The terms $\frac{\partial\Phi}{\partial z}$ and $\frac{\partial\Phi}{\partial x}$ are due to magnetic doublets, whilst the others are due to electric doublets. All these integrals vanish if x', y', z' be on the negative side of (x, y) and $\frac{\partial\Phi}{\partial z}$ and $\frac{\partial^2\Phi}{\partial x\partial z}$ change sign on crossing the plane xy; hence the electric and magnetic doublets each contribute half the total amount of the magnetic force. Putting

$$\lambda = \frac{x'-x}{r}, \quad \mu = \frac{y'-y}{r}, \quad \nu = \frac{z'}{r},$$

we find that at a great distance the electric force contributed by a magnetic doublet is proportional to $(-\nu, 0, \lambda)$, whilst the electric force contributed by an electric doublet is proportional to $[-(\mu^2+\nu^2), \lambda\mu, \lambda\nu]$. If we take the sources as they stand in the integrals above, we find that the electric force contributed by an element of the surface varies as $1+\nu$. This is Prof. Love's expression. If we took only the magnetic doublets, the force would vary as $\sqrt{\nu^2+\lambda^2}$. If we took only the electric doublets, we should get Lord Rayleigh's form $\sqrt{\mu^2+\nu^2}$. Since X, Y, Z are independent of x, y, z, then

$$\frac{\partial X}{\partial z} = 0, \quad \frac{\partial Y}{\partial z} = 0, \quad \frac{\partial Z}{\partial z} = 0.$$

This means that a certain system of electric quadruplets distributed on the surface have a null effect. Combining these sources with the electric doublets, we find Sir G. Stokes's expression $\sqrt{\mu^2+\nu^2}(1+\nu)$. The region of integration is supposed to be an infinite rectangle.

Sets of Intervals on the Straight Line. By W. H. Young.

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The consideration of the theory of linear sets of points leads, in a natural manner, to that of sets of intervals on a straight line. Indeed, in some respects it is more natural to begin with the latter than with the former. For example, every set of non-overlapping intervals defines a set of points, (namely, those points which are not interior to the intervals); whereas the converse is not necessarily true unless the set of points be closed.

A branch of the theory where this latter order seems the more natural, as well as the simpler, is that of the theory of content. We propose in the present paper to investigate certain fundamental theorems on sets of intervals, with the object of subsequently applying our results to the general theory of closed sets of points.

The sequence of thought here presented runs, to some extent, on parallel lines to that of Borel in his *Leçons;** but the mode of presentation is different, and some of the results have, we believe, never been formally stated. The introduction of the explicit distinction between external and semi-external points and the avoidance of the Heine-Borel theorem,† (which is the key-stone of Borel's account), may, perhaps, be said to characterize the present introductory account. The proof of this theorem given by Borel in the *Leçons*, (Borel's second proof), is very elegant in conception; but it can scarcely be said to give the reader an insight into the *raison d'être* of the theorem.

§ 1. We define the *content I_δ of a finite number of intervals* to be the sum of their lengths. With this definition we see at once that

(1) the content I_δ is positive and less than or equal to the length l of the segment (A, B), (supposed finite), of the straight line in which the set lies.

(2) If the content I_δ be less than l, then there exists a complementary set of intervals, whose content I_p is equal to the excess of l over the content of the given set, *i.e.*,

$$I_\delta + I_p = l.$$

(3) If the content I_δ be equal to l, then

(i.) There are no complementary intervals;

(ii.) There are no points of (A, B) exterior to all the intervals;

(iii.) There are no end points (except, of course, A and B), which do not belong to two intervals.

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* Emile Borel, *Lecons sur la Théorie des Fonctions*, 1898.

† § 11, footnote.

§ 2. Next consider an infinite number or set of non-overlapping intervals in the segment (A, B). What are analogous theorems to (1), (2), and (3) of the previous article? Cantor has proved * that in this case the set of intervals is countable; they will therefore have a definite sum * less than or equal to l. *This we define as the content* I_δ. It then follows at once that (1) holds as it stands for any set of non-overlapping intervals.

When $I_\delta = l$ it follows from the meaning of this equation * that no complementary interval can exist. (2), however, falls entirely to the ground; for not only is it not necessary for the complementary points to fill up a set of intervals, but it may even happen that there is no single complementary interval, *i.e.*, no interval whatever of (A, B) free from *interior* points of the intervals δ. Indeed, given any quantity ϵ, however small, it is possible to construct a set of non-overlapping intervals whose content I_δ is less than ϵ, yet such that *no complementary interval* exists at all.

Incredible as this at first sight appears, this is not the only paradoxical circumstance connected with such a set of intervals.

In spite of the absence of even one complementary interval, the relation $I_\delta < l$ will be found sufficient to necessitate the existence of a more than countable set of points *exterior* to every interval δ †; indeed, the potency ‡ of the *exterior* points is that of the linear continuum, provided only $I_\delta < l$, (whether *or no* there are complementary intervals).

It is, perhaps, even more surprising that, whereas this *must* be true if $I_\delta < l$, it *may* be true when $I_\delta = l$. In this case, as already stated, no complementary interval exists; so that (3, i.) holds as it stands; but (3, ii.) falls to the ground, since not only may exterior points exist, but they may even be more than countable in number, (of potency c). This shows that the familiar relations between the interior, end, and exterior points of a finite set of non-overlapping intervals cannot be assumed in dealing with an infinite set. We proceed to investigate these relations in the most general case.

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* See §§ 3 and 4. † § 22 and § 29. ‡ *Mächtigkeit.*

§ 3. *Cantor's Theorem.**

Every set of intervals on a straight line is countable, provided no two overlap.

For let $e_1, e_2, e_3, \ldots$ be any sequence of numbers having zero as limit, and let us consider only the case when the intervals all lie in a finite segment (A, B) of length l. There is in this way no loss of generality, since we can bring the whole infinite straight line into (1, 1)-correspondence with (A, B). The number of intervals of the given set whose magnitude lies between e_r and e_{r+1} must be finite, since the intervals do not overlap. Let these, arranged in any order, be denoted by G_r. Then G_r is finite and the whole set can be arranged in the order $G_1, G_2, \ldots$, and "counted" as they stand; which proves the theorem.

Cor.—From the above it is evident that the intervals can be arranged in order of magnitude $\delta_1, \delta_2, \ldots$, and that, if (A, B) be a finite segment, given any positive quantity ϵ, we can assign an integer m so that, for all values of $n \geq m$, $\delta_n < \epsilon$.

