On a Certain Envelope. *By* Prof. Wolstenholme, M.A., Sc.D.

[*Read Nov. 8th*, 1883.]

If with a point O of the ellipse $a^2y^2+b^2x^2=a^2b^2$ as centre, we describe a circle such that triangles can be inscribed in the ellipse whose sides touch the circle, the envelope of these circles consists of two distinct curves, one being an ellipse, co-asymptotic with the given ellipse, but increased in the linear ratio $a^2+b^2 : a^2-b^2$ $(a>b)$; and the other a curve of the degree 6 and class 6, which osculates the former in four points.

Taking $(a\cos\theta, b\sin\theta)$ for the point O, and forming the discriminant of

$$k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+(x-a\cos\theta)^2+(y-b\sin\theta)^2-r^2,$$

we find that $\Delta=\dfrac{1}{a^2b^2},\quad \Theta=\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}+\dfrac{r^2}{a^2b^2},$

$$\Theta'=r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right),\quad \Delta'=r^2;$$

and, since $\Theta'^2=4\Theta\Delta'$, we get

$$r^2=\frac{4a^2b^2}{(a^2-b^2)^2}(a^2\sin^2\theta+b^2\cos^2\theta),\ \text{or}\ 0.$$

The former value alone is here considered. The equation of the circle is then $(x-a\cos\theta)^2+(y-b\sin\theta)^2=\dfrac{4a^2b^2}{(a^2-b^2)^2}(a^2\sin^2\theta+b^2\cos^2\theta)$; whence for the envelope

$$(x-a\cos\theta)\,a\sin\theta-(y-b\sin\theta)\,b\cos\theta=\frac{4a^2b^2}{a^2-b^2}\sin\theta\cos\theta,$$

or $$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2+\frac{4a^2b^2}{a^2-b^2}=\frac{(a^2+b^2)^2}{a^2-b^2},$$

which is the equation of a normal, at the point whose excentric angle is θ, to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\left(\dfrac{a^2+b^2}{a^2-b^2}\right)^4$.

The equation of this normal, on which the two points of the envelope lie, may be written $\dfrac{a(x-k\,a\cos\theta)}{\cos\theta}=\dfrac{b(y+k\,b\sin\theta)}{\sin\theta}$,

where $k=\dfrac{a^2+b^2}{a^2-b^2}$, and, if we suppose each member of this equation

$= \lambda$, we have

$$x = \left(\frac{\lambda}{a^2}+k\right) a\cos\theta,\quad y = \left(\frac{\lambda}{b^2}-k\right) b\sin\theta\,;$$

and, substituting these values in the equation of the circle,

$$a^2\cos^2\theta\left(\frac{\lambda}{a^2}+k-1\right)^2 + b^2\sin^2\theta\left(\frac{\lambda}{b^2}-k-1\right)^2 = \frac{4a^2b^2}{(a^2-b^2)^2}(a^2\sin^2\theta+b^2\cos^2\theta),$$

a quadratic in λ, whose two roots determine the two points of the envelope. But $k-1=\frac{2b^2}{a^2-b^2}$, and $k+1=\frac{2a^2}{a^2-b^2}$, whence it is obvious that one value of λ is 0, and therefore the other is

$$(2-2k\cos 2\theta) \Big/ \left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right),\quad\text{or}\quad \frac{4a^2b^2}{a^2-b^2}\,\frac{a^2\sin^2\theta-b^2\cos^2\theta}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

Hence one point of the envelope (which denote by P), corresponding to $\lambda=0$, is $x = ka\cos\theta$, $y=-kb\sin\theta$, whose locus is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = k^2 \equiv \left(\frac{a^2+b^2}{a^2-b^2}\right)^2;$$

and the second point (Q) is

$$x = \frac{a\cos\theta}{a^2-b^2}\left(a^2+b^2+4b^2\,\frac{a^2\sin^2\theta-b^2\cos^2\theta}{a^2\sin^2\theta+b^2\cos^2\theta}\right),$$

$$y = \frac{b\sin\theta}{b^2-a^2}\left(b^2+a^2+4a^2\,\frac{b^2\cos^2\theta-a^2\sin^2\theta}{b^2\cos^2\theta+a^2\sin^2\theta}\right),$$

whose locus is therefore a sextic.

