, $s_{ab} - s_a - s_b + s_1$ must be divisible by ab; and, conversely, if Q_a, Q_b, Q_1 are integral and $s_{ab} - s_a - s_b + s_1$ divisible by ab, Q_{ab} is integral.

Again, consider the example

$$s_{a^2b} - s_{ab} - s_{a^2} + s_a,$$

which is
$$= a^2 b Q_{a^2b} + a^2 (Q_{a^2}^b - Q_{a^2}) + ab (Q_{ab}^a - Q_{ab})$$
$$+ a (Q_a^{ab} - Q_a^b - Q_a^a + Q_a)$$
$$+ Q_1^{a^2b} - Q_1^{ab} - Q_1^{a^2} + Q_1^a.$$

The parts involving Q_{a^2b}, Q_{ab}, Q_{a^2}, Q_a are seen to be divisible by a^2b, as in the first example. The part containing Q_1 is divisible by b, for it may be written

$$(Q_1^{a^2})^b - Q_1^{a^2} - \{(Q_1^a)^b - Q_1^a\},$$

and it is also divisible by a^2, for

$$Q^{a^2} \equiv Q^a, \text{ mod. } a^2, \text{ and } Q^{a^2b} \equiv Q^{ab} \text{ mod. } a^2,$$

by reason of the generalized Fermat's theorem.

53. These examples suggest, what is obvious when the suggestion has been made, that, when a is prime, the part involving Q_{ah} in the expression for ahk is (omitting the external multipliers) of the same form as the part involving Q_h in the hk function, and by repeating the reduction it is of the same form as the part involving Q_1 in the critical expression for k. Hence it is only needful to consider the part of the critical expression which involves Q_1; and this part is written down by merely changing the subscripts of the critical expression into indices. When this is done the divisibility by n is seen at once by using the generalization of Fermat's theorem. For let $n = a^s A$, where a is prime and A is prime to a. Suppose one of the divisors of n formed by our rule to be $a^s B$, so that A/B contains no square factor. Then $a^{s-1}B$ is another divisor of n, also furnished by the rule, and in the critical expression we have a pair of terms,

$$\pm (Q_1^{a^s B} - Q_1^{a^{s-1} B}).$$

This is divisible by a^s. As the whole expression can be expressed in pairs of this nature, it also is divisible by every prime-power in n; and therefore by n.

Application to series comprised in log $\mathfrak{P}$. (Arts. 54–57.)

54. I proceed to show that, if $\lambda, \mu, \ldots \nu$ (written for the $\lambda_0, \lambda_1, \ldots$ previously used) be any positive integers whose greatest common measure is unity and whose sum is σ, then

$$s_n = \frac{n\sigma!}{n\lambda!\, n\mu! \ldots n\nu!} \cdot \frac{1}{\sigma}, = \frac{(n\sigma-1)!\, n}{n\lambda!\, n\mu! \ldots n\nu!},$$

satisfies the conditions obtained in Art. 51.

55. It is easy to see that s_n is integral. For it may be written

$$\frac{(n\sigma-1)!}{n\lambda!\, n\mu! \ldots (n\nu-1)!} \cdot \frac{n}{n\nu} = \frac{N}{\nu},$$

where N is some integer. Similarly, it is $= L/\lambda,\ M/\mu, \ldots$ where $L, M, \ldots$ are integers. But the greatest common measure of $\lambda, \mu, \ldots \nu$ is 1, so that the denominator of these equal fractions when in lowest terms must also be 1; that is to say, s_n is an integer.

56. We shall now prove that

$$s_{a^s A}(1+ha^s) = s_{a^{s-1}A}(1+ka^s),$$

where a is prime, A prime to a, and h, k are integers.

This at once gives $\quad s_{a^s A} \equiv s_{a^{s-1}A}$, mod. a^s;

and hence, as in Art. 53, it follows that the critical expression is divisible by n.

It is convenient to write B for $a^{s-1}A$ in some places, and the theorem to be proved may then be written

$$s_{aB}(1+ha^s) = s_B(1+ka^s).$$

Consider the expression $(aB\rho)!$ It contains multiples of a, the product of which is

$$a \,.\, 2a \,.\, 3a \ldots B\rho a = a^{B\rho} \,.\, (B\rho)!$$

The remaining factors may be arranged in $A\rho$ products of which the first is

$$1 \,.\, 2, \ldots (a-1)(a+1) \ldots (a^s-1),$$

the second is formed by adding a^s to each factor of the first; the third by adding $2a^s$, and so on. Now, by Gauss' generalization of Wilson's theorem, the first product

$$= -1 + ma^s,$$

where m is an integer. Clearly the other products are of the same form. Hence the product of all the factors of $(aB\rho)!$ which are prime to a

$$= (-1)^{A\rho} + ma^s,$$

and $$(aB\rho)! = a^{B\rho} . (B\rho)! \left\{-1)^{A\rho} + ma^s\right\},$$

where m is some integer.