§ 4. Now the sum of any number of the intervals cannot be greater than l. There must therefore be an upper limit I_δ, less than or equal to l, such that the sum of any finite number of the intervals is always less than I_δ, but can be made as near as we please to I_δ by taking sufficient of the intervals when arranged in order of magnitude. That is to say, given any ϵ, we can find an integer m such that, for all values of $n \geq m$, $I_\delta - \epsilon < \sum_1^n \delta_r < I_\delta$.

In the usual manner we express this fact in other words by saying that the series

$$\sum_1^\infty \delta \equiv \delta_1 + \delta_2 + \ldots \text{ ad inf.}$$

is convergent, and has I_δ for its sum. *I_δ we call the content of the set of intervals.* This evidently agrees with the definition of the content in the case when the set consists of a finite number of intervals only.

It now follows from the corollary of § 3 that, *given any small positive quantity σ, we can assign a small quantity ϵ such that the sum of all the*

* Stated and proved in precisely this way by Cantor, "Ueber unendliche lineare Punktmannigfaltigkeiten," *Ann.*, Vol. xx., p. 117, 1882.

intervals of the set which are less than ϵ is less than σ. Denoting this sum by $R(\epsilon)$, we have $R(\epsilon) < \sigma$.

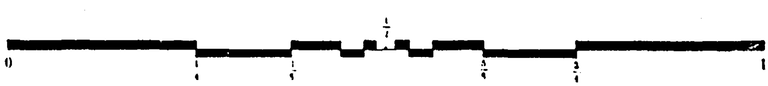
§ 5. Having so defined the content, we can at once prove the following theorem :—

The content I_δ of a set of intervals in a finite segment (A, B) is equal to the sum of the contents in any set of segments inside (A, B) such that each one of the given intervals is interior to some one of the segments.

For let any small quantity σ be assigned ; then we determine ϵ so that $R(\epsilon) < \sigma$. Let the segments be denoted by $D_1, D_2, \ldots$ in order of magnitude, and let us determine m so that, for all values of $n > m$, $D_n < \epsilon$. Then every interval of the given set which is not less than ϵ, lies in $D_1, D_2, \ldots, D_m$, and the sum of these intervals is less than the content I_δ by less than σ. *A fortiori* the sum of the contents of *all* the intervals in $D_1, D_2, \ldots, D_m$ is less than I_δ by less than σ ; which is what we wanted to prove.*

§ 6. We proceed to discuss first the case when $I_\delta = l$. As already stated, (3, i.) holds universally. An example will best prove that (3, ii.) falls to the ground, and it will be seen that the possibility of the presence of external points is due to the fact that it is in the neighbourhood of certain *points* that the intervals δ become smaller than any assignable quantity.

Ex. 1.—In the interval $(0, 1)$ consider the intervals $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{8})$,



$\left(\frac{2^2-1}{2^3}, \frac{2^3-1}{2^4}\right)$, ..., and all the intervals got by reflecting these in the point $\frac{1}{2}$. Then we shall have a (countably) infinite set of intervals in the segment $(0, 1)$, and the sum of them is

$$2\left(\frac{1}{2^2} + \frac{1}{2^3} + \ldots\right) = 1,$$

and yet the point $\frac{1}{2}$ is *exterior* to every interval.

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* The most general form of this theorem, which can be immediately deduced from the results of this paper, is the following :—

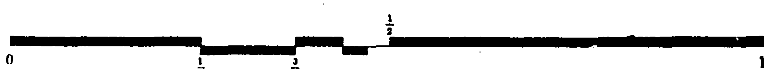
I_δ in (A, B) is equal to the sum of the contents in any set of segments contained in (A, B), provided the sum of those segments and parts of segments which lie inside each interval δ is equal to the content of that interval δ.

In particular, by Theorem 3, this will be the case if there is no external point of the segments which is not exterior to the intervals.

In fact, if we consider the given set as the "limiting set"* G of the finite sets $G_1, G_2, \ldots$, where G_1 consists of the two intervals $(0, \frac{1}{4})$, $(\frac{3}{4}, 1)$; $G_2, \ldots$ of the four intervals

$$\left(0, \frac{1}{2^2}\right), \left(\frac{1}{2^3}, \frac{2^2-1}{2^3}\right), \left(\frac{2^2+1}{2^3}, \frac{2+1}{2^2}\right), \left(\frac{2+1}{2^2}, 1\right);$$

and so on, the complementary set of G_n always consists of one interval, of length successively $\frac{1}{2}, \frac{1}{4}, \frac{1}{2^3}, \ldots$; so that in the limit the complementary interval evanesces, leaving us, however, with the point $\frac{1}{2}$, which is *interior* to the complementary interval of G_n for all values of n.



§ 7. *Ex.* 2.—Instead of reflecting all the intervals to the left of the point $\frac{1}{2}$ in that point, we might take as a new interval $(\frac{1}{2}, 1)$, the intervals to the left of $\frac{1}{2}$ being the same as before. In this case I_δ is still equal to 1; but the point $\frac{1}{2}$ is no longer external to every interval. It is, in fact, the left-hand end point of one interval. On the left of it, however, there is no interval of which it is an end point; so that, although (3, ii.) is not violated, (3, iii.) is so.

We shall find it convenient to use a new term to denote that a point is an end point of one interval only of the set.

DEFINITION.—*A point, other than A or B, is said to be semi-external to a set of non-overlapping intervals in a segment (A, B) when it is an end point of one interval only of the set.* A or B is, however, regarded as only semi-external when it is an end point of *no* interval; otherwise A or B is an ordinary end point. This agrees with the definition if we consider the whole straight line exterior to (A, B) as being, like the intervals, black.

§ 8. Since a semi-external point is an end point of one interval, it follows, by Cantor's theorem, that *the number of semi-external points of a set of non-overlapping intervals is at most countably infinite.*

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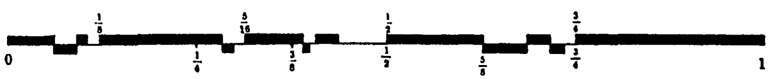
* That is to say, any interval of G occurs as an interval of G_n for all values of n greater than a certain determinable integer m.