If ϕ be the angle which the normal at O makes with the axis of x, and p the perpendicular from the centre upon the tangent,

$$\frac{a\sin\theta}{\sin\phi} = \frac{b\cos\theta}{\cos\phi} = \frac{ab}{p}\,;$$

whence

$$(a^2-b^2)\,x = \frac{a^2}{p}\cos\phi\,(a^2+b^2-4b^2\cos 2\phi),$$

$$(b^2-a^2)\,y = \frac{b^2}{p}\sin\phi\,(b^2+a^2+4a^2\cos 2\phi).$$

From these we deduce

$$(a^2-b^2)^2(x^2+y^2) = 16a^4b^4\frac{\cos^2 2\phi}{p^2} - \frac{8a^2b^2}{p^2}(a^2+b^2)(a^2\cos^2\phi-b^2\sin^2\phi)\cos 2\phi + (a^2+b^2)^2\left(a^2+b^2-\frac{a^2b^2}{p^2}\right);$$

and $2p^2 = a^2+b^2+(a^2-b^2)\cos 2\phi$, whence

$$(a^2-b^2)^2(x^2+y^2)$$

$$= (a^2+b^2)^3+\frac{a^2b^2}{p^2}\{16a^2b^2\cos^2 2\phi-8(a^2+b^2)(a^2\cos^2\phi-b^2\sin^2\phi)\cos 2\phi - (a^2+b^2)^2\}$$

$$= (a^2+b^2)^3+\frac{a^2b^2}{p^2}\{(a^2+b^2)^2+4(a^2+b^2)(a^2-b^2+\overline{a^2+b^2}\cos^2\phi)\cos 2\phi -16a^2b^2\cos^2 2\phi\}$$

$$= (a^2+b^2)^3+-\frac{a^2b^2}{p^2}\{(a^2+b^2)^2+4(a^2+b^2)(a^2-b^2)\cos 2\phi +4(a^2-b^2)^2\cos^2 2\phi\}$$

$$= (a^2+b^2)^3-\frac{a^2b^2}{p^2}(4p^2-a^2-b^2)^2.$$

Hence x^2+y^2 or CQ^2 can never exceed $\frac{(a^2+b^2)^3}{(a^2-b^2)^2}$, and can have this maximum value only when $4p^2 = a^2+b^2$, for the possibility of which $a^2>3b^2$. [If b^2 be negative, and a^2+b^2 positive, these values of CQ will always be possible, but will be minima.]

Again, $$(a^2-b^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

$$= \{a^2\cos^2\phi\,(a^2+b^2-4b^2\cos 2\phi)^2+b^2\sin^2\phi\,(a^2+b^2+4a^2\cos 2\phi)^2\}/p^2$$

$$= (a^2+b^2)^2-\frac{8a^2b^2}{p^2}(a^2+b^2)\cos^2 2\phi+\frac{16a^2b^2\cos^2 2\phi}{p^2}(a^2+b^2-p^2)$$

$$= (a^2+b^2)^2+\frac{8a^2b^2}{p^2}(a^2+b^2)\cos^2 2\phi-16a^2b^2\cos^2 2\phi$$

$$= (a^2+b^2)^2-\frac{8a^2b^2}{p^2}(a^2-b^2)\cos^3 2\phi;$$

or $$(a^2-b^2)^2\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{a^2+b^2}{a^2-b^2}\right)^2\right\} = -\frac{8a^2b^2}{p^2}(a^2-b^2)\cos^3 2\phi;$$

showing that the locus of Q *osculates* the locus of P in the four points for which $\cos 2\phi = 0$; *i.e.*, in the four points of contact of a square circumscribing the locus of P.