Hence, if ρ be replaced by $\sigma, \lambda, \mu, \ldots \nu$, in turn, we shall have

$$s_{aB} = \frac{1}{\sigma} \cdot \frac{(aB\sigma)!}{(aB\lambda)! \ldots (aB\nu)!} = \frac{1}{\sigma} \cdot \frac{(B\sigma)!}{(B\lambda)! \ldots (B\nu)!} \cdot \frac{(-1)^{A\sigma} + ka^s}{(-1)^{A\sigma} + ha^s},$$

for it is at once seen that powers of a cancel. But this is

$$s_{aB} . (1 + ha^s) = s_B . (1 + ka^s),$$

where h, k are positive or negative integers.

This result proves that the congruence

$$s_{aB} \equiv s_B, \text{ mod. } a^s,$$

remains true when the sides are divided by any common factor, whether prime to a or not.

To sum up, we may enunciate the results in the theorem:

If $\lambda, \mu, \ldots \nu$ be any positive integers whose greatest common measure is 1, and whose sum is σ, and

$$s_n = \frac{1}{\sigma} \cdot \frac{\sigma n!}{\lambda n! \ldots \nu n!},$$

then $$\exp. (-\Sigma s_n y^n/n) = 1 + P_1 y + P_2 y^2 + \ldots,$$

where $P_1, P_2, \ldots$ are integers; and if $s_1, s_2, \ldots s_n, \ldots$ have any common factor, say m, then

$$\exp. (-\Sigma s_n y^n/n) = (1 + P_1 y + P_2 y^2 + \ldots)^m,$$

where $P_1, P_2, \ldots$ are still integers.

57. When the greatest common measure of $\lambda, \mu, \ldots \nu$ is p, the theorem of Art. 54 is not true; but in this case ps_n is proved to be integral as in that article. The following articles are not affected by

the change: so that we have the theorem that, when the greatest common measure of $\lambda, \mu, \dots \nu$ is p, then

$$\exp. \left(-\Sigma p\, s_n y^n / n\right) = 1 + P_1 y + P_2 y^2 + \dots,$$

where $P_1, P_2, \dots$ are integers not divisible by p.

The eccentric factor of $\mathfrak{P}$. (Arts. 58, 59.)

58. Let $$\lambda, \mu, \dots \nu$$

be any eccentric system, such that $\lambda, \mu, \dots \nu$ have 1 for their greatest common measure. This system, with its multiples $(n\lambda, n\mu, \dots n\nu)$, contributes to $\log \mathfrak{P}$ the terms

$$-\frac{p}{f} . \Sigma \frac{1}{\sigma} . \frac{n\sigma!}{n\lambda! \dots n\nu!} . \frac{(y^\sigma)^n}{n}.$$

If we combine with this the systems formed from it by cyclic substitutions, the effect is to multiply this by f, and the total contribution is

$$-p . \Sigma \frac{1}{\sigma} . \frac{n\sigma!}{n\lambda! \dots n\nu!} . \frac{(y^\sigma)^n}{n}.$$

This may be regarded as belonging to the point G, which is the common centroid of $(\lambda, \mu, \dots \nu)$ and its multiples.

In $\mathfrak{P}$ there will therefore be a "factor,"

$$(1 - E_1 y^\sigma - E_2 y^{2\sigma} - \&c.)^p,$$

where the E_1, E_2, &c. are integers; and this factor may be considered to belong to the particular point G mentioned above.

59. Every point G, in one sector of the polygon, which is the centroid of a solution of the oft-quoted congruence, contributes a factor of the same kind. For our present purpose it is unnecessary to keep these distinct; and we shall write the "eccentric factor" of $\mathfrak{P}$ in the form

$$\mathfrak{E}^p = (1 - E_k y^k - E_{k+1} y^{k+1} - \dots)^p,$$

where E_k is the first E that does not vanish; so that k is, in fact, the minimum value of σ for an eccentric system.

This may also be written

$$1 - p\,(E_k y^k + E_{k+1} y^{k+1} + \dots),$$

where E_k are still integral; and, in the beginning, positive integers.

The "third factor" of $\mathfrak{P}$. (Arts. 60, 61.)

60. Next consider the case in which $\lambda, \mu, \ldots \nu$ are all multiples of p, and therefore satisfy the congruence

$$\lambda + \mu a + \ldots + \nu a^{f-1} \equiv 0, \text{ mod. } p.$$

But it is supposed that no system of integers

$$\lambda/\delta, \; \mu/\delta, \; \ldots \; \nu/\delta$$

satisfies this congruence. Then the greatest common measure of $\lambda, \mu, \ldots \nu$ must be p.