It is easy to construct examples where the number of semi-external points is actually countably infinite. For instance, as follows:—

Ex. 3.—Take the set of intervals of Ex. 2, viz., $\left(\frac{1}{2}, 1\right)$, $\left(0, \frac{1}{2^2}\right)$, $\left(\frac{1}{2^2}, \frac{2^2-1}{2^3}\right)$, ..., having the point $\frac{1}{2}$ as semi-external point, and divide each of these intervals similarly to the division in Ex. 2 of the segment (0, 1). Then in each of the above intervals the middle



point will be a semi-external point, and there will be no external points. The semi-external points will be $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{8}$, $\frac{5}{16}$, $\frac{21}{32}$, ..., and will be countably infinite in number.

§ 9. Similarly we might generalize Ex. 1, and obtain a set of content l having a countably infinite set of *external* points. We content ourselves here with giving these simple examples illustrating the truth of the statement that when the content $I_\delta = l$ there may be an infinite number of external points, violating (3, ii.), and there may be a countably infinite number of semi-external points, violating (3, iii.). We defer the introduction of examples of a more complicated nature illustrating the other statements till after the enunciation and proof of some general theorems.

§ 10. THEOREM 1.—"*Of External and Semi-External Points.*"

If we have a (countably) infinite set of non-overlapping intervals $\delta_1, \delta_2, \ldots$, *lying in the segment* (A, B) *of length* l, *then, (even when* $I_\delta = l$), *there must be at least one point external or semi-external to the set.*

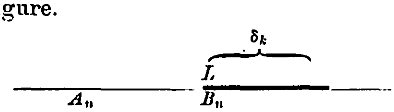
If there be any complementary interval, the theorem is obvious. If not, let the δ's be arranged as in § 3 in order of descending magnitude. Mark the interval δ_1 black. There remain over two segments or one; in any case there remains over at least one segment such that in it there lie intervals δ_k with index k higher than any assignable integer. Let (A_1, B_1) be this segment if determinate, or the left-hand segment if there are two possessing this property.

We treat (A_1, B_1) precisely as we did (A, B), marking black the interval of lowest index in it, and passing on to that one of the segments left in (A_1, B_1) which contains intervals δ_k with index k higher than any assignable integer, and taking, if both segments have this property, the left-hand one.

Proceeding thus, we get an infinite series of segments (A, B), (A_1, B_1), (A_2, B_2), ..., each contained within the preceding and having with it one, and only one, end point common. These segments decrease without limit in length, since there is no interval of (A, B) free of internal points of δ's. They define, therefore, a definite limiting point L, which is either *internal* to *all* the segments (A_r, B_r), or, from and after a fixed integer m, is an end point of every segment (A_r, B_r), $r \geqslant m$.

This point L cannot be *interior* to any interval δ_k; for, if it were, denoting by h the distance of L from the nearer end point of δ_k (or $h = \frac{\delta_k}{2}$, if L lies in the middle), h will be finite, and we can determine an integer m such that for all values of $n \geqslant m$ the length of (A_n, B_n) is less than h. The interval (A_n, B_n), of which L is an interior or end point, will then lie entirely within δ_k, which, since the δ's do not overlap, is contrary to the hypothesis that (A_n, B_n) contained δ's with indices *higher* than k.

L may, however, be an *end* point of a determinate δ_k, while the δ's with indices as high as we please crowd themselves on the other side of L from δ_k. L will then be an end point of every (A_n, B_n) for all values of n from and after a determinable one m. For instance, L might coincide with B_n, $n \geqslant m$, and be the left-hand end point of δ_k, as in the figure.



The point L is therefore either *exterior* to all the black intervals δ or an end point of only one of them, *i.e.*, *semi-exterior*; which proves the theorem.

§ 11. The above proof shows that the point L is such that, at least on one side of L, the intervals δ with indices higher than any assignable quantity crowd themselves together. The lengths of these intervals become, as we saw (§ 3, Cor. 1), indefinitely small; so that the point L is a limiting point of ends of black intervals δ, or, if we please, of the right-hand end points of δ's by themselves, or of the left-hand end points by themselves.

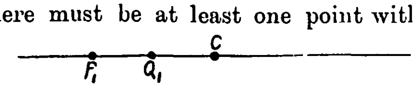
If L is a limiting point on *both* sides, it will certainly be *external* to all the black intervals. The two following theorems will show under what restrictions every point external to all the δ's is a limiting point on both sides of the end points. In the meanwhile we may enunciate the preceding theorem in the following more precise form :— *

An infinite (countable) set of non-overlapping intervals $\delta_1, \delta_2, \ldots$ *has always at least one limiting point of end points which is either external or semi-external to every black interval. If the limiting point be a limit on both sides, it is certainly external to every black interval; otherwise it is semi-external.*

§ 12. Theorem 2.—"*Of External Points.*"

Given a set of non-overlapping intervals, such that no complementary intervals exist, in a segment (A, B), *then, if there be any point* C *of* (A, B) *exterior to the set of intervals,* C *is a limiting point on both sides of intervals.*

For let e_1 be any small positive quantity, and construct on the left of C a segment of length e_1. Then, since no complementary interval exists, there must be at least one point within this segment



which is *not exterior* to every interval. Let P_1 be such a point. Then, since C is exterior to the interval δ containing P_1, there must be in the segment (P_1, C) the right-hand end point of the interval δ to which P_1 is not exterior. Let this point be Q_1, where Q_1 is either identical with P_1 or lies between P_1 and C; then $CQ_1 < e_1$.

* This theorem takes the place of the so-called Heine-Borel theorem, the proof being of the type of Borel's second proof, (*Leçons sur la Théorie des Fonctions*, p. 42), only direct instead of indirect. The enunciation of the Heine-Borel theorem is as follows :—

Given a countable set of intervals, (of course overlapping), such that each point of the closed segment (A, B) *is an internal point of at least one interval, then it is possible to choose out a finite number* N *of these intervals having the same property.*

This theorem may be deduced easily from our theorem. For, if we arrange the Heine-Borel intervals in countable order, then omitting any interval or part of an interval which was contained in any of the preceding intervals, and omitting any parts exterior to (A, B), we get a set of intervals such that *every* internal point of (A, B) is an interior point of one interval, or an end point of two intervals, while A and B are end points each of one interval. By our theorem the number of these intervals cannot be infinite. But each of these is the whole or a part of a definite interval of the given set, and these have the required property.

Now let e_2 be smaller than the length of (C, Q_1). We determine similarly a right-hand end point Q_2 such that $CQ_2 < e_2$. Proceeding thus, we determine a sequence $Q_1, Q_2, \ldots$ of right-hand end points of intervals δ, lying on the left of C and having C as limiting point.