The tangents to the envelope at P, Q will of course intersect each other in a point (T) on the tangent at O, and will be equally inclined to this tangent, and, the tangents at O, P being equally inclined in opposite directions to the axis, we see that the angles which the normals at P, Q make with the axis will be respectively ϕ, 3ϕ (or $3\phi-\pi$). The coordinates of T are determined by the equations

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta = 1,\quad \frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta = \frac{a^2+b^2}{a^2-b^2},$$

whence we see that T is the pole of the normal at O with respect to the given conic. [Putting $a^2+b^2 = 0$, this gives a pretty property of the rectangular hyperbola.] The equation of the tangent to the envelope at Q will then be

$$(a^2-b^2)(x\cos 3\phi+y\sin 3\phi)$$
$$= -\frac{4a^2b^2}{p}\cos^2 2\phi+\frac{a^2+b^2}{p}(a^2\cos\phi\cos 3\phi-b^2\sin\phi\sin 3\phi)$$
$$= -\frac{4a^2b^2}{p}\cos^2 2\phi+\frac{a^2+b^2}{2p}(\overline{a^2-b^2}\cos 2\phi+\overline{a^2+b^2}\cos 4\phi)$$
$$= \frac{(a^2-b^2)^2}{p}\cos^2 2\phi-(a^2+b^2)\frac{(a^2\sin^2\phi+b^2\cos^2\phi)}{p}$$
$$= \frac{(2p^2-a^2-b^2)^2}{p}-\frac{(a^2+b^2)(a^2+b^2-p^2)}{p}=4p^3-3p(a^2+b^2),$$

or, expressing the second member in terms of ϕ,

$$= \sqrt{a^2\cos^2\phi+b^2\sin^2\phi}\,(\overline{a^2-3b^2}\cos^2\phi+\overline{b^2-3a^2}\sin^2\phi);$$

which equation proves that the curve is of the sixth class.

From this equation it appears that part of the *orthoptic locus* for the sextic is the circle

$$(a^2-b^2)^2(x^2+y^2)=(a^2+b^2)^3,$$

which is also the *orthoptic locus* for the ellipse the locus of P. This circle touches the sextic in four points (real only when a^2-3b^2 and a^2+b^2 are both positive), and the normals at these points are both *bitangents* and *binormals*, a peculiarity I do not remember to have met before in any curve.

The *orthoptic locus* is Dr. C. Taylor's name for the locus of the intersection of tangents at right angles to each other.

This curve, by Plücker's formulæ, will therefore have six nodes, six bitangents, four cusps, and four inflexions. When $a^2 > 3b^2$, the six bitangents are all real (two parallel to each axis and two through the centre); but only two of the nodes (on the major axis), there being two acnodes on the minor axis and two at infinity. When $a^2 < 3b^2$, four bitangents and four nodes are real, the others impossible. The cusps and inflexions appear to be always impossible when b^2 is positive. When b^2 is negative and a^2+b^2 positive, there are two real inflexions, but the cusps are impossible. When b^2 and a^2+b^2 are negative, but $5a^2+b^2$ positive, the cusps are all real, but all the other singularities impossible.

If r', p' be the central radius vector (CQ) and perpendicular on the tangent at Q, we have proved that

$$(a^2-b^2)^2r'^2=(a^2+b^2)^3-\frac{a^2b^2}{p^2}(4p^2-a^2-b^2)^2;$$

and
$$(a^2-b^2)\,p'=4p^3-3p\,(a^2+b^2).$$

Hence the radius of curvature at Q

$$= r'\frac{dr'}{dp'} = \frac{1}{a^2-b^2}\left\{16a^2b^2p - \frac{a^2b^2(a^2+b^2)^2}{p^3}\right\} \Big| (12p^2-3a^2-3b^2)$$

$$= \frac{a^2b^2}{(a^2-b^2)p^3}\,\frac{16p^4-(a^2+b^2)^2}{3(4p^2-a^2-b^2)} = \frac{a^2b^2}{3(a^2-b^2)}\,\frac{4p^2+a^2+b^2}{p^3},$$

and the cusps will occur when $4p^2+a^2+b^2=0$. Taking a^2 to be always positive, we see that, since $a^2 > p^2$ at all points of the hyperbola, $5a^2+b^2$ must be positive and a^2+b^2 negative for the cusps to be real; their coordinates being given by

$$x^2(a^2-b^2)^5 = a^4(a^2+5b^2)^3(a^2+b^2),$$

$$y^2(b^2-a^2)^5 = b^4(b^2+5a^2)^3(b^2+a^2);$$

from which the same results might be inferred. I am not quite clear in my mind as to the inflexions. There are two real inflexions when $a^2+b^2=0$, the locus of Q being then a lemniscate of Bernoulli. In general, the radius of curvature is infinite only when p is 0; and when $p=0$, which happens at the points at ∞ on the hyperbola, r' is also infinite, except in the single case when $a^2+b^2=0$. Thus it would seem that there are two real inflexions when the given conic is an hyperbola, but that these are both at infinity; except for the rectangular hyperbola, for which curve they are at the centre.