The contribution to $\log \mathfrak{P}$ made by such a system, and the systems symmetrical with it, and their multiples, is

$$-p \cdot \Sigma \frac{1}{\sigma} \cdot \frac{n\sigma!}{n\lambda! \ldots n\nu!} \cdot \frac{(y^\sigma)^n}{n}.$$

But in reducing from the logarithmic form we find that, as remarked in Art. 57, the factor contributed to $\mathfrak{P}$ is of the form

$$1 - \Sigma . E_\sigma y^\sigma,$$

where the E_σ are integers not divisible by p.

As in the case of the eccentric factor, we express the product of all the individual factors in one; which is the "third factor" of $\mathfrak{P}$.

61. From the way in which the third factor originates, it is clear that all the σ are multiples of p. If the system

$$(\lambda, \mu, \ldots \nu) = (\lambda' p, \mu' p, \ldots \nu' p),$$

and $$\sigma = \sigma' p,$$

it will be seen that the f numbers

$$\lambda', \mu', \ldots \nu'$$

may be any partition of any integer σ', except those which satisfy the fundamental congruence.

The third factor is of the form

$$1 - y^p - \&c.,$$

and the coefficients of the cyclotomic are not affected by the omission of this factor.

The "central factor" of $\mathfrak{P}$. (Arts. 62–66.)

62. The general form of the contribution to $\log \mathfrak{P}$ by a central

system, and its multiples, together with the cyclic transformations, is

$$-\frac{p}{g}\cdot\Sigma\frac{1}{g\sigma}\cdot\frac{(g\sigma n)}{(\lambda n!\ldots\nu n!)^g}\cdot\frac{(y^{g\sigma})^n}{n},$$

where $\sigma=\lambda+\mu+\ldots+\nu$; so that $g\sigma$ corresponds to the σ of the preceding articles, since the sum of the denominator elements is $(\lambda+\mu+\ldots+\nu)\,g$.

63. Now the expression $\dfrac{g\sigma n!}{(\lambda n!\ldots\nu n!)^g}$

is divisible by $(g!)^h$, where h is the number of quantities $\lambda,\mu,\ldots\nu$.

For we have

$$\frac{(k\lambda+1)(k\lambda+2)\ldots(k\lambda+\lambda-1)}{1\quad 2\quad\ldots\quad\lambda-1}\cdot\frac{(k+1)\lambda}{\lambda}$$

$$=\text{an integer}\times(k+1).$$

Putting herein $k=0,1,2,\ldots(g-1)$, and taking the product of the results, we find that

$$\frac{\lambda g!}{(\lambda!)^g}\text{ is divisible by }g!$$

Multiplying by the integer $(\sigma g)!/(\sigma-\lambda)\,g!$ it follows that

$$\frac{\sigma g!}{(\sigma g-\lambda g)!\,(\lambda!)^g}\text{ is divisible by }g!$$

Similarly, $\dfrac{(\sigma g-\lambda g)!}{(\sigma g-\lambda g-\mu g)!\,(\mu!)^g}$ is divisible by $g!$

Continuing the process, and multiplying the results, we have

$$\frac{\sigma g!}{(\lambda!\,\mu!\ldots\nu!)^g}\text{ is divisible by }(g!)^h,$$

where $\sigma=\lambda+\mu+\ldots+\nu$, and h is the number of quantities $\lambda,\mu,\ldots\nu$.

This result is evidently not affected when for $\sigma,\lambda,\mu,\ldots\nu$ we write $n\sigma, n\lambda,\ldots n\nu$, respectively, and the theorem is proved.

64. Hence it follows that

$$\frac{1}{g\sigma}\cdot\frac{(g\sigma n)!}{(\lambda n!\ldots\nu n!)^g},$$

which is integral, is divisible by

$$\frac{1}{\sigma}(g!)^{h-1}(g-1)!$$

if this is an integer, or, generally, by the numerator of this fraction when expressed in lowest terms.

Represent this divisor by d. Then in the contribution to $\log \mathfrak{P}$, namely,

$$-\frac{p}{g}\Sigma\frac{1}{g\sigma}\cdot\frac{(g\sigma n)!}{(\lambda n!\,\ldots\,\nu n!)^{\sigma}}\cdot\frac{(y^{g\sigma})^{n}}{n},$$

the coefficients of $y^{g\sigma n}/n$ are integers or fractions, according as d is or is not a multiple of g. And, accordingly, in $\mathfrak{P}$ there will be a corresponding factor which is an integral or fractional power of a series,

$$1+C_1y^{\sigma}+C_2y^{2\sigma}+\ldots,$$

where the coefficients C_1, C_2, ... are integers. In fact, the denominator of the fractional exponent will be the denominator of the fraction d/g when reduced to its lowest terms.

