ON THE ZEROES OF CERTAIN CLASSES OF INTEGRAL TAYLOR SERIES. PART II.—ON THE INTEGRAL FUNCTION $$\sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}$$ AND OTHER SIMILAR FUNCTIONS

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1. This paper is a continuation of one recently published in these *Proceedings*.† The general object of the two papers is the same, but the methods used and the results obtained are of entirely different types, and I have therefore judged it best to keep the two distinct.

Introductory Remarks.

2. The principal end which I have had in view is *to determine asymptotically the zeroes of the function*

$$F_{a,s}(x) = \sum_0^\infty \frac{x^n}{(n+a)^s n!} \tag{1}$$

[where $(n+a)^s = e^{s \log (n+a)}$, the imaginary part of the logarithm being between $-\pi i$ and πi] for all values of a and s, real or complex, except, of course, negative integral or zero values of a. In endeavouring to obtain the complete solution of this problem I have naturally been led to consider other varieties of functions of similar types, some of which include the particular function (1) as a special case. I have not hesitated to include proofs of these results in this paper, when such developments do not diverge far from the natural course of the analysis required for the discussion of the function (1); but, when this course would have led to a considerable increase in the length and complexity of the paper, I have confined myself to a general indication of the results and of the methods by which they can be proved.

* This part has been entirely rewritten, and some of the results considerably extended.

† *Supra*, p. 332.

In what follows I consider only the half of the plane of the complex variable

(2) $$x = \xi + i\eta = re^{i\theta}$$

for which $\eta \geqslant 0$. The corresponding results for the lower half of the plane may be deduced immediately. By drawing a semicircle whose centre is the origin and whose radius is a sufficiently large fixed quantity R_0, and the radii vectores $\theta = \frac{1}{2}\pi \mp \delta$, where δ is also fixed, but arbitrarily small, we divide the distant part of the plane into three regions

(D) $$r \geqslant R_0, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi - \delta,$$

(D′) $$r \geqslant R_0, \quad \tfrac{1}{2}\pi + \delta \leqslant \theta \leqslant \pi,$$

(E) $$r \geqslant R_0, \quad \tfrac{1}{2}\pi - \delta \leqslant \theta \leqslant \tfrac{1}{2}\pi + \delta,$$

within which the behaviour of the functions which I shall consider is entirely different. It will perhaps be convenient if I state at once the principal results which I obtain concerning the function (1).

I. Throughout D

(3) $$F_{a,s}(x) = \frac{e^x}{x^s}(1+\epsilon_x)$$

where ϵ_x is a function of x which tends uniformly to zero with $1/r$.*

II. Throughout D′

(4) $$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x).$$

In these equations x^s, $(-x)^{-a}$, $\{\log(-x)\}^{s-1}$ are so chosen as to be real when x, a, s are real.

III. Throughout E

(5) $$F_{a,s}(x) = \frac{e^x}{x^s}(1+\epsilon_x) + \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x).$$

In these results a and s may have any values, real or complex, save zero or negative integral values. From III. the nature of the zeroes may be very precisely determined. Thus

IV. If a and s are real, the zeroes of $F_{a,s}(x)$ are given asymptotically† by the equations

(6) $$\begin{cases} \xi = (s-a)\log(2k\pi) + (s-1)\log\log k + \log\dfrac{\Gamma(a)}{\Gamma(s)}, \\ \eta = (2k+1)\pi + \tfrac{1}{2}(s+a)\pi, \end{cases}$$

where k is a large positive integer.

* I use ϵ_x generally in this sense, sometimes omitting the suffix. Of course ϵ_x is not the same in different equations.

† For a precise definition of what is meant by this see Part I., *supra*, p. 333.

If a and s are complex, these formulæ require a slight modification.

So far, zero or negative integral values of a and s have been excluded altogether. If a is zero or a negative integer, $F_{a,s}(x)$ is no longer defined. On the other hand, if s is zero or a negative integer,

V. The function $F_{a,s}(x)$ reduces to the product of e^x by a polynomial, and has but a *finite* number of zeroes. It will be seen in the sequel that greater precision can be given to some of these results, notably to I.

The paper is divided into four sections. In Section I. (§§ 3–13) I consider the region D, in which the functions under consideration have no zeroes; and in Section II. (§§ 14–22) the region D′, of which the same is true. I have in Section I. considered the function $F_{a,s}(x)$ as a particular case of a certain class of functions, and I have endeavoured to make my method as direct and elementary as possible, avoiding the use of contour integrals and other contrivances, which, although very powerful aids to the obtaining of particular results, are apt to obscure the basis on which they rest. In Section III. (§§ 23–31) I consider the region E. The analysis in this Section (as, in a less degree, in Section II.) is more difficult and indirect, the problem being inherently less simple, and I have not attempted to deal with more than a few special functions besides the function $F_{a,s}(x)$. Finally, Section IV. (§§ 32–37) is taken up with a brief discussion of several matters naturally arising out of the previous work.

It will be found in the case of the asymptotic expansions discussed in Sections I. and II. that to each of the regions D and D′ corresponds a special function whose asymptotic expansion is of a particularly simple character, reducing, in fact, to *one term*. For D the function is

$$1+\frac{x}{s+1}+\frac{x^2}{(s+1)(s+2)}+\dots,$$

and for (D′) it is

$$1+\frac{x}{(1+a)\,1!}+\frac{x^2}{(2+a)\,2!}+\dots.$$

The results, so far as they concern these functions, are particular cases of results which have already been arrived at from a different point of view—that of the theory of linear differential equations—by Horn, Jacobsthal, and others. Horn has also considered the question as to the nature of the places in which the integrals of the equations assume assigned values, and, in particular, of their zeroes,* but his approximations are much less

* *Crelle*, Bd. cxx., p. 1.

precise than mine. The general types of functions considered here are, of course, not solutions of linear differential equations at all.

I. THE REGION D ($r \geqslant R_0$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi - \delta$).

The Function $\Sigma \frac{\Gamma(s)}{\Gamma(s+n+1)} x^n$.

3. It will be found that there are certain functions whose behaviour in D (or D′) can be specified in a particularly simple manner. In the case of D the function

$$(7)\qquad f_s(x) = \sum_{n=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s+n+1)} x^n = \frac{1}{s} + \frac{x}{s(s+1)} + \frac{x^2}{s(s+1)(s+2)} + \dots$$

is such a function.

Let us suppose first that $\mathrm{R}(s) > 0$. Then, by the help of the formula

$$(8)\qquad \int_0^1 u^n (1-u)^{s-1} du = \frac{\Gamma(n+1)\Gamma(s)}{\Gamma(n+1+s)},$$

we find that*

$$(9)\qquad f_s(x) = \int_0^1 e^{xu} (1-u)^{s-1} du,$$

where $(1-u)^{s-1} = e^{(s-1)\log(1-u)}$, the logarithm being real. Hence

$$(10)\qquad f_s(x) = e^x \int_0^1 e^{-x\omega} \omega^{s-1} d\omega = \phi_s(x) - \psi_s(x)$$

where

$$(11)\qquad \phi_s(x) = e^x \int_0^\infty e^{-x\omega} \omega^{s-1} d\omega = \Gamma(s) x^{-s} e^x$$

(x^{-s} being defined by $x^{-s} = e^{-s\log x}$ where the logarithm is real with x) and

$$(12)\qquad \psi_s(x) = e^x \int_1^\infty e^{-x\omega} \omega^{s-1} d\omega.$$

Thus

$$(13)\qquad f_s(x) = \Gamma(s) x^{-s} e^x - e^x \int_1^\infty e^{-x\omega} \omega^{s-1} d\omega$$

when $\mathrm{R}(s) > 0$. But it is easy to see that the right-hand side of (12) represents an analytic function of s regular for all values of s, save negative integral (including zero) values. The equation (13) consequently holds for all values of s, with these exceptions. Now, we find easily by

* The term by term integration is easily justified.

integration by parts that

(14) $$\psi_s(x) = \frac{1}{x} + \frac{s-1}{x^2} + \ldots + \frac{(s-1)(s-2)\ldots(s-\nu+2)}{x^{\nu-1}} + \frac{(s-1)(s-2)\ldots(s-\nu+1)}{x^{\nu-1}} e^{x} \int_1^{\infty} e^{-x\omega}\,\omega^{s-\nu} d\omega.$$

But

$$\left| e^{x} \int_1^{\infty} e^{-x\omega}\,\omega^{s-\nu} d\omega \right| < e^{r\cos\theta} \int_1^{\infty} e^{-r\omega\cos\theta} |\omega^{s-\nu}|\, d\omega < \int_0^{\infty} e^{-rt\cos\theta} |1+t|^{s-\nu} dt < K_\nu.\text{*}$$

Hence, changing ν into $\nu+1$,

(15) $$f_s(x) = \Gamma(s)x^{-s}e^{x} - \sum_{\mu=1}^{\nu-1} \frac{(s-1)(s-2)\ldots(s-\mu+1)}{x^{\mu}} + R$$

where $$|R| < \frac{K_\nu}{r^\nu}.$$

That is to say, *the asymptotic expansion of* $f_s(x)$ *in* D *is*

(16) $$\Gamma(s)x^{-s}e^{x} - \frac{1}{x} - \frac{s-1}{x^2} - \frac{(s-1)(s-2)}{x^3} - \ldots,$$

and the asymptotic expansion of $e^{-x}f_s(x)$ *is*

(17) $$\Gamma(s)\,x^{-s}$$

simply; for $|e^{-x}f_s(x) - \Gamma(s)\,x^{-s}|$ decreases with $1/r$ more rapidly than any power of $1/r$.