When the given conic is a parabola, the locus of Q is a curve of degree 4, and class 4, having a real node and real bitangent, and two impossible cusps. This case I have, however, fully investigated in the answer to a question in the *Educational Times*.

I have found the figure of the locus of Q tolerably easy to draw for the two essentially different forms when the given conic is an ellipse, (according as $a^2 >$ or $< 3b^2$); but cannot make a decent figure for the cases when b^2 and a^2+b^2 are negative. There are two points of a high singularity when $5a^2+b^2=0$,

$$x^2 = a^2\,\frac{24^3\times 4}{6^5},\ y=0;\ \ i.e.,\ \ x = \pm\frac{8a}{3},\ y=0.$$

I have thought the investigation of this envelope might be interesting, as the cases of envelopes breaking up in this way are rare. Of course it would be easy to devise envelopes for which it must happen: *e.g.*, a circle with its centre on a parabola and touching the axis will have for the remainder of its envelope a curve (tricusped quartic) which is the involute starting from the vertex of the first negative pedal of the parabola with respect to the focus; but, in the case considered in this paper, there seems no *à priori* reason why the envelope should sever.

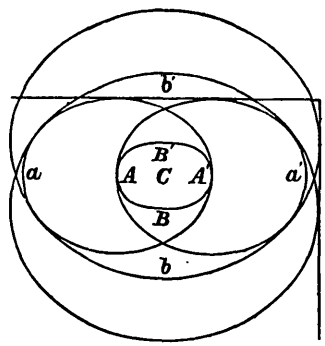


FIG. 1.

(1) a^2-b^2 positive, a^2-3b^2 negative; $a^2=2b^2$. The nodes

$$y=0,\ x=\pm\frac{(a^2+b^2)^{\frac{3}{2}}}{a^2-b^2}\sqrt{\frac{3a^2-b^2}{3a^2+b^2}},\quad x=0,\ y=\pm\frac{(a^2+b^2)^{\frac{3}{2}}}{a^2-b^2}\sqrt{\frac{3b^2-a^2}{3b^2+a^2}}$$

are real; and the bitangents

$$x=\pm\frac{a^2}{a^2-b^2}\sqrt{a^2+3b^2},\quad y=\pm\frac{b^2}{a^2-b^2}\sqrt{b^2+3a^2}$$

are real; the circular points at infinity are the two remaining nodes; and the bitangents through the centre $y=\pm x\frac{b^2}{a^2}\sqrt{\frac{3a^2-b^2}{a^2-3b^2}}$ are impossible. The four cusps and four inflexions are all impossible, AA', BB' the axes of the given ellipse; aa', bb', the axes of the elliptic envelope, osculating the sextic in four points.

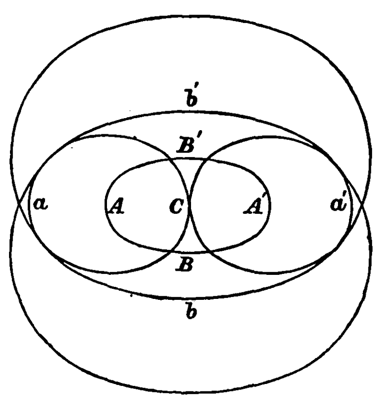


FIG. 2.

(2) $a^2=3b^2$. The nodes and bitangents as in (1) are real, but the nodes on the axis of y coincide in the centre; and the bitangents through the centre become real and coincident.