65. We can now indicate some of the characters of the product of the central factors of $\mathfrak{P}$: and firstly in the case when f is prime. The central systems are all multiples of (111 ... 1), so that

$$g=f,\quad h=1,\quad \sigma=1,$$

and therefore $$d=(f-1)!$$

Thus the central part of $\log \mathfrak{P}$ is

$$-\frac{p}{f}\cdot(f-1)!\,\Sigma s'_n y^{nf}/n,$$

where the Σ covers the logarithm of a series, say

$$1+C_1y^{f}+C_2y^{2f}+\ldots,$$

with integral coefficients. The central part of $\mathfrak{P}$ is this series raised to the power

$$-\frac{p}{f}\cdot(f-1)!,\quad =-e\,.\,(f-1)!-\frac{(f-1)!+1}{f}-\frac{1}{f},$$

whereof the last term only is a fraction.

It appears at once that the central factors of $\mathfrak{P}$ are functions of e only in the form $e\,.\,(f-1)!$ In the tables which follow will be found verifications of this.

It also appears that $\mathfrak{P}$, which has all its coefficients integral, contains two factors, (and no more),

$$(1-fy)^{-1/f} \text{ and } (1+C_1y^f+C_2y^{2f}+\ldots)^{-1/f},$$

whose coefficients are not all integers. The product of these must, therefore, have integral coefficients. Therefore

$$(1-fy)^{-1/f}(1+C_1y^f+C_2y^{2f}+\ldots)^{-1/f}=(1+y+B_2y^2+\ldots),$$

where for B_1 is written its obvious value, 1. It will be found that, if we replace

$$1+C_1y^f+C_2y^{2f}+\ldots$$

by its equivalent

$$(1-fy)^{-1}(1+y+B_2y^2+\ldots)^{-f},$$

the asymptotic factor of

$$\mathfrak{P}=(1-fy)^e(1+y+B_2y^2+\ldots)^p,\text{*}$$

and other forms may be obtained in which both the C and the B series appear. In the tables only one combination is shown, viz., the product of all the central factors into the factor $(1-fy)^{-1/f}$.

66. The significance of the result of Art. 64 is seen in dealing with the cases in which f is or contains a power (higher than first) of a prime. It will suffice to give a particular example. Take $f=9$. Then $g=9, 3$, $h=1, 3$. (The value $g=1$ is excluded because in a central system on a nonagon the weights cannot all be different; every triangle must have equal weights at its corners). The corresponding values of σ are 1 and an indeterminate number respectively. For when $g=9$ the system must be a multiple of $(1)^9$, but when $g=3$ the system is a multiple of $(a, b, c)^3$, where a, b, c are any positive integers. The values of d/g in the two cases are $8! \div 9$ and $3!\,3!\,2! \div 3\sigma = 24/\sigma$. The first is an integer, and the second would also be integral but for the σ: and $\mathfrak{P}$ would contain one factor only, viz., $(1-9y)^{-\frac{1}{3}}$ raised to a fractional power.

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* From this a well-known theorem relative to cyclotomics is easily deduced. The proof for unrestricted values of f may be given, as it is very short. We have

$$\mathfrak{P}=(1-fy)^{-1/f}H^{p/f}\,.\,T,$$

where H and T are functions such as $1+H_1y+H_2y^2+\ldots$ with integral coefficients, and T_1, $T_2\ldots$ are multiples of p. Hence, separating the fractional-power factors,

$$(1-fy)^{-1/f}\,.\,H^{1/f}=K,$$

another function of the same kind as H.

Eliminating H, $\mathfrak{P}=(1-fy)^e\,.\,K^p\,.\,T.$

Hence $\mathfrak{P}\equiv(1-fy)^e$, mod. p,

or $\eta^e+\eta^{e-1}+P_2\eta^{e-2}+\ldots\equiv(\eta-f)^e$, mod. p, $(\eta=1/y)$.

Example of Calculation of the Asymptotic Factor.

67. An abstract of the calculation of the asymptotic factor of 𝔓 for the case $f = 3$ follows. This will serve to point out the interesting character of the coefficients Q (Arts. 7, 50). It will also indicate the origin of the form of the successive coefficients in the asymptotic factor described in Art. 3. The proof that this form is general, comes at once from the observation that s_h is linear in p (or e) when n is not prime to f; and does not contain p (or e) when n is prime to f. The equation (s, Q) and (P, Q) of Art. 50 then give P, the coefficients in the asymptotic factor, in the form described (Art. 3).

The results of the calculation are entered, as they are obtained, in a table such as the following:—

k	s_k	Q_k	P_k
1	-1	-1	1
2	-3	-2	2
3	$-9+2p, = -7+6e$	$-2+2e$	$4-2e$
4	-27	-9	$11-2e$
5	-81	-16	$29-4e$
6	$-243+30p$ $= -213+90e$	$-35+19e-2e^2$	$73-23e+2e^2$
7	-729	-104	$207-37e+2e^2$
8	-2187	-318	$574-88e+4e^2$
9	$-6561+560p,$ $= -6001+1680e$	$\frac{1}{3}(-1992+536e$ $+24e^2-8e^3)$	$\frac{1}{3}(4626-1178e$ $+114e^2-4e^3)$