I may remark that the extension of (13) to general values of s, which was inferred from the principle of analytic continuity, may be deduced directly from the formulæ

(18) $$\int_0^1 \left\{u^n - \sum_{\nu=0}^{p} (-)^\nu \binom{n}{\nu} (1-u)^\nu \right\} (1-u)^{s-1} du = \frac{\Gamma(n+1)\Gamma(s)}{\Gamma(n+1+s)} - \sum_{\nu=0}^{p} \frac{(-)^\nu \binom{n}{\nu}}{s+\nu},$$

(19) $$\int_0^{\infty} \left\{ e^{-x\omega} - \sum_{\nu=0}^{p} (-)^\nu \frac{(x\omega)^\nu}{\nu!} \right\} \omega^{s-1} d\omega = x^{-s}\Gamma(s),$$

which hold for a wider range of values of s than the ordinary Eulerian formulæ;† in each of them p is the greatest integer in $\mathbf{R}(-s)$, which is supposed not to be integral.

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* For an explanation of my use of K see Part I., p. 336 (footnote). By using K_ν I imply that the limits of K_ν may depend on ν, but not on x.

† The formula (19) is attributed by Mr. Whittaker to Saalschütz (see *Modern Analysis*, p. 184); but it really dates from Cauchy ("Sur un nouveau genre d'Integrales," *Exercices de Math.*, t. I., p. 57).

The Function $\int_0^1 e^{xu}(1-u)^{s-1}\psi(1-u)\,du.$

4. I shall now consider the more general function

$$(20)\qquad \Psi_s(x)=\int_0^1 e^{xu}(1-u)^{s-1}\psi(1-u)\,du$$

where $\mathrm{R}(s)>0$. I suppose that $\psi(\omega)$ is a function expansible in a Taylor series

$$c_0+c_1\omega+c_2\omega+\dots$$

whose radius of convergence is *at least* equal to unity, and that, if

$$(21)\qquad \overline{\psi}(\omega)=\Sigma\,|c_\nu|\,\omega^\nu,$$

$\int^1 \overline{\psi}(\omega)\,d\omega$ is convergent.*

I propose first to show that under these circumstances $\Psi_s(x)$ can be expanded in a series

$$(22)\qquad \sum_0^\infty c_\nu f_{s+\nu}(x)$$

convergent throughout D.

5. Consider the integral

$$(23)\qquad I(\mu,\mu')=\int_0^1 e^{xu}(1-u)^{s-1}du\sum_\mu^{\mu'} c_k(1-u)^k.$$

We divide the range of integration into the two parts $(0,\epsilon)$, $(\epsilon, 1)$. We can determine a positive quantity K, independent of μ and μ', and greater than the maximum of

$$|\,e^{xu}(1-u)\,|^{s-1}$$

in $(0,\delta)$, δ being any small fixed quantity greater than ϵ. We can then choose ϵ so small that the modulus of the integral over $(0,\epsilon)$ is less than any given small quantity σ; for it is obviously

$$<K\int_0^\epsilon du\sum_\mu^{\mu'}|\,c_k\,|\,(1-u)^k<K\int_0^\epsilon\overline{\psi}(u)\,du.$$

When ϵ is fixed the series $\Sigma c_k(1-u)^k$ is *uniformly* convergent in $(\epsilon, 1)$. Hence μ can be so chosen that

$$\left|\int_\epsilon^1 e^{xu}(1-u)^{s-1}du\sum_\mu^{\mu'}c_k(1-u)^k\right|<\sigma$$

* A more stringent condition than that which merely asserts the *absolute* convergence of $\int^1\psi(\omega)\,d\omega$.

for all values of $\mu' > \mu$, and hence so that

$$|\,\mathrm{I}(\mu, \mu')\,| < 2\sigma$$

for all values of $\mu' > \mu$. It follows that

(24) $$\Psi_s(x) = \sum_{\nu=0}^{\infty} c_\nu f_{s+\nu}(x),$$

by a deduction so obvious that I need not set it out in detail.*

6. Again, if μ is any positive integer and

(25) $$\Psi_{s,\mu}(x) = \sum_{\nu=0}^{\mu-1} c_\nu f_{s+\nu}(x).$$

(26) $$\Psi_s(x) = \psi_{s,\mu}(x) + \int_0^1 e^{xu}(1-u)^{s-1}\,du \sum_{\mu}^{\infty} c_k(1-u)^k.$$

Let us divide the range of integration into the two parts

$$(0,\ 1-\delta), \qquad (1-\delta,\ 1)$$

where $$\delta = \xi^{-\lambda} \qquad (0 < \lambda < 1).$$

Then

(27) $$\left|\int_0^{1-\delta}\right| < Ke^{(1-\delta)\xi}\int_0^1 \bar{\psi}(u)\,du < Ke^{\xi-\xi^{1-\lambda}}.$$

Moreover, it is plain that throughout $(1-\delta,\ 1)$

$$\left|\sum_{\mu}^{\infty} c_k(1-u)^k\right| < K(1-u)^\mu < \delta^\mu = \xi^{-\lambda\mu},$$

and so

(28) $$\left|\int_{1-\delta}^1\right| < Ke^\xi\xi^{-\lambda\mu}.$$

Thus $$\left|\int_0^1\right| < Ke^\xi(e^{-\xi^{1-\lambda}} + \xi^{-\lambda\mu}) < Ke^\xi\xi^{-\lambda\mu}$$

(for sufficiently large values of r), or

(29) $$< K\left|\frac{e^x}{x^{\lambda\mu}}\right|,$$

since $\xi > Kr$ throughout D.

* Generally, if

$$\int_a^{A-\epsilon} \Theta(u)\,\Sigma\,\phi_n(u)\,du = \Sigma\int_a^{A-\epsilon} \Theta\phi_n\,du$$

for any $\epsilon > 0$, we may replace ϵ by 0 if (i.) Θ is continuous up to A, and (ii.) $\int^A \bar{\phi}(u)\,du$ is convergent, where $$\bar{\phi}(u) = \Sigma\,|\phi_n(u)|.$$

This set of sufficient conditions for the integration of an infinite series is often useful in practice, as it covers certain cases which frequently occur and are excluded by the ordinary tests.

Again, we can choose R so that, if $r > R$,

$$\Psi_{s,\mu}(x) = e^x \sum_0^{\mu-1} \frac{c_\nu \Gamma(s+\nu)}{x^{s+\nu}} + \rho$$

where

(30) $$|\rho| < K_\mu \left| \frac{e^x}{x^{\lambda\mu}} \right|.$$

Hence $$\Psi_s(x) = e^x \sum_0^{\mu-1} \frac{c_\nu \Gamma(s+\nu)}{x^{s+\nu}} + \rho,$$

where ρ again satisfies an equation of the form (30). Now $\lambda\mu$ tends to infinity with μ, and so, finally, putting $m = [\lambda\mu]$,

(31) $$\Psi_s(x) = e^x \sum_0^{m} \frac{c_\nu \Gamma(s+\nu)}{x^{s+\nu}} + \rho$$

where $|\rho| < K_m |x^{-m} e^x|$. It is easy to see that we can, if we like, replace this last inequality by $|\rho| < K_m |x^{-(s+m+1)} e^x|$: and we may sum up the result by saying that *the function*

$$x^s e^{-x} \Psi_s(x)$$

possesses an asymptotic expansion

(32) $$\sum \frac{c_\nu \Gamma(s+\nu)}{x^\nu}$$

valid throughout D.

Extension to General Values of s.

7. Throughout §§ 5, 6 it was supposed that $\mathrm{R}(s) > 0$. In fact the integral by means of which $\Psi_s(x)$ was defined is evidently divergent when $\mathrm{R}(s) \leqslant 0$. We might generalise our result by the use of contour integrals instead of line integrals, but for my present purpose it is more convenient to proceed as follows :—

We shall, in the sequel, be occupied with functions $\Psi_s(x)$ which (i.) are analytic functions of s for all values of s save (possibly) negative integral or zero values; (ii.) are expressible, when $\mathrm{R}(s) > 0$, in the forms (20) and (22).