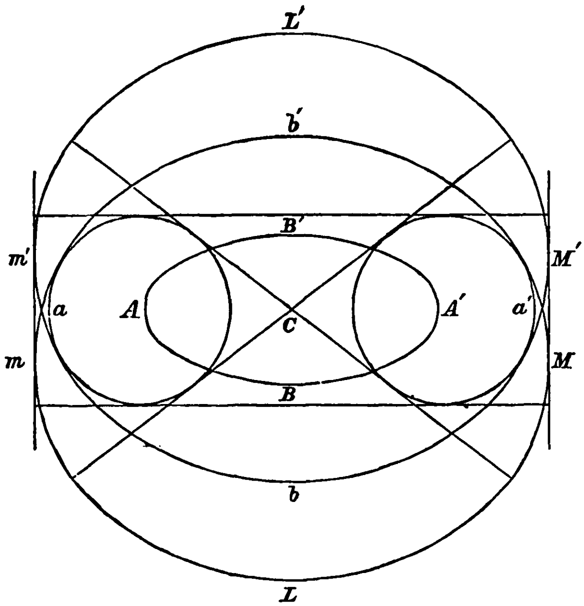


Fig. 3.

(3) $a^2 = 4b^2$. Six bitangents and two nodes are real,

$$AC = CA' = a, \quad BC = CB' = b = \tfrac{1}{2}a;$$
$$aC = Ca' = \tfrac{5}{3}a, \quad bC = Cb' = \tfrac{7}{3}b;$$
$$LC = CL' = \tfrac{11}{3}b.$$

The bitangents through the centre are also binormals. The figure is drawn for $a = 2b$, but these properties hold for all ellipses in which $a^2 > 3b^2$. The distance of the nodes from the centre is

$$\frac{(a^2+b^2)^{\frac{3}{2}}\sqrt{3a^2-b^2}}{(a^2-b^2)\sqrt{3a^2+b^2}};$$

that of the bitangents parallel to the minor axis is

$$\frac{a^2}{a^2-b^2}\sqrt{a^2+3b^2};$$

and that of the bitangents parallel to the major axis is

$$\frac{b^2}{a^2-b^2}\sqrt{3a^2+b^2}.$$

The points of contact M, M', m, m' should be nearer to the transverse axis than is shown in the figure, the distance being in general $\dfrac{b^2\sqrt{3}}{\sqrt{a^2+3b^2}}$.

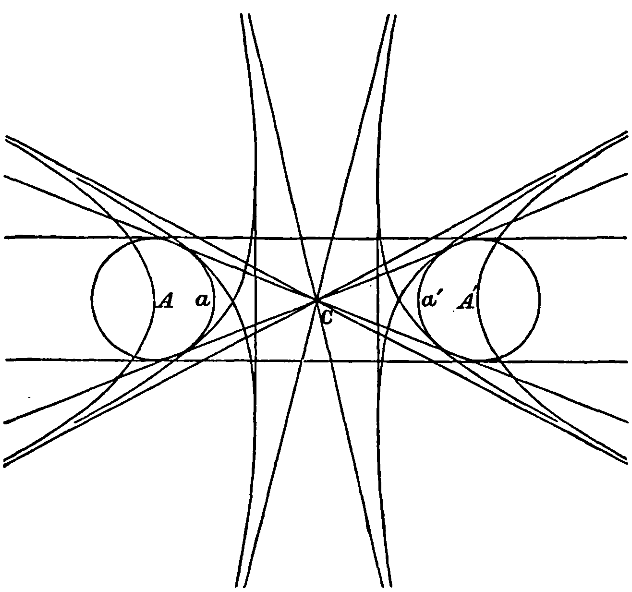


Fig. 4.

(4) b^2 negative, $a^2 + 3b^2$ positive. The locus of Q has six real bitangents and two real nodes; two real asymptotes. The bitangents through the centre are also binormals. $AC = CA' = a$; a, a' the vertices of the locus of P. As $a^2 + 3b^2$ approaches zero, the bitangents parallel to the conjugate axis and the asymptotes tend to coincidence in the conjugate axis. The figure is drawn for $a^2 + 4b^2 = 0$.