To show the mode of forming this table, take the last line. s_9 consists of two parts, viz., $-3^8, = -6561$, and a part arising from the central system (3, 3, 3) of weight 9. The latter is

$$\frac{p}{3} \cdot \frac{9!}{3!\,3!\,3!} = 560p = 560+1680e.$$

Thus $$s_9 = -6561+560p = -6001+1680e.$$

Q_9 is given by the equation

$$s_9 = 9Q_9+3Q_3^3+Q_1^9,$$

therefore

$$\begin{aligned}
9Q_0 = \quad s_9 \quad &= -6001+1680e \\
-3Q_3^3 \quad &+ \quad 24- \quad 72e+72e^2-24e^3 \\
-\ Q_1^9 \quad &+ \quad 1 \\
&= -5976+1608e+72e^2-24e^3,
\end{aligned}$$

or

$$Q_0 = \tfrac{1}{3}(-1992+536e+24e^2-8e^3)$$

Finally,

$$\begin{aligned}
P_9 = -Q_0 \quad &= \tfrac{1}{3}(1992- \ 536e- \ 24e^2+8e^3) \\
+Q_1Q_8 \quad &+ \ 318 \\
+Q_2Q_7 \quad &+ \ 208 \\
+Q_3Q_6 \quad &+ \quad 70- \ 108e+ \ 42e^2-4e^3 \\
+Q_4Q_5 \quad &+ \ 144 \\
-Q_1Q_2Q_6 \quad &+ \quad 70- \ 38e+ \ 4e^2 \\
-Q_1Q_3Q_5 \quad &+ \quad 32- \ 32e \\
-Q_2Q_4Q_0 \quad &+ \quad 36- \ 36e \\
&= \tfrac{1}{3}(4626-1178e+114e^2-4e^3)
\end{aligned}$$

Tables.

Appended are some tables for $f = 3, 4, 5, 6, 7, 8, 9$ illustrative of the preceding paper. They are similar in arrangement.

Under I. are given the formulæ for the coefficients of y^0, y^1, y^2, in the asymptotic cyclotomic. The coefficients are ranged in order down the several columns. They are expressed in terms of $e = (p-1)/f$.

Under II., the same coefficients are expressed in terms of ϵ, a suitable multiple of e.

Under III. are given the numerical values of the coefficients for values of p less than 100. The end of the cyclotomic is marked by a | : and the first coefficient that is affected by the eccentric factor is underlined.

Under IV. are given the eccentric factors corresponding to the same values of p. The product of III. into IV. gives the cyclotomic (as far as y^{p-1}).

In the last column of III. references are given, such as R. p. 7, to the page in Reuschle's *Tafeln*, on which the corresponding cyclotomic is given.

$f = 3.$

I. $1, \; -(2e-4), \;\; 2e^2-23e+73, \;\; -\tfrac{1}{3}(4e^3-114e^2+1178e-4626),$

$1, \; -(2e-11), \;\; 2e^2-37e+207, \;\; -\tfrac{1}{3}(4e^3-156e^2+2297e-13305),$

$2, \; -(4e-29), \;\; 4e^2-88e+574, \;\; -\tfrac{1}{3}(8e^3-354e^2+5869e-37707).$

$$\epsilon = 2e.$$

II. $1, \; -(\epsilon-4), \;\; \tfrac{1}{2}(\epsilon^2-23\epsilon+146), \;\; -\tfrac{1}{6}(\epsilon^3-57\epsilon^2+1178\epsilon-9252),$

$1, \; -(\epsilon-11), \;\; \tfrac{1}{2}(\epsilon^2-37\epsilon+414), \;\; -\tfrac{1}{6}(\epsilon^3-78\epsilon^2+2297\epsilon-26610),$

$2, \; -(2\epsilon-29), \;\; \tfrac{1}{2}(2\epsilon^2-88\epsilon+1148), \;\; -\tfrac{1}{6}(2\epsilon^3-177\epsilon^2+5869\epsilon-75414),$

III.

p													
7	1,	1,	2 ∣	0,	$\underline{7}$,	21,	35,	141,	414,	898,	3101,	9107,	R. p. 7
13	1,	1,	2,	− 4,	3 ∣	$\underline{13}$,	13,	91,	286,	494,	2119,	6461,	R. p. 13
19	1,	1,	2,	− 8,	− 1,	5,	7 ∣	$\underline{57}$,	190,	266,	1425,	4503,	R. p. 24
31	1,	1,	2,	−16,	− 9,	−11,	43,	$\underline{37}$,	94,	82,	645 ∣	2139,	R. p. 41
37	1,	1,	2,	−20,	−13,	−19,	85,	51,	94,	− 2,	$\underline{431}$,	1477,	R. p. 50
43	1,	1,	2,	−24,	−17,	−27,	143,	81,	$\underline{126}$,	− 166,	249,	991,	R. p. 65
61	1,	1,	2,	−36,	−29,	−51,	413,	267,	414,	−1778,	− 745,	− 691,	R. p. 88
67	1,	1,	2,	−40,	−33,	−59,	535,	361,	574,	−2902,	−1439,	$-\underline{1753}$,	
73	1,	1,	2,	−44,	−37,	−67,	673,	471,	766,	−4426,	$-\underline{2421}$,	−3279,	
79	1,	1,	2,	−48,	−41,	−75,	827,	597,	990,	−6414,	−3775,	−5457,	
97	1,	1,	2,	−60,	−53,	−99,	1385,	1071,	1854,	−15802,	−10509,	−16983,	

IV.