Now suppose $\mathrm{R}(s) \leqslant 0$. We can choose k so that $\mathrm{R}(s+k) > 0$. This being so, it can be shown, by the method of §§ 5, 6, that the series

(33) $$\Psi_s^{(k)}(x) = \sum_k^\infty c_\nu f_{s+\nu}(x)$$

is convergent, and that $x^s e^{-x} \Psi_s^{(k)}(x)$ possesses the asymptotic expansion

$$\sum_k c_\nu x^{-\nu} \Gamma(s+\nu).$$

Moreover, it appears that the function (33) is an analytic function of s for all values of s for which $\mathrm{R}(s+k)$ is positive. The equation

$$\Psi_s(x) = \sum_0^{k-1} c_\nu f_{s+\nu}(x) + \Psi_s^{(k)}(x)$$

is therefore valid for all such values of s save $0, -1, \ldots, -(k-1)$. From this it follows at once that the conclusion of § 6 holds for the function $\Psi_s(x)$ for all values of s save zero or negative integral values.

Examples.

8. Before proceeding further I shall illustrate this result by some examples.

(i.) Suppose that we write a for s, and that

$$\psi(1-u) = u^{\beta-1} = \{1-(1-u)\}^{\beta-1}$$

where $R(\beta) > 0$. Then

$$c_\nu = \frac{(1-\beta)(2-\beta)\ldots(\nu-\beta)}{1.2\ldots\nu}.$$

It is easy to see (by a glance at a figure) that, however small κ may be,

$$|\nu-\beta| < \nu - \mathrm{R}(\beta) + \kappa,$$

after a certain value of ν. Hence it follows that, after a certain value of ν, $|c_\nu|$ is less than the coefficient of x^ν in the expansion of

$$K(1-x)^{-[1+\kappa-\mathrm{R}(\beta)]}.$$

Now κ can be so chosen that $1+\kappa-\mathrm{R}(\beta) < 1$, in which case

$$\int^1 \frac{dx}{(1-x)^{[1+\kappa-\mathrm{R}(\beta)]}}$$

is convergent. The condition of § 5 relative to $\bar{\psi}(u)$ is therefore satisfied. We find easily that

$$\Psi_a(x) = \frac{\Gamma(a)\Gamma(\beta)}{\Gamma(a+\beta)}\left\{1 + \frac{\beta}{1.(a+\beta)}x + \frac{\beta(\beta+1)}{1.2.(a+\beta)(a+\beta+1)}x^2 + \ldots\right\}$$

and the asymptotic expansion of $e^{-x}\Psi_a(x)$ is

$$\Sigma \frac{\Gamma(a+\nu)\Gamma(\nu+1-\beta)}{\Gamma(1-\beta)\Gamma(\nu+1)} x^{-a-\nu},$$

provided that $\mathrm{R}(\beta) > 0$, and that neither a nor $a+\beta$ is a negative integer.

The first restriction is easily removed by the help of an obvious recurrence formula for $\Psi_a(x)$, unless β is a negative integer. The cases in which β or $a+\beta$ is a negative integer are obviously trivial. If a is a negative integer, the function $1+\dfrac{\beta}{1.a+\beta}x+\dots$ may be easily reduced to the form

$$\Sigma P(\nu)\frac{x^\nu}{\nu!},$$

where $P(\nu)$ is a polynomial. This is one of a class of functions which may be reduced to the product of e^x by a polynomial.

The asymptotic expansion found for $\psi_a(x)$ in D has been otherwise obtained by W. Jacobsthal* from the point of view of the theory of linear differential equations.

9. (ii.) Suppose that

$$\psi(\omega) = (1-\omega)^{a-1}\left\{\frac{1}{\omega}\log\left(\frac{1}{1-\omega}\right)\right\}^{s-1}, \tag{34}$$

a and s having their real parts positive. Then

$$\Psi_s(x) = \int_0^1 e^{xu}u^{a-1}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}du = \sum_{\nu=0}^{\infty}\frac{x^\nu}{\nu!}\int_0^1 u^{a+\nu-1}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}du \tag{35}$$

$$= \Gamma(s)\sum_0^\infty \frac{x^\nu}{(\nu+a)^s\nu!} = \Gamma(s)F_{a,s}(x),$$

if $u^a = e^{a\log u}$ and $\{\log(1/u)\}^{s-1} = e^{(s-1)\log\log(1/u)}$, $\log u$ and $\log\log(1/u)$ being real, while $(\nu+a)^s$ is defined as in § 2. If a and s are real, it is almost obvious that, from a certain ν, c_ν is positive. Otherwise it may be verified by an extension of the argument used in the preceding section, that the condition concerning $\bar{\psi}(\omega)$ is satisfied. Thus we find the asymptotic expansion

$$\Gamma(s)x^s e^{-x}F_{a,s}(x) = \Sigma\frac{c_\nu\Gamma(s+\nu)}{x^\nu} \tag{36}$$

where c_ν is the coefficient of ω^ν in $(1-\omega)^{a-1}\left\{\dfrac{1}{\omega}\log\left(\dfrac{1}{1-\omega}\right)\right\}^{s-1}$; so that, in particular,

$$c_0 = 1,\quad c_1 = \tfrac{1}{2}(1+s)-a,\quad \dots. \tag{37}$$

This is valid throughout D if the real parts of a and s are positive. This

* "Asymptotische Darstellung von lösungen linearer Diff.-gleichungen," *Math. Annalen*, Vol. LVI., p. 129.

restriction is not, however, essential. The restriction as to s may be removed by an argument similar to that of § 7, notwithstanding that the coefficients c_ν depend upon s. The restriction as to a may be removed if we choose k so that $\mathbf{R}(a+k)$ is positive, and consider the function

$$(38)\quad F_{a,s}(x)-\sum_{\mu=0}^{k-1}\frac{x^\mu}{(a+\mu)^s\,\mu!}=\int_0^1\left\{e^{xu}-\sum_{\mu=0}^{k-1}\frac{(xu)^\mu}{\mu!}\right\}u^{a-1}\left\{\log\left(\frac{1}{1-u}\right)\right\}^{s-1}du.$$

But I do not propose to go into the details of this here; for when I come to consider the region E, which it is difficult to deal with satisfactorily by the comparatively simple and direct methods of this part of the paper, I shall have to apply to the function $F_{a,s}(x)$ a different and less elementary treatment which leads with greater ease to the desired extension. The only exceptional cases are those in which a or s is zero or a negative integer, the cases which were indicated as exceptional in § 2.

10. When x, a, and s are real and x *positive*, the dominant terms in the asymptotic equations for $f_s(x)$, $F_{a,s}(x)$ may be deduced very easily from a formula given by M. le Roy, who has proved* that, if

$$f(x)=\sum_0^\infty a_n x^n,\qquad a_n=e^{-\phi(n)},$$

where $\phi(n)$ is a positive function such that it and its first derivate $\phi'(n)$ tend steadily to ∞ for $n=\infty$, while $\phi''(n)$ tends steadily to 0, then for large values of x

$$f(x)\sim\sqrt{(2\pi)}\,\frac{e^{\xi\phi'(\xi)-\phi(\xi)}}{\sqrt{\{\phi''(\xi)\}}}$$

where ξ is defined by the equation $\phi'(\xi)=\log x$.

11. The general form of the coefficient a_n in the Taylor's expansion of $\Psi_s(x)$ is easily seen to be

$$a_n=\sum_{\nu=0}^\infty c_\nu\frac{\Gamma(s+\nu)}{\Gamma(s+n+\nu+1)}.$$

Thus, for instance, in the first example of § 8,

$$a_n=\frac{1}{\Gamma(1-\beta)}\sum_{\nu=0}^\infty\frac{\Gamma(1-\beta+\nu)\,\Gamma(a+\nu)}{\Gamma(1+\nu)\,\Gamma(a+n+\nu+1)}$$

$$=\frac{\Gamma(a)}{\Gamma(a+n+1)}F(1-\beta,\ a,\ a+n+1,\ 1).$$

* *Bull. des Sciences Math.*, t. XXIV., p. 245.

We saw otherwise that

$$a_n = \frac{\Gamma(a)\,\Gamma(\beta+n)}{\Gamma(n+1)\,\Gamma(a+\beta+n)};$$

and the two results agree in virtue of a well known property of the hypergeometric series.* A more general form of a_n which we might take would be

$$a_n = \frac{\Gamma(a)}{\Gamma(a+n+1)} F(1-\beta,\, a,\, a+n+1,\, t),$$

corresponding to $\quad \psi(1-u) = \{1-t(1-u)\}^{\beta-1}.$

12. Instead of starting with the function $f_s(x)$, we might have started with the function

$$f_{s,t}(x) = \Sigma \frac{1.2\ldots n}{s(s+1)\ldots(s+n)\,t(t+1)\ldots(t+n)} x^n$$

defined, when $\mathbf{R}(s)$ and $\mathbf{R}(t)$ are positive, by the double integral

$$\int_0^1\int_0^1 e^{xuv}(1-u)^{s-1}(1-v)^{t-1}\,du\,dv;$$

or from other more general functions which suggest themselves immediately.† But, as I said in § 2, I shall content myself for the present with indicating these generalisations.