Similarly we construct a sequence of left-hand end points of intervals δ, lying on the right of C and having C as limiting point. Thus C is a limiting point on both sides of end points of intervals.

Q. E. D.

Cor.—*If* $I_\delta = l$, *then, if there be any exterior point, it is a limiting point on both sides of intervals.* For in this case, (§ 4), no complementary interval can exist.

§ 13. In the case, then, when the complementary set of *intervals* does not exist, the whole continuum (A, B) is made up of the given set of non-overlapping intervals of content $I_\delta \leqslant L$ and the limiting points, (if any), on *both* sides, (exterior points). As to these exterior points, since they are not, like the semi-exterior points, each bound to some particular interval, we cannot assert that they are not more than countably infinite.

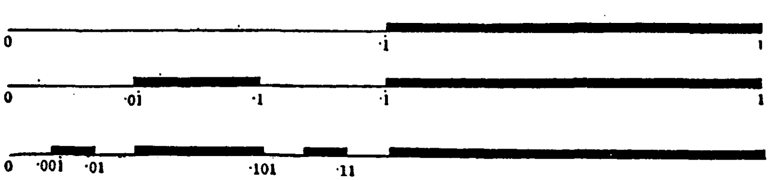
In fact, though there may be none at all, or only a finite or countably infinite number of them, they may be more than countably infinite, as we will now show by examples.

The first of the following examples where, although $I_\delta = l$, (so that certainly no complementary interval exists), the exterior points have the potency of the linear continuum is of classic interest.

§ 14. Consider the segment (0, 1), and let us take as right-hand end points of our intervals the point 1, and all those points whose numbers expressed as fractions in the ternary scale involve a finite number of 0's and 1's, and do not involve the figure 2.

The number corresponding to the left-hand end point of any interval is got from the number corresponding to the right-hand end point by changing the final one into $0\dot{1}$.

Thus, in order of magnitude, the first few intervals are $(\cdot\dot{1}, 1)$, $(\cdot 0\dot{1}, \cdot 1)$, $(\cdot 00\dot{1}, \cdot 01)$, and $(\cdot 10\dot{1}, \cdot 11)$; or, which is the same thing,



using the everyday notation, $(\frac{1}{2}, 1)$, $(\frac{1}{6}, \frac{1}{3})$, $(\frac{1}{18}, \frac{1}{9})$, and $(\frac{1}{3}+\frac{1}{18}, \frac{1}{3}+\frac{1}{9})$. It is evident from the numbers that these intervals do not overlap nor abut anywhere, and that there are no complementary intervals, so that *every end point is a semi-external point*. These semi-external points correspond, therefore, by construction, to all the terminating ternary fractions, involving only 0's and 1's, (right-hand end points), and all the simple recurrers, involving only 0's and 1's and ending in $\dot{1}$, (left-hand end points).

Also it is easy to show that *all the interior points of black intervals correspond to ternary fractions involving at least one* 2, *and vice versa.* For, if N denote any combination of n figures, 0's and 1's only, and M any combination of 0's, 1's, and 2's, the ternary fraction $\cdot N02M$, as well as, (for all integral values of p), $N01^p2M$, lies, whatever M may be, between $\cdot N0\dot{1}$ and $\cdot N1$). These numbers correspond therefore to internal points of a determinate black interval, $(\cdot N0\dot{1}, N1)$. *Vice versa*, any number lying in this interval is expressible in one of the two given forms.

The remaining (exterior) points consist therefore of all the non-terminating ternary fractions involving only 0's *and* 1's, *other than the simple recurrers ending in* $\dot{1}$. Every such point is, as follows, easily shown to be a limiting point on both sides of the semi-external points.

Given any such non-terminating fraction, we can form a sequence of semi-external points, (right-hand end points), having that fraction as limiting point, by stopping in succession at each 1 inclusive. A similar sequence on the other side of the point in question is determined by stopping short of each 0 and appending a 1.

It is easy to see that *these external points are more than countably infinite* in number; indeed, the potency is actually c, (that of the linear continuum). For we only have to interpret these ternary fractions in the binary scale and we have set up a (1, 1)-correspondence between these points and the whole continuum from 0 to 1, with the exception of a countable number of binary points, which do not, of course, affect the potency.

We can, moreover, show that *the sum of the black intervals is actually* 1. For, since N consists of n figures, 0's and 1's, the number of semi-external points $\cdot N1$, (for the same n), is 2^n. Each of these is the right-hand end point of an interval of length $\frac{1}{2}\frac{1}{3^{n+1}}$.

Hence the sum of all the intervals is

$$\tfrac{1}{2}\left(1+\frac{1}{3}+\frac{2}{3^2}+\frac{2^3}{3^3}+\ldots\right)=\tfrac{1}{2}\left(1+\frac{1}{3}\frac{1}{1-\frac{2}{3}}\right)=1.$$

It has therefore been proved that *in the segment* (0, 1) *the points corresponding to all the ternary fractions involving the figure* 2 *fill up a set of intervals of content* 1,* *which do not overlap nor abut anywhere, so that the semi-external points are countably infinite in number. The external points are more than countably infinite, and yet are not dense in any interval, however small.*

§ 15. The mode in which we have treated this example was suggested by the context as throwing light on the preceding theorems and statements about external and semi-external points. Historically this example—the first of its kind—must be referred to H. J. S. Smith,† who, however, treated the subject purely geometrically and from a somewhat different point of view. H. J. S. Smith's set of intervals are not in *all* essentials identical with those given above,‡ since each of our black intervals is replaced in his treatment by an infinite set of abutting intervals exactly filling it up. Thus, a number of points which in our set were internal points become end points, and a number of semi-exterior points, (left-hand end points), become external points.

The law of formation of these abutting intervals in the interval ($\cdot N01, N1$) is that the ends of the abutting intervals consist of the sequence $\cdot N02$, $\cdot N012$, $N0112$, ..., $N01^p2$, ..., (for all integral values of p), the limiting point of which is $N0\dot{1}$. The intervals are plotted down by H. J. S. Smith successively in order of magnitude.

On account of its historical interest, and the light the method throws on the general lie of the set in its relation to the continuum, and also because the method lends itself easily to a form of generalization which will be exceedingly valuable to us in the sequel, we shall now give a more detailed discussion of H. J. S. Smith's set of intervals from his point of view.