$p = 31$	$1-31\,(y^7+36y^{10}+42y^{11}+0\,.\,y^{12}+\dots)$
37	$1-37\,(12y^{10}+30y^{11}+y^{12}+\dots)$
43	$1-43\,(y^8+45y^{11}+132y^{13}+1430y^{14}+\dots)$
61	$1-61\,(55y^{13}+\dots)$
67	$1-67\,(5y^{11}+286y^{14}+\dots)$
73	$1-73\,(y^{10}+66y^{13}+\dots)$
79	$1-79\,(22y^{13}+\dots)$
97	$1-97\,(26y^{14}+\dots)$

$f=4$.

I. 1, $(2e-7)(e-3)$, $\tfrac{1}{6}(4e^4-124e^3+1511e^2-8693e+20262)$,

1, $2e^2-23e+77$, $\tfrac{1}{6}(4e^4-164e^3+2723e^2-22189e+76896)$.

$-(2e-2)$, $-\tfrac{1}{3}(4e^3-66e^2+380e-771)$,

$-(2e-7)$, $-\tfrac{1}{3}(4e^3-96e^2+851e-2889)$,

$$\epsilon=2e.$$

II. 1, $\tfrac{1}{2}(\epsilon-6)(\epsilon-7)$, $\dfrac{1}{4!}(\epsilon^4-62\epsilon^3+1511\epsilon^2-17386\epsilon+81048)$,

1, $\tfrac{1}{2}(\epsilon^2-23\epsilon+154)$, $\dfrac{1}{4!}(\epsilon^4-82\epsilon^3+2723\epsilon^2-44378\epsilon+307584)$..

$-(\epsilon-2)$, $\tfrac{1}{6}(\epsilon^3-33\epsilon+380\epsilon-1542)$,

$-(\epsilon-7)$, $\tfrac{1}{6}(\epsilon^3-48\epsilon+851\epsilon-5778)$,

III.

<table>
<tr><td>p = 5</td><td>1,</td><td>1, |</td><td>0,</td><td>$\underline{5}$,</td><td>10,</td><td>56,</td><td>151,</td><td>710,</td><td>2160,</td><td>9545,</td><td>R. p. 2</td></tr>
<tr><td>13</td><td>1,</td><td>1,</td><td>− 4,</td><td>1 |</td><td>0,</td><td>$\underline{26}$,</td><td>39,</td><td></td><td></td><td></td><td>R. p. 15</td></tr>
<tr><td>17</td><td>1,</td><td>1,</td><td>− 6,</td><td>− 1,</td><td>1 |</td><td>$\underline{17}$,</td><td></td><td></td><td></td><td></td><td>R. p. 19</td></tr>
<tr><td>29</td><td>1,</td><td>1,</td><td>−12,</td><td>− 7,</td><td>28,</td><td>14,</td><td>− 9,</td><td>$\underline{88}$ |</td><td></td><td></td><td>R. p. 35</td></tr>
<tr><td>37</td><td>1,</td><td>1,</td><td>−16,</td><td>−11,</td><td>66,</td><td>32,</td><td>− 73,</td><td>$\underline{30}$,</td><td>44,</td><td>741 |</td><td>R. p. 51</td></tr>
<tr><td>41</td><td>1,</td><td>1,</td><td>−18,</td><td>−13,</td><td>91,</td><td>47,</td><td>−143,</td><td>− 7,</td><td>72,</td><td>$\underline{551}$,</td><td>R. p. 58</td></tr>
<tr><td>53</td><td>1,</td><td>1,</td><td>−24,</td><td>−19,</td><td>190,</td><td>116,</td><td>−601,</td><td>−246,</td><td>738,</td><td>$\underline{427}$,</td><td>R. p. 79</td></tr>
<tr><td>61</td><td>1,</td><td>1,</td><td>−28,</td><td>−23,</td><td>276,</td><td>182,</td><td>−1193,</td><td>−592,</td><td>2307,</td><td>956,</td><td>R. p. 89</td></tr>
<tr><td>73</td><td>1,</td><td>1,</td><td>−34,</td><td>−29,</td><td>435,</td><td>311,</td><td>−2659,</td><td>−1539,</td><td>7838,</td><td>3867,</td><td></td></tr>
<tr><td>89</td><td>1,</td><td>1,</td><td>−42,</td><td>−37,</td><td>707,</td><td>543,</td><td>−6079,</td><td>−3987,</td><td>28512,</td><td>16237,</td><td></td></tr>
<tr><td>97</td><td>1,</td><td>1,</td><td>−46,</td><td>−41,</td><td>861,</td><td>687,</td><td>−8543,</td><td>−5845,</td><td>48069,</td><td>28796,</td><td></td></tr>
</table>

IV.