13. Before leaving this part of the subject I may point out an interesting application of these results to the theory of multiform functions, defined by Taylor's series, with finite radii of convergence.

Application to the Function $\phi(x) = 1+\dfrac{\beta+1}{a+1}x+\dfrac{(\beta+1)(\beta+2)}{(a+1)(a+2)}x^2+\ldots.$

If

$$\phi(x) = \frac{\Gamma(a+1)}{\Gamma(\beta+1)} \sum_{\nu=0}^{\infty} \frac{\Gamma(\beta+1+\nu)}{\Gamma(a+1+\nu)} x^\nu, \tag{39}$$

it is easy to see that $\phi(x) = \dfrac{a}{\Gamma(\beta+1)}\displaystyle\int_0^\infty e^{-t}\,t^\beta f_a(tx)\,dx$

if $|x| < 1$ and $R(\beta) > -1$. If x approaches $x=1$ along any path which does not meet the circle of convergence, $tx = u$ approaches infinity along a path lying entirely within the region D in the u-plane. Hence

$$f_a(tx) = \Gamma(a)\,\frac{e^{tx}}{(tx)^a} + R$$

where $\quad |R| < K.$

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* Forsyth, *Differential Equations*, p. 199.

† The dominant term of $f_{s,t}(x)$ is easily proved to be $\Gamma(s)\,\Gamma(t)\,x^{-s-t}\,e^x$.

Now, if $\beta-a+1$ has its real part positive,

$$\Gamma(a)\int_0^\infty e^{-t+tx}t^\beta(tx)^{-a}\,dt=\frac{\Gamma(a)\,\Gamma(\beta-a+1)}{x^a(1-x)^{\beta-a+1}}.$$

Moreover, $$\left|\int_0^\infty e^{-t}t^\beta R\,dt\right|<K.$$

It follows that

$$\phi(x)=\frac{\Gamma(a+1)\,\Gamma(\beta-a+1)}{\Gamma(\beta+1)\,x^a(1-x)^{\beta-a-1}}+\Theta(x) \tag{40}$$

where $\Theta(x)$ remains numerically below a finite limit as x approaches $x=1$ along any path lying inside the circle of convergence. This result may be verified by means of the relations between the particular integrals of the hypergeometric differential equation.

The condition $\mathbf{R}(\beta)>0$ may be removed without difficulty, either by a recurrence formula or by the use of a contour instead of a line integral. If $\mathbf{R}(\beta-a+1)<0$, the series for $\phi(1)$ is convergent. If $\mathbf{R}(\beta-a+1)=0$, the result still holds unless $\beta-a+1=0$, in which case the part of $\phi(x)$ which becomes infinite is easily found to be

$$\frac{a}{x}\log\left(\frac{1}{1-x}\right).$$

And, obviously, similar results may be obtained for such functions as

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+a)^s}. \tag{41}$$

It is not difficult to determine the limit of $\Theta(x)$ in (40), and of the corresponding term in the similar formula for the function (41); but to enter into this would carry me too far from my subject.

II.—The Region D′ ($r\geqslant R_0$, $\frac{1}{2}\pi+\delta\leqslant\theta\leqslant\pi$).

14. The functions which we have been considering belong to a class of which it may roughly be said that they exhibit their most characteristic behaviour in the region D; and, notably, for *real positive* values of x. An obvious illustration is provided by the function $e^x-P(x)$, where P is a polynomial. The dominant term of all such functions is *the same* in D; in D′ it depends on the particular polynomial chosen. It is then not to be expected that the easy analysis of I. will be be equally effective now.

In this section I shall consider the function $\Phi_{a,s}(x)$ defined (when $\mathbf{R}(a)$ and $\mathbf{R}(s)$ are positive) by the equation

$$(42)\qquad \Phi_{a,s}(x) = \int_0^1 e^{xu} u^{a-1} \left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} \psi(u)\, du,$$

where $\psi(u)$ is a function of u subject to certain conditions. I shall not, however, treat the function in its most general form, but I shall consider only two cases: (i.) the case in which $\psi(u) \equiv 1$, (ii.) the case in which $s = 1$.

The Function $\int_0^1 e^{xu} u^{a-1} \psi(u)\, du$.

15. I shall consider first the function which is defined, when $\mathbf{R}(a) > 0$, by the equation

$$(43)\qquad \Phi_a(x) = \int_0^1 e^{xu} u^{a-1} \psi(u)\, du,$$

where $\psi(u)$ is a function satisfying the same conditions as $\psi(\omega)$ in § 4. The simplest case is that in which $\psi(u) \equiv 1$, *i.e.*,

$$(44)\qquad \Phi_a(x) = \sum_0^\infty \frac{x^n}{(a+n)\, n!} = F_{a,1}(x);$$

this function more or less fulfils the *rôle* of "simple element" fulfilled by $f_s(x)$ in I.

It is evident that

$$(45)\qquad F_{a,1}(x) = \frac{\Gamma(a)}{(-x)^a} - \int_1^\infty e^{xu} u^{a-1}\, du$$

where $(-x)^a = e^{a \log(-x)}$, the logarithm being real for real negative values of x; and this formula holds for all values of a save negative integral values (including zero). We easily find that

$$(46)\qquad \int_1^\infty e^{xu} u^{a-1}\, du = e^x \chi(x)$$

where $\chi(x)$ is a function which possesses the asymptotic expansion

$$(47)\qquad \frac{1}{x} + \frac{1-a}{x^2} + \frac{(1-a)(2-a)}{x^3} + \dots;$$

so that

$$(48)\qquad F_{a,1}(x) = \frac{\Gamma(a)}{(-x)^a} + \rho$$

where $|\rho|$ is for sufficiently large values of r less than any power of $1/r$; in other words, we may say that *the complete asymptotic expansion of* $F_{a,1}(x)$ *is*

$$(-x)^{-a}\Gamma(a).$$

16. Now consider the more general function $\Phi_a(x)$ of (43). We can prove, as in §§ 5, 6, that, if μ is a sufficiently large positive integer and

(49) $$\Phi_{a,\mu}(x) = \sum_{\nu=0}^{\mu-1} c_\nu F_{a+\nu,1}(x),$$

then

(50) $$\Phi_a(x) = \Phi_{a,\mu}(x) + \int_0^1 e^{xu} u^{a-1}\, du \sum_\mu^\infty c_\nu u^\nu.$$

We divide the range of integration into the two parts $(0, \delta)$, $(\delta, 1)$ where

$$\delta = (-\xi)^{-\lambda} \qquad (0 < \lambda < 1),$$

and we prove by analysis similar to that of § 6 that

$$\left|\int_0^1\right| \leqslant \left|\int_0^\delta\right| + \left|\int_\delta^1\right| < Ke^{\xi\delta} + K\delta^{\mu+a-1} < K\left\{e^{-(-\xi)^{1-\lambda}} + (-\xi)^{-\lambda(\mu+a-1)}\right\},$$

finally deducing that, throughout D′, $\Phi_a(x)$ possesses the asymptotic expansion

(51) $$\sum \frac{c_\nu \Gamma(a+\nu)}{(-x)^{a+\nu}}.$$

Thus, for example, if $a = \beta$, and

$$\psi(u) = (1-u)^{a-1},$$

where $\mathrm{R}(a) > 0$, we obtain for the function (i.) of § 8 the asymptotic expansion

$$\frac{1}{\Gamma(1-a)} \sum \frac{\Gamma(\beta+\nu)\,\Gamma(1-a+\nu)}{\Gamma(\nu+1)} (-x)^{-\beta-\nu}.$$

In particular, if $\beta = 1$, we obtain for the function $f_a(x)$ the expansion

$$\sum \frac{(1-a)(2-a)\dots(\nu-a)}{(-x)^{1+\nu}}.$$

This again agrees with a result of Herr Jacobsthal's,* and the restriction on a is easily removed.

The Function $F_{a,s}(x)$.

17. I come now to the question of the behaviour of $F_{a,s}(x)$ in D′; and it is at this point that we begin to feel the need of more powerful analytical machinery.

I start from the equation

$$F_{a,s}(x) = \sum \frac{x^\nu}{\nu!\,(\nu+a)^s} = \frac{1}{\Gamma(s)} \int_0^1 e^{xu} u^{a-1} \left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} du,$$

* *L. c.*

valid when the real parts of a and s are positive. To obtain a formula for $F_{a,s}(x)$, valid for other values of a and s, I consider the integral

$$(52) \qquad \int e^{xu} u^{a-1} \left\{ \log \left(\frac{1}{u} \right) \right\}^{s-1} du$$

taken round the contour in the plane of $u = \sigma e^{i\phi}$ formed by (i.) the positive real axis from ρ to $1-\rho$ and from $1+\rho$ to R, ρ being small and R large; (ii.) the radius vector $\phi = \pi - \theta$, from $\sigma = R$ to $\sigma = \rho$; (iii.) arcs of circles whose centres are at the origin and whose radii are ρ and R; (iv.) a small semicircle of radius ρ described around and above the point $u = 1$ (see Fig. 1).