§ 16. *H. J. S. Smith's Ternary Set of Intervals of the First Kind.*

Divide first the segment (0, 1) into three equal parts, and blacken the right-hand part. At the second division divide each of the two unblackened parts into three equal parts and blacken the two right-hand parts. The two black intervals which abut, if amalgamated, would

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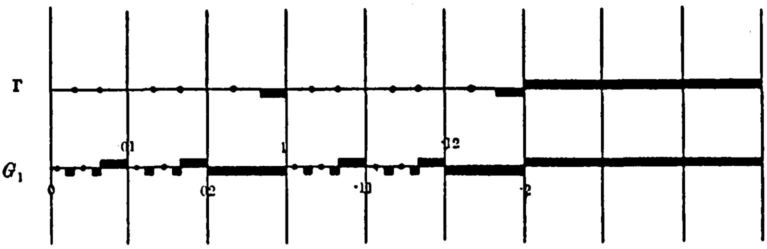
* So that certainly no complementary interval exists.

† *Proc. Lond. Math. Soc.*, Vol. VI., 1870.

‡ *Cf.* W. H. Young, "On the Density of a Linear Set of Points," *Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 286, footnote.



form the germ of the first of the black intervals of the preceding section. At the third division divide each of the 2^2 unblackened segments, of length $\frac{1}{3^2}$, into three equal parts, and blacken the 2^2 right-hand parts. Generally at the $(n+1)$-th division divide each of the 2^n unblackened segments, of length $\frac{1}{3^n}$, into three equal parts and blacken the 2^n right-hand parts. (See G_1 in Fig.)

That this set of intervals is dense everywhere is almost immediately evident from the mode of construction, as are also the facts that there are a countably infinite set of simple end points, (left-hand),* and another countably infinite set of semi-external points, (right hand). It is not, however, easy to see geometrically that the external points are more than countable.†

These facts did not interest H. J. S. Smith at the moment; the question with him was one of content pure and simple.

The sum of the intervals blackened in the first n processes of division is

$$1+\frac{2}{3}+\ldots+\frac{2^{n-1}}{3^n}=\frac{1}{3}\frac{1-(\frac{2}{3})^n}{1-\frac{2}{3}}=1-\left(\frac{2}{3}\right)^n.$$

If, therefore, we continue the division *ad infinitum*, we shall get a set of black intervals of content 1.

§ 17. This method of H. J. S. Smith is capable of a natural modification which does not suggest itself when we work numerically. We are furnished in this way with the following example, among others, given also by H. J. S. Smith, without use of number, *loc. cit.* This example serves to show the fallacy of the naïve assumption that when I_s is less than l at least one complementary interval must exist.

* Whose numbers all terminate with a 2, and involve otherwise (a finite number of) 0's and 1's only.

† For instance, the reasoning in Borel, p. 44, is only valid when $I_s < l$.

§ 18. *H. J. S. Smith's Ternary Set of Intervals of the Second Kind.*

As in the former example, divide the segment (0, 1) into three equal parts, and blacken the right-hand part (·2, 1), *omitting it from* further division. The origin together with the point of division ·1, which lies outside the omitted segment, we shall denote by G_1.

At the second division we shall divide each of the two segments not blackened, (0, ·1) and (·1, ·2), into 3^2 equal parts, and blacken the right-hand segment. (See Γ in Fig. of p. 257.)

Denoting by G_2 the set of points consisting of G_1 and all the fourteen points of division which lie outside the blackened parts, we see that the numbers of G_2 are completetely characterized as being all the numbers of the form $\cdot e_1 e_2 e_3$, where

$$A_2 \begin{cases} (1) \ e_1 \text{ is not } 2, \\ (2) \ e_2 \text{ and } e_3 \text{ are not } \textit{both } 2. \end{cases}$$

At the third division we divide each of the $(3-1)(3^2-1)$ segments not already blackened into 3^3 parts and blacken the right-hand segment in each.

Denoting by G_3 the set of points consisting of G_2 and all the $(3-1)(3^2-1)(3^3-2)$ points of division which lie outside the blackened parts, we see that the numbers G_3 are completely characterized as being all the numbers $\cdot e_1 e_2 e_3 e_4 e_5 e_6$, where

$$A_3 \begin{cases} (1) \ e_1 \text{ is not } 2, \\ (2) \ e_2 \text{ and } e_3 \text{ are not both } 2, \\ (3) \ e_4, \ e_5, \text{ and } e_6 \text{ are not } \textit{all } 2; \end{cases}$$

that is, where, in addition to the conditions A_2, the condition $A_3\,(3)$ is satisfied; and these conditions together are denoted by A_3.

It is now obvious how in turn G_4, G_5, ... will be constructed and will become in turn the right-hand end points of black intervals. At the n-th division we divide each of the $(3-1)(3^2-1) \dots (3^{n-1}-1)$ segments not already blackened into 3^n equal parts, and omit in each the right-hand segment. G_n will then consist of G_{n-1} and all the $(3-1)(3^2-1)(3^3-1) \dots (3^{n-1}-1)(3^n-2)$ points of division outside the blackened parts, and the numbers of G_n are therefore completely characterized as being all the numbers $\cdot e_1 e_2 \dots e_{\frac{1}{2}[n(n+1)]}$, where, in addition to the conditions A_{n-1}, we have the condition $A_n(n)$, viz., $e_{\frac{1}{2}[(n-1)n]+1}, e_{\frac{1}{2}[n(n-1)]+2}, \dots, e_{\frac{1}{2}[n(n+1)]}$ are not all to be 2. These conditions we denote by A_n. The limiting set * of G_1, G_2, we denote by G.

* That is, that set G such that any assigned point of G has been a point of every G_n, from and after an assignable index m.

§ 19. Each point of G is a right-hand end-point of a set of abutting black intervals. If we consider all abutting intervals as amalgamated, and continue these intervals up to their limiting points on the left, we obtain a set of intervals analogous to those of § 14. Those limiting points, which were external and have now become semi-external, can be assigned as follows :—

Suppose $\cdot P$ to be a point of G which belongs to G_n,* but not to G_{n-1}, and let $\cdot Q$ be the point of G_n immediately to the left of $\cdot P$.† The left-hand end point of the interval of which $\cdot P$ is the right-hand end point is obtained by appending to $\cdot Q$, if necessary, so many 0's as to make the number of figures $\frac{n(n+1)}{2}$, and then appending λ_n, where

$$\lambda_1 = 21221222122221\ldots$$

and

$$\lambda_n = 2^n 1 2^{n+1} 1 2^{n+2} 1 \ldots$$

the number of 2's between consecutive 1's increasing each time by one.