$p=29$	$1-29\,(3y^7+\dots)$
37	$1-37\,(y^7+36y^9+\dots)$
41	$1-41\,(14y^9+y^{10}+\dots)$
53	$1-53\,(4y^9+165y^{11}+3960y^{13}+\dots)$
61	$1-61\,(42y^{11}+\dots)$
73	$1-73\,(15y^{11}+\dots)$
89	$1-89\,(99y^{13}+\dots)$
97	$1-97\,(55y^{13}+\dots)$

$f = 5.$

$$
\text{I.}\quad
\begin{array}{lll}
1, & -(24e-180), & 288e^2-15660e+331282, \\
1, & -(24e-796), & 288e^2-30444e+1544418, \\
3, & -(72e-3532), & 864e^2-118788e+7211960, \\
11, & -(264e-15906), & 3168e^2-506484e+33850952, \\
44, & -(1056e-72490), & 12672e^2-2238720e+159612948.
\end{array}
$$

$$\epsilon = 4!\, e.$$

$$
\text{II.}\quad
\begin{array}{lll}
1, & -(\epsilon-180), & \tfrac{1}{2}(\epsilon^2-1305\epsilon+662564), \\
1, & -(\epsilon-796), & \tfrac{1}{2}(\epsilon^2-2537\epsilon+3088836), \\
3, & -(3\epsilon-3532), & \tfrac{1}{2}(3\epsilon^2-9899\epsilon+14423920), \\
11, & -(11\epsilon-15906), & \tfrac{1}{2}(11\epsilon^2-42207\epsilon+67701904), \\
44, & -(44\epsilon-72490), & \tfrac{1}{2}(44\epsilon^2-186560\epsilon+319225896).
\end{array}
$$

III.

$$
\begin{array}{r|ll}
p = 11 & 1, 1, 3 \mid \underline{11}, 44, \; 132, 748, \; 3388, 15378, 70378, 301114, & \text{R. p. 10} \\
31 & 1, 1, 3, \; 11, 44, \; 36, \underline{652} \mid 3100, & \text{R. p. 42} \\
41 & 1, 1, 3, \; 11, 44, \; -\underline{12}, 604, \; 2956, 13794 \mid 41\times 1562, & \text{R. p. 59} \\
61 & 1, 1, 3, \; 11, \underline{44}, \; -108, 508, \; 2668, 12738, 59818, 184834, & \\
 & \qquad 1220562, 5910920, & \text{R. p. 90} \\
71 & 1, 1, 3, \; 11, 44, \; -156, \underline{460}, \; 2524, 12210, 57706, 168490, & \\
 & \qquad 1174650, 5718272, 27381104, 130754580. & \text{R. p. 111}
\end{array}
$$

$$
\text{IV.}\quad
\begin{array}{r|l}
p = 31 & 1-31\,(20y^6+80y^7+\dots), \\
41 & 1-41\,(y^5+10y^6+63y^7+231y^8+988y^9+\dots), \\
61 & 1-61\,(y^4+30y^7+148y^8+848y^9+882y^{10}+10980y^{11}+57917y^{12}+\dots), \\
71 & 1-71\,(10y^6+15y^7+105y^8+616y^9+792y^{10}+13660y^{11} \\
 & \qquad +50790y^{12}+242328y^{13}+1149635y^{14}+\dots).
\end{array}
$$

$f = 6.$

$$
\text{I.}\quad
\begin{array}{lll}
1, & \tfrac{1}{2}(e-3)(9e-44), & \tfrac{1}{8}(27e^4-1254e^3+20469e^2-146818e+397952), \\
1, & \tfrac{1}{3}(33e^2-297e+670), & \\
-(3e-3), & -\tfrac{1}{2}(9e^3-202e^2+1467e-3472), & \\
-(7e-14), & -\tfrac{1}{2}(45e^3-961e^2+7160e-18454). &
\end{array}
$$