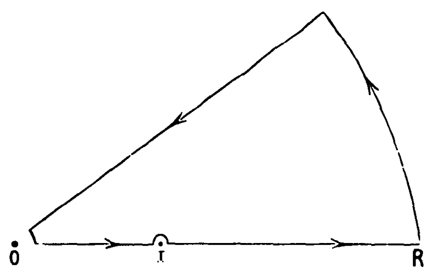

FIG. 1.

It is easy to see that the contributions of all the curvilinear parts of this contour tend to zero when ρ tends to zero and R to infinity.

We start from ρ towards $1-\rho$ with

$$u^{a-1} = e^{(a-1)\log u}, \qquad \left\{ \log \left(\frac{1}{u} \right) \right\}^{s-1} = e^{(s-1)\log\log(1/u)},$$

$\log u$ and $\log\log(1/u)$ being real. If $u = 1-\rho e^{i\psi}$,

$$\log \frac{1}{1-\rho e^{i\psi}} = \rho e^{i\psi} + \dots, \qquad \log\log\left(\frac{1}{u}\right) = \log\rho + i\psi + \dots .$$

When u is at $1-\rho$, $\psi = 0$, and as u goes round the small semicircle ψ decreases to $-\pi$. When u is at $1+\rho$, $\log\log(1/u) = \log\rho - i\pi + \dots$, and so the value of $\{\log(1/u)\}^{s-1}$ along the line $(1+\rho, R)$ is defined by

$$\left\{ \log \left(\frac{1}{u} \right) \right\}^{s-1} = e^{-(s-1)\pi i} (\log u)^{s-1},$$

where $$(\log u)^{s-1} = e^{(s-1)\log\log u},$$

$\log\log u$ being real. Thus the contribution of (i.) is ultimately

$$(53) \qquad \int_0^1 e^{xu} u^{a-1} \left\{ \log \left(\frac{1}{u} \right) \right\}^{s-1} du + e^{-(s-1)\pi i} \int_1^\infty e^{xu} u^{a-1} (\log u)^{s-1} du,$$

with the above assumptions as to the values of the many-valued functions involved.

Again it is easy to see that the contribution of (ii.) is

$$(54)\qquad -e^{a(\pi-\theta)i}\int_0^\infty e^{-\sigma r}\sigma^{a-1}\left\{\log\left(\frac{1}{\sigma e^{i\phi}}\right)\right\}^{s-1}d\sigma,$$

where the path of integration is real, $\sigma^{a-1}=e^{(a-1)\log\sigma}$, $\log\sigma$ being real, and

$$\left\{\log\left(\frac{1}{\sigma e^{i\phi}}\right)\right\}^{s-1}=e^{(s-1)\log[-\log\sigma+i(\theta-\pi)]},$$

that branch of $\log\{-\log\sigma+i(\theta-\pi)\}$ being taken whose imaginary part is very small with σ. Thus, by Cauchy's theorem, we arrive at the equation

$$(55)\quad \Gamma(s)F_{a,s}(x)= e^{a(\pi-\theta)i}\int_0^\infty e^{-\sigma r}\sigma^{a-1}\{-\log\sigma+i(\theta-\pi)\}^{s-1}d\sigma$$

$$-e^{(s-1)\pi i}\int_1^\infty e^{xu}u^{a-1}(\log u)^{s-1}du$$

$$= e^{a(\pi-\theta)i}\,r^{-a}\int_0^\infty e^{-t}t^{a-1}\{\log r-\log t+i(\theta-\pi)\}^{s-1}dt$$

$$-e^{x-(s-1)\pi i}\int_0^\infty e^{xt}(1+t)^{a-1}\{\log(1+t)^{s-1}\}\,dt,$$

on transforming the two integrals by the substitutions $\sigma r=t$ and $u=1+t$. This formula, which I shall write in the form

$$(56)\qquad \Gamma(s)F_{a,s}(x)=e^{a(\pi-\theta)i}r^{-a}A-e^{x-(s-1)\pi i}B,$$

is valid if the real parts of a and s are positive.

Introduction of Loop Integrals.

20. This formula is easily generalised so as to cover all values of s save negative integral values. For consider the integral

$$\int e^{xt}(1+t)^{a-1}\{\log(1+t)\}^{s-1}dt,$$

taken round the contour C shown in the figure (Fig. 2), including the

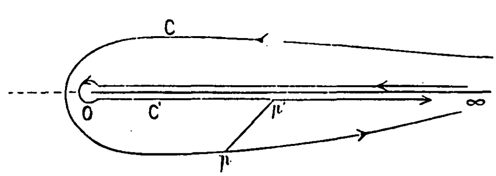

FIG. 2.

positive half of the real axis, but excluding the point $t=-1$. The values of the many-valued functions are to be so chosen that, if t moves along pp', they assume at p' the values already assigned to them in B. Then, if $\mathrm{R}(s)>0$, the contour C may be transformed into the limit of the contour C' (see the figure) and it is easy to see that

(57) $$\int_{(C)} = \{1-e^{-2\pi i(s-1)}\}\, B,$$

$\{\log(1+t)\}^{s-1}$ being multiplied by $e^{2\pi i(s-1)}$ by a positive circuit round the origin. Thus

(58) $$\Gamma(s)F_{a,s}(x) = e^{a(\pi-\theta)i}\, r^{-a}A - \frac{e^{x-(s-1)\pi i}}{1-e^{-2\pi i(s-1)}}\int_{(C)}$$

$$= e^{a(\pi-\theta)i}\, r^{-a}A - e^{x-(s-1)\pi i}B', \text{ say.}$$

This formula is valid for all non-integral values of s; while (56) is valid for all values of s whose real part >0. Thus *one* of (56) or (57) is valid for all values of s save negative integral or zero values of s. In both, however, $\mathrm{R}(a)>0$.

21. Now it is easy to see that throughout D′

(59) $$|B'| < K(-\xi)^{\gamma},$$

where γ is a real constant. For, if we take C' to be formed by two lines practically coinciding with the real axis, and a small circle of radius ρ, then along the circle

$$|e^{-xt}| < e^{-\xi\rho}, \qquad |(1+t)^{a-1}\{\log(1+t)\}^{s-1}| < K\rho^{s'-1},$$

where $s'=\mathrm{R}(s)$; so that the modulus of the contribution of the circle is less than $K\rho^{s'}e^{-\xi\rho}$. If we take $\rho=-1/\xi$, this is less than $K(-\xi)^{-s'}$. Again, the contribution of the rectilinear parts is in absolute value

$$< K\int_{\rho}^{\delta} t^{s'-1}\,dt + Ke^{\delta\xi},$$

where δ is any small quantity $>\rho$. It is easy to see that, if we take $\delta=\log(-\xi)/(-\xi)$, the first of these terms is less than $K(-\xi)^{1-s'}$, and the second less than $K/(-\xi)$. Hence the second term of (58) is in absolute value less than $Ke^{\xi}(-\xi)^{\gamma}$, or, what is the same thing, less than $Ke^{\xi}r^{\gamma}$.

22. Again

$$A = (\log r)^{s-1}\int_0^{\infty} e^{-t}t^{a-1}\left\{1-\frac{\log t - i(\theta-\pi)}{\log r}\right\}^{s-1}dt,$$

and it is easy to see that the limit of this integral for $r=\infty$ is

$$\int_0^{\infty} e^{-t}t^{a-1}\,dt = \Gamma(a).$$

I omit the formal proof of this, which is a little tedious, and in no way particularly interesting. Hence we arrive at the following conclusion:— for all values of θ such that $\frac{1}{2}\pi+\delta \leqslant \theta < \pi$

$$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}(\log r)^{s-1}(1+\epsilon_x) \tag{60}$$

where ϵ_x is a function of x which tends to 0 for $r=\infty$, and that *uniformly* for all values of θ in question, being, in fact, numerically less than $K/\log r$. Here

$$(-x)^{-a} = r^{-a}e^{a(\pi-\theta)i},$$

which is real for $\theta = \pi$. Again $(\log r)^{s-1} = \{\log(-x)\}^{s-1}(1+\epsilon)$. Finally, from the uniformity of the convergence of lim ϵ_x, we infer that the equation (60) is valid also for $\theta=\pi$ (as may be proved independently). We have thus proved theorem II. of § 2 with the sole restriction that $\mathrm{R}(a) > 0$.