Thus, for example, the black interval whose right-hand end point is $\cdot 0\ 12\ 001\ 02$ has for left-hand end point $\cdot 0\ 12\ 001\ 0122\ 2^4\ 12^5\ 12^6\ 1\ \ldots$. These numbers show us that the left-hand end points are, as was asserted, semi-external points, since, stopping at any figure in one of these numbers, we get a number obeying the conditions A_n for a certain definite integer n, and therefore a sequence of numbers of G, having the given left-hand end point as limit on the right. The right-hand end points are also semi-external, since we can add on at the end of any one of these numbers any number of 0's followed by a 1, and so get a sequence of points of G, having the desired right-hand end point as limit on the left.

Both the left- and the right-hand end points being semi-external, it is evident that *no complementary interval can exist.* This could be deduced from the geometrical construction.

The content of the set of intervals is, however, no longer 1. It is,

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* Given $\cdot P$, it is quite easy practically to determine n: for instance,

$$\cdot P = \cdot 1\ 02\ 101\ 2121\ 00011\ 12$$

belongs to G_6, but not to G_5.

† $\cdot Q$ is known at once as soon as $\cdot P$ is given, since

$$\cdot P = \cdot Q - \frac{1}{3^{\frac{1}{2}[n(n+1)]}}.$$

in fact, evident from the geometrical construction that the content is

$$\frac{1}{3}+\frac{(3-1)}{3^{2+1}}+\frac{(3-1)(3^2-1)}{3^{3+2+1}}+\ldots+\frac{(3-1)(3^2-1)\ldots(3^{n-1}-1)}{3^{1+2+\ldots+n}}+\ldots,$$

which is

$$\frac{1}{3}+\frac{1}{3^2}\left(1-\frac{1}{3}\right)+\frac{1}{3^3}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3^2}\right)+\ldots,$$

whence $$\tfrac{1}{3}<I_s<\tfrac{1}{2}.$$

Any point which violates the condition $A_n(n)$ for any value of n is evidently an interior point of a black interval which we can at once assign. The right-hand end point of this black interval is obtained by stopping at the first $\frac{n(n-1)}{2}$ figures of the number in question: *e.g.*, the points whose numbers begin with the figures 1 02 001 2222 ... are interior to the black interval (1 02 000 2^31 2^41 ..., .1·02 001).

It was asserted that when $I_s<l$, (that is, here <1), there must be a more than countable set of exterior points. The only ternary fractions at present unaccounted for are all the infinite ternary fractions which do not violate the conditions A_n for any value of n, (other than those ending in λ_n). These are the exterior points of the set. That they have the potency of the continuum is evident when we consider that all the non-terminating ternary fractions involving only 0's and 1's are among them (§ 14).

Numerically we have proved the following theorem:—

Let $\cdot e_1e_2e_3\ldots e_{\frac{1}{2}[n(n-1)]+1}\,e_{\frac{1}{2}[n(n-1)]+2}\ldots e_{\frac{1}{2}[n(n+1)]}$ *denote any ternary fraction with* $\frac{n(n+1)}{2}$ *figures, and let A_n denote the conditions that for no positive integral value of* $r\leqslant n$ *all the figures* $e_{\frac{1}{2}[r(r-1)]+1},\ldots,e_{\frac{1}{2}[r(r+1)]}$ *should be 2's. Then all the ternary fractions which violate the conditions A_n for some value of n fill up a set of intervals of content I_s, where*

$$\tfrac{1}{3}<I_s<\tfrac{1}{2},$$

*which do not overlap nor abut anywhere, so that the semi-external points are countably infinite in number. The external points are more than countably infinite, (of potency c), and yet are not dense in any interval, however small.**

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* The numbers corresponding to the simple end points of H. J. S. Smith's ternary set of the second kind are evidently those terminating fractions which violate some condition A_n, but only in the final stage $A_n(n)$.

§ 20. *H. J. S. Smith's Sets with Base m.*

Instead of the base 3, we might take any base $m \geqslant 3$ and perform the construction of § 16, or that of § 18, *mutatis mutandis*. The former series of sets, in which at each division we divide the unblackened parts of the straight line into m equal parts, gives us nothing particular of interest beyond the set for $m = 3$. But the latter series of sets, in which at the n-th division we divide the unblackened parts into m^n parts, presents this new and highly interesting feature, that

$$\frac{1}{m} < I_s < \frac{1}{m-1},$$

the other properties of the set being the same as in § 18.

Hence we see how it is possible to construct a set of intervals whose content is less than any assigned magnitude, and having a more than countable set of exterior points, (of potency c), which, however, are not dense in any interval, however small.

§ 21. Similarly by using any base $m \geqslant 3$, we can, as in §§ 14 and 18, construct sets of *non-abutting* intervals, with contents ranging from L down to *less than any assignable quantity*, and yet dense everywhere.

The numerical statements of these facts are interesting.

(1) *In the segment* (0, 1), *the points corresponding to all the m-ary fractions involving the figure* $(m-1)$ *fill up a set of intervals of content* 1, *which do not overlap or abut anywhere; so that the semi-external points are countably infinite in number. The external points are more than countably infinite, (of potency c), and yet are not dense in any interval, however small.*

These semi-external points correspond to all the terminating fractions not involving the figure $(m-1)$ and all the non-terminating fractions ending in "$(m-2)$ circulating." The external points correspond to all the other non-terminating fractions not involving the figure $(m-1)$.

(2) *Let* $\cdot e_1 e_2 e_3 \ldots e_{\frac{1}{2}[n(n-1)]+1} \ldots e_{\frac{1}{2}n(n+1)}$ *denote any m-ary fraction, and let* A_n *denote the conditions that for no positive integral value of* $r \leqslant n$ *all the figures* $e_{\frac{1}{2}[r(r-1)]+1}, \ldots, e_{\frac{1}{2}r(r+1)}$ *should be* $(m-1)$'s. *Then all the m-ary fractions which violate the conditions* A_n *for some value of n fill up a set of intervals of content* I_s, *where*

$$\frac{1}{m} < I_s < \frac{1}{m-1},$$

which do not overlap nor abut anywhere; so that the semi-external points are countably infinite in number. The external points are more than countably infinite, (of potency c), and yet are not dense in any interval, however small.