$p = 7$ | 1, 1 | 0, $\underline{7}$, 35, 203, 1099, 6105, 33797, R. p. 4

13 | 1, 1, − 3 | 0, $\underline{26}$, 104, 637, 3809, 22074, R. p. 16

19 | 1, 1, − 6, − 7 | 0, $\underline{38}$, 323, 2209, 13756, R. p. 26

31 | 1, 1, −12, −21, 1, 5 | $\underline{31}$, 527, 4464, R. p. 44

37 | 1, 1, −15, −28, 15, 38, − 1 | $\underline{185}$, 2257, R. p. 52

43 | 1, 1, −18, −35, 38, 104, 7, − $\underline{6}$, | 989, R. p. 66

61 | 1, 1, −27, −56, 161, 500, 1, −1023, − 916, R. p. 92

67 | 1, 1, −30, −63, 220, 698, −101, −1960, −1758, R. p. 102

73 | 1, 1, −33, −70, 288, 929, −298, −3421, −2921, R. p. 122

79 | 1, 1, −36, −77, 365, 1193, −617, −5541, −4414, R. p. 135

97 | 1, 1, −45, −98, 650, 2183, −2576 −17205, −9748,

IV. $p = 43$ | $1-43\,(y^7+22y^8+\ldots)$,

61 | $1-61\,(14y^9+210y^{10}+1062y^{11}+\ldots)$,

67 | $1-67\,(4y^9+93y^{10}+1260y^{11}+\ldots)$,

73 | $1-73\,(y^9+\ldots)$,

79 | $1-79\,(12y^{10}+\ldots)$,

97 | $1-97\,(15y^{11}+\ldots)$.

$f = 7$.

I., II.

$$\begin{aligned}
&1, && -720e+23412 &&= -\epsilon+23412,\\
&1, && -720e+146865 &&= -\epsilon+146845,\\
&4, && -2880e+930385 &&= -4\epsilon+\ldots,\\
&20, && -14400e+5955040 &&= -20\epsilon+\ldots,\\
&110, && -79200e+38439040 &&= -110\epsilon+\ldots,\\
&638, && -459300e+249861680 &&= -638\epsilon+\ldots,\\
&3828, && -2756160e+1633746320 &&= -3828\epsilon+\ldots.
\end{aligned}$$

$(\epsilon = 6!\,e)$.

III. $p = 29$ | 1, 1, 4, 20, $\underline{110}$ | R. p. 36

43 | 1, 1, 4, $\underline{20}$, 110, 638, 3828 | R. p. 67

71 | 1, 1, 4, 20, $\underline{110}$, 638, 3828, 16212, 139645, 901585, 5811040 | R. p. 112

IV. $p = 29$ | $1-29\,(3y^4+19y^5+\ldots)$,

43 | $1-43\,(y^4+12y^5+26y^6+230y^7+\ldots)$,

71 | $1-71\,(3y^4+4y^5+35y^6+135y^7+1245y^8+7916y^9+48363y^{10}+\ldots)$

$f = 8.$

I. $1, \quad 8e^2 - 58e + 152,$

$1, \quad 8e^2 - 142e + 1034,$

$-(4e-4), \quad -\tfrac{1}{8}(32e^3 - 600e^2 + 4888e - 20943).$

$-(4e-25),$

$$\epsilon = 4e.$$

II. $1, \quad \frac{1}{2}(\epsilon^2 - 29\epsilon + 304),$

$1, \quad \frac{1}{2}(\epsilon^2 - 71\epsilon + 2068),$

$-(\epsilon - 4), \quad -\frac{1}{3!}(\epsilon^3 - 75\epsilon^2 + 2444\epsilon - 41886),$

$-(\epsilon - 25),$

III. $p = 17$	1,	1,	− 4 |	$\underline{17}$,					R. p. 20
41	1,	1,	−16,	5,	$\underline{62}$,	524 |	2501,		R. p. 60
73	1,	1,	−32,	−11,	278,	$\underline{404}$,	741,		R. p. 123
89	1,	1,	−40,	−19,	482,	$\underline{440}$,	− 939,		R. p. 149
97	1,	1,	−44,	−23,	608,	$\underline{482}$,	−2203,		R. p. 162

IV. $p = 41 \mid 1 - 41\,(y^4 + 12y^5 + 65y^6 + \ldots),$

$73 \mid 1 - 73\,(6y^5 + 10y^6 + \ldots).$

$f = 9.$

I. $1, \quad -(6e-31), \quad 18e^2 - 591e + 12510,$

$1, \quad -(6e-221), \quad 18e^2 - 1731e + 98618,$

$5, \quad -(30e-1637), \quad 90e^2 - 11847e + 789617,$

$$\epsilon = 6e.$$

II. $1, \quad -(\epsilon - 31), \quad \tfrac{1}{2}(\epsilon^2 - 197\epsilon + 25020),$

$1, \quad -(\epsilon - 221), \quad \tfrac{1}{2}(\epsilon^2 - 577\epsilon + 197236),$

$5, \quad -(5\epsilon - 1637), \quad \tfrac{1}{2}(5\epsilon^2 - 3949\epsilon - 1578334).$

III. $p = 19$	1,	1,	5 |	$\underline{19}$,			R. p. 27
37	1,	1,	5	7,	$\underline{197}$ |		R. p. 53
73	1,	1,	5,	−17,	$\underline{173}$,	1397,	R. p. 124