This last restriction also may be removed unless a is zero or a negative integer. For, if x_0 is a fixed point in D′ and $|x|$ is large, and the path of integration is rectilinear,

$$\int_{x_0}^{x} F_{a,s}(x)\,dx = \sum_{0}^{\infty}\frac{x^{\nu+1}}{(\nu+1)!\,(\nu+a)^s}+C = F_{a-1,s}(x)-1+C,$$

where C is independent of x. Now

$$\int_{x_0}^{x}(-x)^{-a}\{\log(-x)\}^{s-1}dx$$
$$= \left[-\frac{(-x)^{1-a}}{1-a}\{\log(-x)\}^{s-1}\right]_{x_0}^{x} + \frac{s-1}{1-a}\int_{x_0}^{x}(-x)^{-a}\{\log(-x)\}^{s-2}dx.$$

The first term may, if $0 < \mathrm{R}(a) \leqslant 1$, be put in the form

$$-\frac{(-x)^{1-a}}{1-a}\{\log(-x)\}^{s-1}(1+\epsilon_x),$$

while the second is in absolute value less than

$$K\int_{r_0}^{r} r^{-a}(\log r)^{s-2}dr < Kr^{1-a}(\log r)^{s-2}.$$

Finally, it is easy to deduce from the inequality $|\epsilon_x| < K/\log r$ that

$$\left|\int_{x_0}^{x}(-x)^{-a}\{\log(-x)\}^{s-1}\epsilon_x\,dx\right| < Kr^{1-a}(\log r)^{s-2}.$$

Thus, finally, $F_{a-1,s}(x) = \dfrac{\Gamma(a-1)}{\Gamma(s)}(-x)^{1-a}(\log r)^{s-1}(1+\epsilon_x)$.

If we write a for $a-1$, the range of (60) is extended to all values of a other than $a=0$, for which $\mathrm{R}(a) > -1$. Repeating this process of extension, we arrive finally at the complete proof of II.

III. THE REGION E: THE ZEROES OF $F_{a,s}(x)$.

23. It follows from the results of I. and II. that there are infinitely many zeroes of $F_{a,s}(x)$ within the region E. In order to determine them more precisely it is necessary to determine an asymptotic formula for $F_{a,s}(x)$ valid within this region. We must distinguish three cases—the cases in which $\xi \lesseqgtr 0$.

The case $\xi < 0$.

24. The analysis which led to (58) assumed only that $\xi < 0$, and the formula is therefore valid for all such points of E. The same is true of the reduction of the first term on the right-hand side of (58) to the form

$$\Gamma(a)(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_r).$$

But we must now consider the second term more precisely. We therefore turn our attention to the integral

$$I = \int_{(C)} e^{xt}(1+t)^{a-1}\{\log(1+t)\}^{s-1}\,dt. \tag{61}$$

The real part of x being negative, it is easy to see that we may replace the contour of integration by a similar contour C_1 enclosing the origin and the straight line for which $t = \tau e^{i\phi}$, $\phi = \pi - \theta$.

This contour we replace by a contour C_1' similar to the contour C' of § 21, taking the radius of the small circle to be $1/r$. Now

$$I = \int_{(C_1)} e^{xt} t^{s-1}\,dt + R = \{1 - e^{-2\pi i(s-1)}\}\frac{\Gamma(s)}{(-x)^s} + R,$$

where $(-x)^s = r^s e^{(\theta-\pi)si}$ and

$$R = \int_{(C_1)} e^{xt}[(1+t)^{a-1}\{\log(1+t)\}^{s-1} - t^{s-1}]\,dt. \tag{62}$$

We can prove, as in § 21, that the absolute value of the contribution of the circular part of the contour is less than $Kr^{-s'-1}$. The absolute value of the remaining part of R is less than

$$K\int_{1/r}^{\infty} e^{-r\tau}\,|\,(1+\tau e^{i\phi})^{a-1}\log(1+\tau e^{i\phi})^{s-1} - \tau^{s-1}e^{(s-1)i\phi}\,|\,d\tau = K\int_{1/r}^{\delta} + \int_{\delta}^{\infty}$$

where δ is less than unity. The first term is less than

$$K\int_{1/r}^{\delta} \tau^{s'}d\tau < K(\delta^{s'+1} - r^{-s'-1}),$$

and the second than $\qquad Ke^{-r\delta}.$

If we take $\delta = \kappa \log r/r$, where $\kappa > s'+1$, both terms are small in comparison with $r^{-s'}$, and so

(63) $$\lim_{r=\infty} r^{s'} R = 0,$$

and that uniformly for all values of x whose real part is negative. Hence

(64) $$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x) + \frac{e^x}{x^s}(1+\epsilon_x')$$

where $(-x)^{-a} = e^{-a\log(-x)}$, and $x^s = e^{-s\log(x)}$, the logarithms being real on the negative and positive halves of the real axis respectively, and ϵ_x, ϵ_x' are quantities which tend uniformly to zero with $1/r$ for all values of x whose real part is negative. Owing to the uniformity of the convergence of the limit in (63) the formula is also valid when $\mathrm{R}(x) = 0$. The extension to values of a whose real part is less than 0 is much the same as before. We have only to make the almost obvious additional remark that when r is large

$$\int_{x_0}^{x} \frac{e^x}{x^s}\,dx = \frac{e^x}{x^s}(1+\epsilon_x)+C$$

where C is independent of x.

The case $\xi > 0$.

25. It is of importance for our present purpose to prove that the formula (64) is valid also for those points of E whose real part is positive. The proof of this is so similar to the preceding analysis that I shall merely indicate the principal steps in it.

We start from the formula

(65) $$\Gamma(s)\cdot F_{a,s}(x) = e^x \int_0^1 e^{-xu}(1-u)^{a-1}\left\{\log\left(\frac{1}{1-u}\right)\right\}^{s-1} du,$$

valid, like (35), so long as the real parts of a and s are positive; and we consider the integral

$$\int e^{-xu}(1-u)^{a-1}\left\{\log\left(\frac{1}{1-u}\right)\right\}^{s-1} du$$

taken round a contour which only differs from that of § 17 in that the radius vector (ii.) is defined by $\phi = -\theta$ and that the semicircle (iv.) is turned downwards. By arguments similar to those of § 17, we arrive at the formula

(66) $$\Gamma(s)F_{a,s}(x) = e^{x-i\theta}\int_0^\infty e^{-\sigma r}(1-\sigma e^{-i\theta})^{a-1}\left\{\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right)\right\}^{\sigma-1} d\sigma$$
$$- e^{x+(a-1)\pi i}\int_1^\infty e^{-xu}(u-1)^{a-1}\{-\log(u-1)-\pi i\}^{s-1} du$$

where in the first integral $(1-\sigma e^{-i\theta})^{a-1} = e^{(a-1)\log(1-\sigma e^{-i\theta})}$, the logarithm vanishing for $\sigma = 0$, and

$$\left\{\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right)\right\}^{s-1} = \exp\left[(s-1)\log\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right)\right],$$

wherein

$$\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right) = \sigma e^{i\theta}+\ldots, \quad \log\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right) = \log\sigma - i\theta + \ldots$$

when σ is small; while in the second integral $(u-1)^{a-1} = e^{(a-1)\log(u-1)}$, the logarithm being real, and

$$\{-\log(u-1)-\pi i\}^{s-1} = \exp[(s-1)\log\{-\log(u-1)-\pi i\}],$$

$\log(u-1)$ being real and $\log\{-\log(u-1)-\pi i\}$ having its imaginary part small when $(u-1)$ is small.

We transform each of these integrals as in § 19, obtaining

(67) $\Gamma(s)F_{a,s}(x)$

$$= e^{x-si\theta}r^{-s}\int_0^\infty e^{-t}t^{s-1}\left(1-\frac{t}{r}e^{-i\theta}\right)^{a-1}\left\{\frac{re^{i\theta}}{t}\log\left(\frac{1}{1-\frac{t}{re^{i\theta}}}\right)\right\}^{s-1}dt$$

$$-e^{(a-1)\pi i}\int_0^\infty e^{-xt}t^{a-1}\{-\log t-\pi i\}^{s-1}dt$$

where in the first integral the last bracket, when expanded in powers of t, starts with the term $1+\ldots$.

The first of these integrals must be replaced by a loop integral when $\mathbf{R}(s) \leqslant 0$, as in § 20. Finally, by arguments similar to those of §§ 22–24, we arrive at the asymptotic formula (64).

The Zeroes of $F_{a,s}(x)$.

26. We have then the asymptotic formula*

(64) $$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x)+\frac{e^x}{x^s}(1+\epsilon'_x),$$

valid for all points of E, and all values of a and s other than negative integral values. If $x = \xi+i\eta$ is a zero of $F_{a,s}(x)$,

$$e^{\xi+i\eta-si\theta}r^{-s} = -\frac{\Gamma(a)}{\Gamma(s)}(\log r)^{s-1}r^{-a}e^{-(\theta-\pi)ia}(1+\epsilon).$$

* In this section I suppose, for simplicity, that a and s are real. The necessary modifications when they are not are easily made.