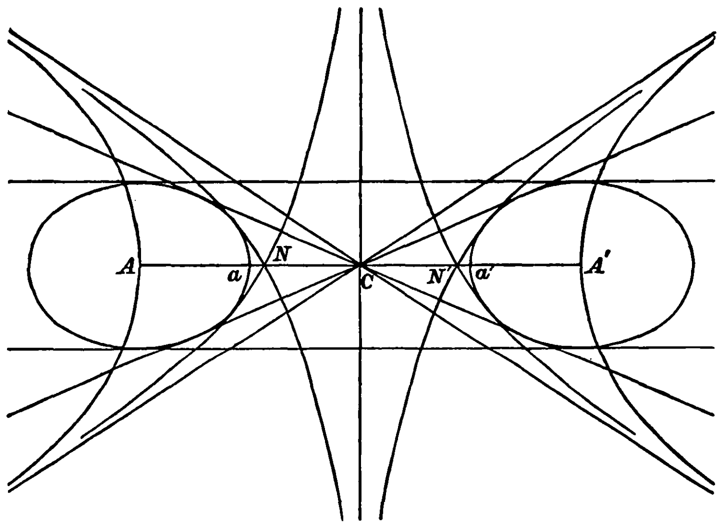
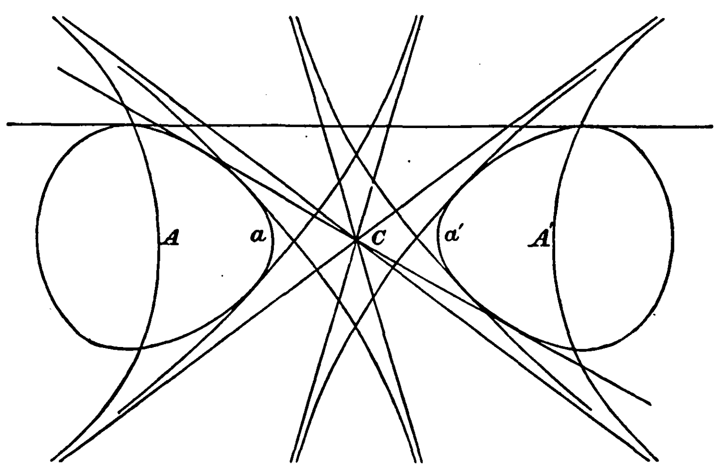


FIG. 5.

(5) $a^2 + 3b^2 = 0$. The bitangents parallel to the conjugate axis coalesce, their points of contact going to infinity; the other bitangents are real, and those through the centre are also binormals, $AC = CA' = a$, $aC = Ca' = \frac{1}{3}a$; $NC = CN' = a\sqrt{\frac{5}{24}}$.



FIGS. 6 & 7.

(6) $a^2 + 3b^2$ negative, $a^2 + b^2$ positive. Four nodes real; four bitangents real, the two parallel to the conjugate axis having become impossible, $AC = CA' = a$, $aC = Ca' = \frac{1}{3}a$. The bitangents through the centre real (only one is shown in the figure), and also binormals. The figure is drawn for $a^2 + 2b^2 = 0$.

(7) $a^2 + b^2 = 0$. There is no need of a figure in this case, the locus of P degenerating into the asymptotes, and the locus of Q being a lemniscate of Bernoulli whose foci coincide with the foci of the hyperbola. The infinite branch disappears and the curve degenerates into a quartic. When $a^2 + b^2$ becomes negative, four real cusps appear, and continue so long as $5a^2 + b^2$ is positive; and a complete system of figures would be for $b^2 + 2a^2 = 0$, $b^2 + 3a^2 = 0$, $b^2 + 4a^2 = 0$, $b^2 + 5a^2 = 0$, $b^2 + 6a^2 = 0$.

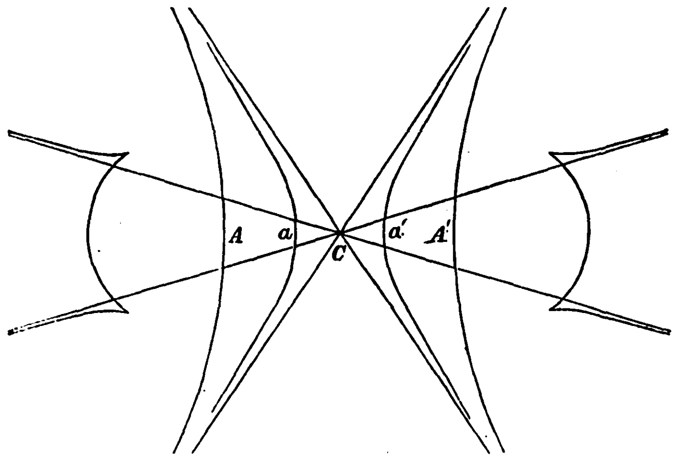


FIG. 8.