§ 22. The properties of the sets which we have denoted as H. J. S. Smith's sets of the second kind are of the most general character possible, as is indicated by the following theorem :—

THEOREM 3.—*If we have a set of non-overlapping intervals $\delta_1, \delta_2, \ldots$, whose content I_δ is less than l, lying in a segment (A, B) of length l, then the points of (A, B) which are* external *to all the intervals δ form an infinite non-countable set.** (It will appear from the sequel that this set has the potency c of the linear continuum.)

If any complementary interval exists, the theorem is obvious. If not, every external or semi-external point C will be a limiting point of intervals δ, and there will be at least one such point. The number of semi-external points is at most countably infinite.

This being the case, let us, if possible, arrange all the external and semi-external points in countable order $C_1, C_2, \ldots$. Round C_1 describe an interval d_1 of length $\frac{e}{4}$ with C_1 as middle point, where

$$e = l - I_\delta.$$

Let C_i be the next point of the series $C_1, C_2, \ldots$ which is exterior to d_1, and describe similarly an interval of length $\frac{e}{2^{i+1}}$ with C_i as middle point. If this overlap or abut with d_1, we amalgamate them into one interval; if not, we call it d_2.

Proceeding in order with the points C which are not already interior or end points of intervals d, we get a finite or countably infinite set of non-overlapping and non-abutting intervals d, whose content is not greater than

$$e \sum_2^\infty \frac{1}{2^r} = \frac{e}{2},$$

containing all the points C. Adding to these such intervals δ, or parts of intervals δ, as are external to the intervals d, and amalgamating them with the intervals d where they overlap with them, we get a set of non-overlapping intervals such that no point of (A, B) is exterior or semi-exterior to them. By Theorem 1, "Of External and Semi-

* Borel, *Leçons*, p. 41.

external Points," the number of these intervals can only be finite. *A fortiori* the number of intervals d is finite. Let them be denoted by $d_1, d_2, \ldots, d_n$.

Marking these black, we are left with a finite number ($\leqslant n-1$) of segments (A, B) whose content also lies between $l-\frac{e}{2}$ and l. In each such unblackened segment there is no external or semi-external point of the intervals δ. Hence, by Theorem 1, in each, and therefore in all, of these segments there is a *finite* number of intervals δ entirely filling them up, and whose content therefore also lies between $l-\frac{e}{2}$ and l. But this is inconsistent with the fact that $I = l-e$. Hence the assumption was false, and, by a *reductio ad absurdum*, we are obliged to deny the possibility of counting the points C.

Q.E.D.

§ 23. *The Typical Ternary Set of Intervals.*

We have seen that, given any set of intervals, we have a perfectly determinate countable set of semi-external points, and that any end point of an interval where it does not abut on another interval is either (1) a semi-external point or (2) not a limiting point at all. In the latter case the set of intervals is *not* dense everywhere.

The question of content is obviously unaffected if we amalgamate all abutting intervals, in which case the end points of the new intervals will consist entirely of the two classes of points above referred to, and, if the original set of intervals be dense everywhere, will consist exclusively of semi-external points.

This demonstrates the importance of considering in detail the properties of sets of intervals whose end points are *all* semi-external points, that is, *sets of non-abutting intervals which are dense everywhere.*

The sets considered in §§ 14 and 18 belong to this class; but we shall find it desirable to modify them and take another set* as typical of the class, its numerical equivalent being the most convenient, as giving us a system of indices for the δ's in the general case, by means of which any two sets of the class are arranged at once in (1, 1)-correspondence.

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* *Cf.* G. Cantor, *Math. Ann.*, Vol. XXI., p. 590, and Schoenflies's *Bericht*, p. 102.

§ 24. Take the segment (0, 1) of the x axis and divide it into three equal parts and blacken the middle one: this is (·1, ·2), or, as we prefer to write it, $(\cdot 0\dot{2}, \cdot 2)$. This we denote by δ_1. In each of the two unblackened segments repeat the process: we get two new black intervals,

$$\delta_{01} = (\cdot 00\dot{2},\ \cdot 02),$$

$$\delta_{11} = (\cdot 20\dot{2},\ \cdot 22).$$

In each of the unblackened segments repeat the process, and so on.

Denoting by N any combination of n figures, 0's and 1's, and by $(2N)$ the number got by multiplying N by 2, we see that any black interval obtained by our process may be denoted by δ_{N1}, and characterized by the symbolic equation

$$\delta_{N1} = [\cdot(2N)\,0\dot{2},\ \cdot(2N)\,2].$$

We notice that the order of the intervals is precisely that of the binary fractions $\cdot N1$. The internal points of the black intervals are characterized by the appearance of a proper 1, not $\dot{2}$, in their ternary factions.

Given any ternary fraction with a proper 1 in it, we can at once assign the black interval to which it belongs. For example, the number ·020022021 ..., (where the dots denote *any* subsequent figures), lies between $\cdot 020022020\dot{2}$ and ·020022022, and therefore is interior to the black interval $\delta_{010011011}$.

The only numbers unaccounted for are the non-terminating ternary fractions other than the simple recurrers ending in $\dot{2}$ involving only the figures 0 and 2. These must then represent the exterior points of our set of black intervals. Dividing each such number by 2, and interpreting in the scale of 2, we evidently get the whole continuum, with the exception of a countable set; so that the potency of this set of numbers is c. Hence *the potency of the exterior points is c.*

§ 25. If we are given *any* set of non-overlapping and non-abutting intervals, dense everywhere in an interval (A', B'), then A' and B' themselves may be ordinary end points or may be semi-external points. In the former case there are black intervals (A, A'), (B', B) belonging to the set, and A' and B' are not semi-external points. Such intervals, if they exist, do not materially affect the character of the set; they may therefore be conveniently omitted from consideration in the present section. We are working, then, in a segment (A, B) where A and B are semi-external points of our set of intervals. We now propose to set up a (1, 1) correspondence, *maintaining the order* between the given set and the typical set of intervals, § 24.

§ 26. Divide (A, B) into three equal parts at C and D. Then, since the set is dense everywhere, either (C, D) forms part of a determinate black interval or else there is a black interval inside (C, D), possibly coinciding with it. Let us then choose some particular black interval, for instance, the largest possible in (C, D), and denote it by δ_1, and, in the first case, let δ_1 be the black interval of which (C, D) forms a part.

Since A and B, being semi-external points, are not the end points of any black interval, there will be two unblackened segments left after we have blackened δ_1. The end points of each of these two segments are semi-external points; hence we may repeat the process in each of these two segments and choose out two new intervals of our set, one in each of these segments, and these we may call δ_{01} and δ_{11}. The order of the three intervals δ_{01}, δ_1, δ_{11} is evidently the same as in the typical case, (§ 24), and the same as that of the binary fractions ·01, ·1, ·11.