Now $r = \eta(1+\epsilon)$, $\log r = \log \eta(1+\epsilon)$, and $\theta = \frac{1}{2}\pi+\epsilon$. Hence

$$e^{\xi+i\eta-\frac{1}{2}s\pi i-\epsilon i} = -\frac{\Gamma(a)}{\Gamma(s)}(\log \eta)^{s-1}\eta^{s-a}e^{\frac{1}{2}\pi i a}(1+\epsilon). \tag{68}$$

Equating the moduli of the two sides, we find

$$e^{\xi} = \frac{\Gamma(a)}{\Gamma(s)}(\log \eta)^{s-1}\eta^{s-a}(1+\epsilon), \tag{69}$$

$$\xi = (s-a)\log \eta+(s-1)\log\log \eta+\log\frac{\Gamma(a)}{\Gamma(s)}+\epsilon. \tag{70}$$

Dividing (68) by (69),

$$e^{(\eta-\frac{1}{2}s\pi-\epsilon)i} = -e^{\frac{1}{2}a\pi i}, \tag{71}$$

or

$$\eta = \tfrac{1}{2}(a+s)\pi+(2k+1)\pi+\epsilon, \tag{72}$$

k being a positive integer. From (70) and (71) we deduce the asymptotic formulæ

$$\begin{cases} \xi = (s-a)\log(2k\pi)+(s-1)\log\log k+\log\dfrac{\Gamma(a)}{\Gamma(s)}+\epsilon, \\ \eta = (2k+1)\pi+\tfrac{1}{2}(a+s)\pi+\epsilon. \end{cases} \tag{73}$$

Thus, the zeroes of $F_{a,s}(x)$ are associated with some or all of the points obtained by giving k any large positive integral value in the above formulæ. The real part of the zeroes is therefore ultimately positive if $s > a$, negative if $s < a$. If $s = a$, its sign depends on that of $s-1$. If $s = a = 1$,

$$F_{a,s}(x) = \sum_0^\infty \frac{x^n}{(n+1)!} = \frac{e^x-1}{x},$$

and the zeroes are all purely imaginary.

27. It still remains to be proved that *one, and only one*, zero of $F_{a,s}(x)$ corresponds to each of the points (73). The proof of this is not difficult, though a little tedious. I shall only indicate the argument briefly; it is as follows:—

In the first place, it is easy to show that the function

$$\Theta_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}+\frac{e^x}{x^s} \tag{74}$$

vanishes, when k is large, once, and only once, in the immediate neighbourhood of each of the points (73). To prove this, we have only (following a line of argument which I have employed on several occasions in the papers already referred to) to draw the portions of the curves

$$\left|\frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}\right| = \left|\frac{e^x}{x^s}\right|,$$

$$\operatorname{am}\frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1} = \operatorname{am}\left(-\frac{e^x}{x^s}\right),$$

which lie in the part of the plane in question, and to satisfy ourselves that there is in fact just one intersection near each of the points (73).

Now let δ be a fixed, but fairly small, positive quantity (such as $\frac{1}{10}$). Let us surround each of the points (73) by a closed contour, say a square with its sides parallel to the coordinate axes and all at unit distance from the point. First we prove that for all points on this square

(75) $$|\Theta_{a,s}(x)| > \delta\left|\frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}\right|.$$

Then we have only to show that for points on the square the ratio of the moduli of the two terms of $\Theta_{a,s}(x)$ lies between certain fixed limits in order to satisfy ourselves that along the contour of the square

(76) $$F_{a,s}(x) = \Theta_{a,s}(x)(1+\epsilon),$$

where ϵ is small. It follows that $F_{a,s}(x)$ has within the square the same number of zeroes as $\Theta_{a,s}(x)$, that is to say, *one*.

The Zeroes of the Two Simple Functions $f_s(x)$ and $F_{a,1}(x)$.

28. When $s = 1$ we obtain, for the zeroes of the function $F_{a,1}(x)$ which served as "simple element" in D′, the asymptotic formula

(77) $$\xi = (1-a)\log(2k\pi)+\log\Gamma(a), \qquad \eta = (2k+1)\pi+\tfrac{1}{2}(a+1)\pi.$$

29. The corresponding investigation for the function $f_s(x)$, which served as our simple element in D, is simpler, and I shall not set it out in detail, as the formula

(78) $$f_s(x) = \Gamma(s)x^{-s}e^x(1+\epsilon)-x^{-1}(1+\epsilon)$$

is already known.* From this we deduce the asymptotic formula for the zeroes, viz.,

(79) $$\xi = (s-1)\log(2k\pi)-\log\Gamma(s), \qquad \eta = 2k\pi+\tfrac{1}{2}(s-1)\pi.$$

It may be shown, as above, that one and only one zero is associated with each of these points.

* See *e.g.*, Jacobsthal, *loc. cit.*

30. I do not propose to attempt a similar discussion for the more general functions considered in I. and II. It is obvious that in order to apply the preceding methods assumptions would have to be made not only as regards the behaviour of the arbitrary functions ψ along the line (0, 1), but also as regards their analytic nature for complex values of u. To take a simple example, consider the function defined by the integral

$$G_a(x) = \int_0^1 e^{xu} u^{a-1} \psi(u)\, du \quad (a > 0)$$

and its continuation in the a-plane, $\psi(u)$ being real and expansible in a Taylor's series which converges for $u = 1$. Then the dominant terms of the asymptotic expressions for $G_a(x)$ in D and D′ respectively are $x^{-1}e^x\psi(1)$ and $\Gamma(a)(-x)^{-a}\psi(0)$ respectively. It is natural to suppose after what has preceded that in E

$$G_a(x) = \frac{e^x}{x}\psi(1)(1+\epsilon) + \Gamma(a)(-x)^{-a}\psi(0)(1+\epsilon),$$

in which case the zeroes are, when a is real, given by the points

$$(1-a)\log(2k\pi) + \log\Gamma(a) + \log\frac{\psi(0)}{\psi(1)} + i\{(2k+1)\pi + \tfrac{1}{2}(a+1)\pi\}.$$

But I do not intend now to attempt to investigate the conditions regarding $\psi(u)$ which are sufficient to establish the truth of this.

The case in which $s = 1$ and a is a Positive Integer.

31. If $s = 1$ and a is a positive integer, we can obtain an easy and interesting verification of our results. In fact, in this case,

$$(80)\quad F_{a,s}(x) = \int_0^1 e^{xu} u^{a-1} du = \frac{e^x}{x} \sum_0^{a-1} \frac{(-)^\nu (a-1)\dots(a-\nu)}{x^\nu} + \frac{(-)^a (a-1)!}{x^a},$$

as is easily found by repeated integration by parts. In the first place, this verifies the formula (64). Again, the equation $F_{a,s}(x) = 0$ takes the form

$$e^x = (-)^{a-1}(a-1)!/x^{a-1} - \dots,$$

and it follows from results which I have proved elsewhere* that the

* *Quarterly Journal*, Vol. xxxv., p. 261.

asymptotic solution of this equation is given by

$$\xi = (1-a)\log(2k\pi)+\log\Gamma(a), \qquad \eta = (2k+1)\pi+\tfrac{1}{2}(a+1)\pi,$$

which is in agreement with the general result. The case in which $a = 1$ has been already disposed of (§ 26, end).

IV.

32. I shall conclude this paper by a short discussion of one or two points of a miscellaneous character.

The Function $F_{a,\,-n}(x)$.

In all the preceding analysis it has been assumed that neither a nor s is a negative integer. If a is one, $F_{a,s}(x)$ is no longer defined. But the case in which s is a negative integer $-n$ is of considerable interest. In fact, in this case $F_{a,s}(x)$ reduces to the product of e^x by a polynomial $P_n(x)$ of degree n. For

$$F_{a,\,-n}(x) = \sum_0^\infty \frac{(\nu+a)^n x^\nu}{\nu!}$$

is the coefficient of t^n in the expansion of

$$n!\sum_0^\infty \frac{e^{(\nu+a)t}x^\nu}{\nu!} = n!\, e^{at+xe^t};$$

so that

$$F_{a,\,-n}(x) = \left[\left(\frac{d}{dt}\right)^n e^{at+xe^t}\right]_{t=0},$$

which is easily seen to be of the form*

(81) $$e^x P_n(x).$$

From the method of formation of the polynomials P_n it is easy to deduce the recurrence formula

(82) $$P_{n+1}(x) = (x+a)P_n(x)+x\frac{dP_n(x)}{dx};$$

so that

(83) $$P_0(x) = 1, \quad P_1(x) = x+a, \quad P_2(x) = x^2+(2a+1)x+a^2, \; \dots .$$

If a is real and positive, the roots of $P_n(x)$ are all real and negative, and separated by those of $P_{n-1}(x)$. This is easily proved by induction.

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* A result substantially equivalent to this was proposed as a problem in the Mathematica Tripos for 1903.

Another interesting property of these polynomials is that

$$\int_{-\infty}^{0} e^x P_n(x)\,dx = (a-1)^n. \tag{84}$$

The Equation $F_{a,s}(x) = c$.