(8) a^2+b^2 negative, $3a^2+b^2$ positive; $(2a^2+b^2=0)$. The four cusps all real, the other singularities impossible; the two asymptotes

$$\frac{x}{a(a^2+3b^2)} = \pm \frac{y}{b\sqrt{-1}(b^2+3a^2)} \text{ real.}$$

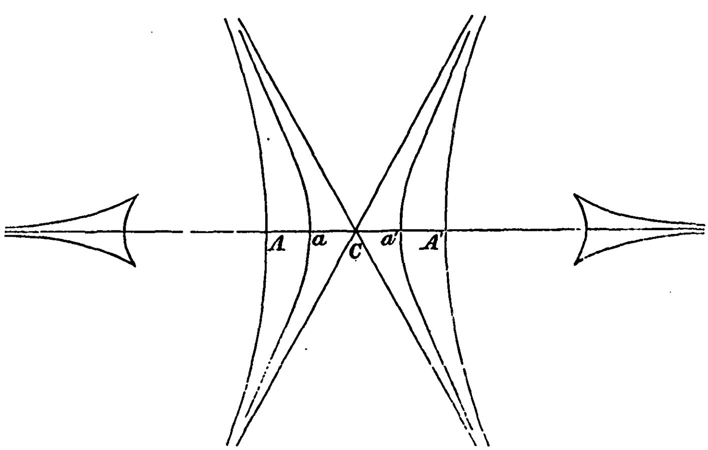


FIG. 9.

(9) $3a^2+b^2=$. Asymptotes become coincident with the axis of x.

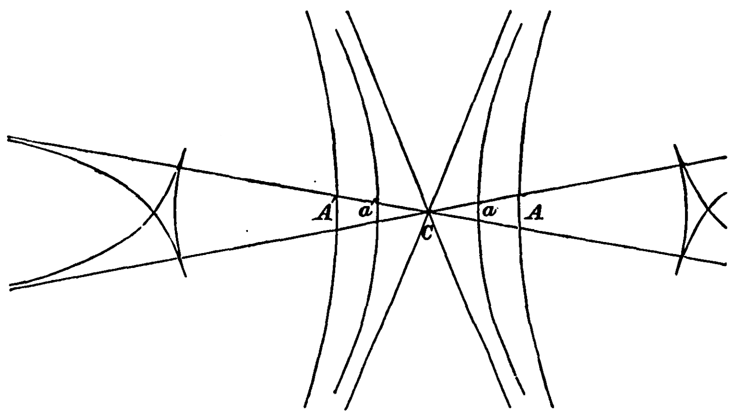


Fig. 10.

(10) $3a^2+b^2$ negative, $5a^2+b^2$ positive; $(b^2+4a^2=0)$. The four cusps real, and the nodes on the axis of x real.

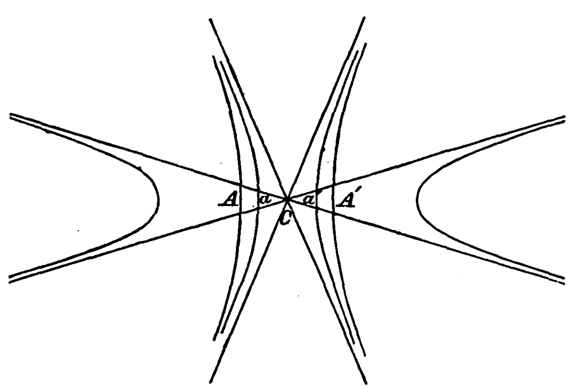


Fig. 11.

(11) $5a^2+b^2=0$. Two cusps and a node coincide at each vertex on the axis of x; each of which points is similar to the point on the curve $y^4=mx^3$ at the origin. From $b^2=-5a^2$ to $b^2=-\infty$, the visible features of the sextic are not much altered.