We are now left with four segments, whose end points are all semi-external to our set of black intervals; so that we can repeat our process in each of them, and choose in each a black interval of our set. These we denote by δ_{001}, δ_{011}, δ_{101}, δ_{111} in order from left to right; so that the order of the binary fractions is maintained.

Proceeding thus, we can evidently use the terminating binary fractions, (omitting the point), as a general system of indices, not merely proving the countableness of our set of intervals, but also indicating exactly their order in relation to the continuum. This sets up, *ipso facto*, a (1, 1)-correspondence, maintaining the order, between the general set of this class and the typical set of § 24, enabling us to solve many problems for the general set by means of the known properties of the typical set.

§ 27. One consequence of the mode adopted for determining the indices is that, *given any positive quantity* ϵ, *we can determine an integer* m *such that, for all values of* $n \geqslant m$, $\delta_{N1} < \epsilon$ (n being, as always, the number of figures in N). For, by the construction, the two segments left after the blackening of δ_1 are each less than $\frac{2}{3}$ of (A, B), and at each stage a similar statement can be made as to the length of each unblackened segment. Thus we have only to determine m so that $(\frac{2}{3})^m (A, B) < \epsilon$, and this m will certainly satisfy our requirements.

§ 28. Since the order has been maintained, it follows from the above that any sequence of intervals of the given set, defining a single limiting point, will correspond to a sequence of intervals of the typical set, defining a single limiting point, and *vice versa*. We can most easily express this correspondence between the limiting points by denoting the left- and right-hand end points of any black interval of δ_{N1} by $P_{N0\dot{2}}$, and P_{N2}, and any exterior point by P, with, as index, the ternary number denoting the limiting point of the corresponding intervals of the typical set.

We see that, the semi-external points being in (1, 1)-correspondence by themselves, the exterior points will be so also. *Hence it follows that the potency of the external points is c.*

§ 29. We have proved by our correspondence that *the external points of a set of non-abutting intervals have the potency of the linear continuum*, it being unnecessary to postulate that the set should be dense everywhere, since, when at least one complementary interval exists, the theorem is obvious. The following can now be deduced :—

THEOREM 3′.—*Of the Potency of the External Points.*

If we have a set of non-overlapping intervals $\delta_1, \delta_2, \ldots,$ *whose content* $I_\delta < l$, *lying in a segment of length* l, *then the points which are* external *to all the intervals* δ *form an infinite non-countable set of potency c.*

If the set is not dense everywhere, the theorem is obvious. We assume, therefore, that the set is dense everywhere.

We shall, for reasons which will immediately be evident, denote as a *point of arrest* a point such that in any segment, however small, containing it as internal point, there is a more than countable set of external points.

Let (A', B') be the segment in which the set exists. If A' and B' are not points of arrest, we will show how to replace (A', B') by a segment (A, B), lying within it, so that the potency of the external points is unaltered, while A and B are points of arrest.

Let us bisect (A', B') at M. Then, since, by Theorem 3, there is a more than countable set of external points in (A', B'), this must be the case in one or both of (A', M), (M, B'). We determine whether or no (A', M) has this property. Next we bisect again, and determine that segment which (i.) contains a more than countable set of external points, and (ii.) lies nearest to A'. Continuing this process, since there is nothing to prevent our doing so, *ad infinitum*, we have a series of segments, each contained within the preceding, and of half its length. This determines a limiting point, which will evidently coincide with A', if A' be a point of arrest, and will otherwise be the first point of arrest on the right of A', so that between it and A' there is at most a countable set of external points. This point of arrest we denote by A, and blacken the whole interval (A', A). In amalgamating all the intervals between A' and A in this way we have, *at most*, affected a countable set of external points.

We notice, then, that the content of the amalgamated intervals is, by Theorem 3, equal to that of (A', A). Similarly, working from right to left, we obtain B from B'.

Now, as in § 26, divide (A, B) into three equal parts and determine a black interval of the given set, which either coincides with the middle segment, or contains it, or is contained in it. This interval we subject to the already described process of amalgamation, blackening up to and including the first point of arrest on the right and left respectively.

During our process of amalgamation we have, as before, affected at most a countable set of exterior points, and the content of the amalgamated intervals is the same as that of the new interval we have constructed and blackened. This new interval does not abut with (A', A) nor (B', B) nor with the remaining intervals of the given set, and, if we denote it by (A_1, B_1), each of the segments (A, A_1) and (B_1, B) has its end points points of arrest, and the length of either of them is less than $\frac{2}{3}(A, B)$.

We now proceed separately with (A, A_1) and (B_1, B), as we did with (A, B). Subsequently we repeat the same process separately with the 2^2 segments left after the amalgamation and blackening have been carried out in (A, A_1) and (B_1, B); and so on.

After the n-th stage we shall have blackened

$$1+2+2^2+\ldots+2^{n-1} = 2^n-1$$

distinct intervals, obtained by amalgamation, without affecting more than a countable set of external points. The 2^n segments left over are such that the length of each is less than $(\frac{2}{3})^n (A, B)$, while the end points of each are points of arrest.

Thus, after a countably infinite series of steps, we determine a definite new set of black intervals. By our mode of construction these black intervals are dense everywhere in (A, B) and abut nowhere; while, with the possible exception of a finite or countably infinite set of points, our new set of black intervals has the same external points as the given set. By § 28 the potency of the external points is therefore c, as was asserted.

§ 30. We notice, further, that in each of the $2^n - 1$ new black intervals at the end of the n-th stage the content of the amalgamated intervals is equal to that of the new black interval in which they lie; while, since in the remaining segments there is no interval of the given set greater than $(\frac{2}{3})^n (A, B)$, (which may be made as small as we please by choosing n sufficiently large), the sum of the intervals of the given set in the remaining segments may be made as small as we please by choosing n sufficiently large; for the same reason the sum of all the intervals of the new set which lie in one of these segments can be made as small as we please. It follows, therefore, that *the new set of black intervals has the same content as the given set.** Thus we have incidentally proved the following important theorem:—

THEOREM 4.—*Of the Ultimate Set.*

Any set of non-overlapping intervals determines uniquely, (after a finite or countably infinite series of steps of the type described), an ultimate set of non-abutting *intervals having the same content and, (with the possible exception of a finite or countably infinite set of points), the same external points.*

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* *Cf.* § 5.