33. The question is naturally suggested whether the functions $F_{a,s}(x)$ possess the property that for *one* value of the constant c the distribution of the roots of $F_{a,s}(x) = c$ is abnormal. It is easy to see that in certain cases they do, though the peculiarity is far less marked than in the case in which $s = 0$ (or, more generally, s is a negative integer). Suppose, for simplicity, that $s = 1$, and that a and c are real. Then we have to satisfy the equation

$$\frac{e^x}{x}(1+\epsilon)+\Gamma(a)(-x)^{-a}(1+\epsilon) = c.$$

It is easy to infer from this that ξ must be positive and large (though small in comparison with η), whatever be the value of a. If $a < 0$, we approximate to the roots by taking

$$e^x = \Gamma(a)(-x)^{1-a}(1+\epsilon),$$

and the value of c is indifferent. But, if $a > 0$, we must take

$$e^x = cx(1+\epsilon),$$

i.e., $$\xi = \log(2k\pi)+\log c+\epsilon, \qquad \eta = (2k+\tfrac{1}{2})\pi,$$

unless $c = 0$, in which case the approximation (77) still holds. Thus the case of $c = 0$ is abnormal, provided $a > 0$ [and, more generally, provided $R(a) > 0$].

The Function $F_s(x) = \sum_1^\infty \frac{x^n}{n^s n!}$.

34. In the case in which a is zero or a negative integer the definition of $F_{a,s}(x)$ by means of a series fails. But, if, for instance, $a = 0$, it is natural to define $F_s(x) = F_{0,s}(x)$ as

$$\lim_{a=0}\{F_{a,s}(x)-a^{-s}\} = \sum_1^\infty \frac{x^n}{n^s n!}. \tag{85}$$

If $\mathbf{R}(s) > 0$, $$\Gamma(s)F_s(x) = \int_0^1 \frac{e^{xu}-1}{u}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} du$$

and $$\Gamma(s)F_s'(x) = \int_0^1 e^{xu}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} du = \Gamma(s)F_{1,s}(x).$$

The asymptotic expressions in D and D′ for $F_{1,s}(x)$ are

$$x^{-s}e^{x} \quad \text{and} \quad -\frac{1}{\Gamma(s)}\frac{\{\log(-x)\}^{s-1}}{x},$$

and it may be shown, in the first place, that the dominant terms in the expressions for $F_s(x)$ are dominant terms in the integrals of these expressions, namely

$$x^{-s}e^{x} \quad \text{and} \quad -\frac{\{\log(-x)\}^{s}}{\Gamma(s+1)};$$

and, in the second place, that the equation $F_s(x) = 0$ is equivalent to

$$x^{-s}e^{x} = \frac{\{\log(-x)\}^{s}}{\Gamma(s+1)}(1+\epsilon),$$

from which we deduce as an asymptotic formula for the zeroes

(86) $\xi = s\log(2k\pi)+s\log\log k-\log\Gamma(s+1), \qquad \eta = (2k+\tfrac{1}{2}s)\pi.$

In the particularly interesting case in which $s = 1$, so that

(87) $$F_1(x) = \sum_{1}^{\infty}\frac{x^n}{n.n!} = \text{li}(e^x)-\log(-x)-\gamma,$$

where γ is Euler's constant, the asymptotic expressions are*

(88) $$e^x/x, \quad -\log(-x),$$

and the formula for the zeroes is

(89) $$\log(2k\pi)+\log\log k+i(2k+\tfrac{1}{2})\pi.$$

Functions analogous to the Sine Function.

35. All the functions which have been considered so far are in many ways analogous to the ordinary exponential function. Their increase is substantially that of e^r, and the distribution of their zeroes is substantially similar to that of the zeroes of e^x-c $(c\neq 0)$. Even in the case of those functions whose zeroes have not been approximated to by the methods of Section III., the asymptotic expressions obtained in I. and II. show that the zeroes ultimately lie inside any small angle issuing from O and including the imaginary axis.

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* See Barnes, "On Integral Functions," *Phil. Trans.* (A), Vol. 199, p. 411, and Horn, *Crelle*, Bd. cxx., p. 1, where complete asymptotic expansions of this function are obtained.

By means of combinations of these functions we can form a variety of functions similarly related to the simple function $\sin x$.

Consider, for instance, the function

$$(90)\qquad \psi_a(x) = \int_0^1 \sin(xu)u^{a-1}du \quad [\mathrm{R}(a) > 0]$$

$$= \frac{1}{2i}\{F_{a,1}(ix) - F_{a,1}(-ix)\} = \sum_{n=0}^{\infty} \frac{(-)^n x^{2n+1}}{(2n+1+a)(2n+1)!}.$$

We easily find that in the domain D_1 for which $0 < \delta \leqslant \theta \leqslant \pi - \delta < \pi$

$$(91)\qquad \psi_a(x) = -\frac{e^{-xi}}{2x}(1+\epsilon),$$

while within the corresponding domain below the real axis

$$(92)\qquad \psi_a(x) = -\frac{e^{xi}}{2x}(1+\epsilon).$$

Thus $\psi_a(x)$ possesses the property of $\sin x$ that its modulus tends to infinity along any line issuing from the origin and going to infinity save along the real axis.

On the other hand, if ξ and ξ/η are large and ξ positive, $\psi_a(x)$ may be expressed in the form

$$(93)\qquad -\frac{e^{xi}+e^{-xi}}{2x}(1+\epsilon) + \frac{\Gamma(a)}{2i}\{(-xi)^{-a} - (xi)^{-a}\}(1+\epsilon),$$

where $(-xi)^{-a}$ has an argument nearly equal to $\frac{1}{2}\pi i a$ and $(xi)^{-a}$ one nearly equal to $-\frac{1}{2}\pi i a$. From this formula an asymptotic formula for the zeroes may be deduced. If, *e.g.*, a is real, positive, and less than unity, ξ, η, and ξ/η are all large and positive and

$$\frac{e^{-i\xi+\eta}}{\xi} = \frac{\Gamma(a)\sin\frac{1}{2}\pi a}{\xi^a}(1+\epsilon),$$

and so

$$(94)\qquad \xi = 2k\pi + \epsilon, \qquad \eta = (1-a)\log(2k\pi) + \log\{\Gamma(a)\sin\tfrac{1}{2}\pi a\} + \epsilon.$$

In the special case in which $a = 1$,

$$\psi_1(x) = \int_0^1 \sin xu\,du = \frac{1-\cos x}{x},$$

so that all the zeroes are real; in fact, $\xi = 2k\pi$, $\eta = 0$, which agrees with the general result. The close analogy between $\psi_a(x)$ and $\sin x$ is now apparent.

Functions analogous to the Function $\sum_0^\infty \frac{x^n}{\Gamma(an+1)}$.

36. Prof. Mittag-Leffler has defined a function

$$E_a(x) = \sum_0^\infty \frac{x^n}{\Gamma(an+1)},$$

and has summarily indicated some of its properties, which are in many ways analogous to those of the exponential

$$e^{x^{1/a}}.$$

It is natural to suppose that the function

$$F_{a,\,a,\,s}(x) = \sum_0^\infty \frac{x^n}{(n+a)^s\,\Gamma(an+1)}$$

would be, to some extent at any rate, amenable to analysis similar to that of this paper. But, as Prof. Mittag-Leffler's extended memoir on the subject has not yet appeared, I shall not discuss this question further at present.

Conclusion.

37. The behaviour in D of the series

$$\Sigma \frac{c_n x^n}{n!}$$

where
$$c_n = \frac{1}{(n+a)^s}\left(b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \ldots\right),$$

the series $b_0 + b_1/n + \ldots$ being convergent for $n \geqslant 1$, may be determined in certain cases by means of the results of Section I. But the corresponding investigations for D′ and E seem to present serious difficulties, the nature of which I have to some extent already indicated. And, if, instead of postulating the entire analytic nature of the coefficients c_n, we confine ourselves to the information furnished by inequalities, however precise, we find at once that very little progress can be made. Suppose, for instance, that we consider the function

$$F(x) = \Sigma c_n \frac{x^n}{n!}$$

where
$$c_n = \frac{1}{n+a} + \rho_n, \qquad |\rho_n| < \frac{K}{(n+a)^2}.$$

Then
$$F(x) = F_{a,\,1}(x) + \phi(x)$$

where $$|\phi(x)| < K\Sigma \frac{r^n}{(n+a)^2 n!} < K\frac{e^r}{r^2}.$$

Thus at a zero of $F(x)$

$$|F_{a,1}(x)| < K\frac{e}{r^2},$$

which, if, *e.g.*, $\xi > 0$, gives

$$\frac{e^{\xi}}{r} < K\frac{e^r}{r^2}, \qquad \xi < K + r - \log r,$$

an inequality which conveys very little information indeed. And this is only as it should be. Consider, for instance, the case in which

$$c_n = \frac{1}{n+a} + \frac{(-i)^n}{(n+a)^2}, \qquad F(x) = F_{a,1}(x) + F_{a,2}(-ix).$$

The modulus of this function becomes exponentially infinite when x approaches infinity along any radius vector situated in the angle $(\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta)$; and its zeroes are distributed over the plane in a manner entirely different from that of the zeroes of $F_{a,1}(x)$.